K. Khashyarmanesch

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On the Free Resolutions of Locally Cohomology Modules with Respect to an Ideal Generated by a U.S. D-Sequence

KAZEM KHASHYARMANESH

Mashad

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Let $\mathfrak{a}$ be an almost complete intersection ideal of a commutative Noetherian local ring R and r be the number of elements of a minimal generating set of $\mathfrak{a}$. Suppose that the ith local cohomology module $H^i_{\mathfrak{a}}(R)$ is finitely generated for all $i < r$. We show that there exists a sequence $\mathrm{x} = x_1, \ldots, x_r$ of elements in $\mathfrak{a}$ which is both an $\mathfrak{a}$-filter regular and u.s.d-sequence on R and

$$\Omega^{r-1}_R(H^{r-1}_{\mathfrak{a}}(R)) \cong \Omega^{r+1}_R(R/(\mathrm{x}))$$

where, for an R-module M, $\Omega^i_R(M)$ is the ith syzygy of M.

1. Introduction

Let R be a commutative ring and M be an R-module. For an ideal $\mathfrak{a}$ of R, we denote the ith local cohomology functor with respect to $\mathfrak{a}$ by $H^i_{\mathfrak{a}}(-)$. Also, for a minimal free resolution F of M, we set the ith syzygy of M by $\Omega^i_R(M)$, that is $\mathrm{Coker}\partial^F_{i+1}$.

There have been some works on the study of the syzygies of different modules. But there are not many papers concerning the syzygies of local cohomology modules. In this paper, under certain circumstances, we obtain some syzygies of local cohomology modules of ideal generated by u.s.d-sequence $x_1, \ldots, x_n$, in terms of the syzygies of $R/(x_1, \ldots, x_n)$. Clearly, if R is a d-dimensional Cohen-Macaulay local ring and $\mathrm{x} = x_1, \ldots, x_d$ is a system of parameters for R, a minimal free resolution $R/(\mathrm{x})$ is determined by a Koszul complex of d elements. Recently, in [8], Rahmati

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Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

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E-mail address: Khashyar@ipm.ir

proved that if R is a d-dimensional local ring with maximal ideal $\mathfrak{m}$, $d - \operatorname{depth} R \leq 1$ and $H_{\mathfrak{m}}^{d-1}(R)$ is finitely generated, then there exists an integer n such that for every system of parameters x for R contained in $\mathfrak{m}^n$,

$$\Omega_R^{d-1}(H_{\mathfrak{m}}^{d-1}(R)) \cong \Omega_R^{d+1}(R/(\mathrm{x})).$$

In this paper, by using natural generalizations of regular sequence which are called d-sequence and filter regular sequence, we show the following theorem.

Theorem: Let $\mathfrak{a}$ be an ideal of a local ring R such that $\operatorname{grade}(\mathfrak{a}, R) \geqslant r - 1$ where r is the number of elements of a minimal generating set of $\mathfrak{a}$. Also, suppose that $H_{\mathfrak{a}}^i(R)$ is finitely generated for all $i < r$. Then there exists a sequence $\mathrm{x} = x_1, \dots, x_r$ of elements in $\mathfrak{a}$ which is both an $\mathfrak{a}$-filter regular and u.s.d-sequence on R and

$$\Omega_R^{r-1}(H_{\mathfrak{a}}^{r-1}(R)) \cong \Omega_R^{r+1}(R/(\mathrm{x})).$$

So our result provides some information about the minimal free resolution of $H_{\mathfrak{a}}^{r-1}(R)$. Our original goal of this paper is to show that an u.s.d-sequence in $\mathfrak{a}$ which is an $\mathfrak{a}$-filter regular sequence is an excellent analogue of standard system of parameters and can be used for studying the syzygies of certain modules.

Throughout this paper, R will denote a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ an ideal of R. We shall use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers. Also M will denote a finitely generated R-module. Our terminology follows the textbook [1] on local cohomology.

2. The Results

Recall that a sequence $x_1, \dots, x_n$ of elements of R is called a d-sequence on M if, for each $i = 0, 1, \dots, n-1$, the equality

$$(\Sigma_{j=1}^{i} Rx_j)M :_M x_{i+1}x_k = (\Sigma_{j=1}^{i} Rx_j)M :_M x_k$$

holds for all $k \geqslant i + 1$ (this is actually a slight weakening of Huneke's definition in [5]); it is an unconditioned strong d-sequence (u.s.d-sequence) on M if $x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$ is d-sequence on M in any order for all $\alpha_1, \dots, \alpha_n \in \mathbb{N}$.

Also, we need another natural generalization of regular sequences which is called filter regular sequences. We say that a sequence $x_1, \dots, x_n$ of elements of $\mathfrak{a}$ is an $\mathfrak{a}$-filter regular sequence on M, if

$$\operatorname{Supp}_R\left(\frac{(x_1, \dots, x_{i-1})M :_M x_i}{(x_1, \dots, x_{i-1})M}\right) \subseteq V(\mathfrak{a})$$

for all $i = 1, \dots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$. The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the one of a filter regular sequence which has been studied in [11], [12], [6] and has led to some interesting results. Both concepts coincide if $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of a local ring with maximal ideal $\mathfrak{m}$. Note that $x_1, \dots, x_n$ is a weak M-sequence if and only if it is an R-filter regular sequence on M. It is easy to see that the analogue of [12, Appendix

2(ii)] holds true whenever R is Noetherian, M is finitely generated and $\mathfrak{m}$ replaced by $\mathfrak{a}$; so that, if $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M, then there is an element $y \in \mathfrak{a}$ such that $x_1, \ldots, x_n, y$ is an $\mathfrak{a}$-filter regular sequence on M. Thus, for a positive integer n, there exists an $\mathfrak{a}$-filter regular sequence on M of length n.

Lemma 2.1 (See [6, Proposition 1.2].) *Let* $\mathrm{x} = x_1, \ldots, x_n$ $(n > 0)$ *be an* $\mathfrak{a}$*-filter regular sequence on* M*. Then there are the following isomorphisms*

$$H^i_{\mathfrak{a}}(M) \cong \begin{cases} H^i_{(\mathrm{x})}(M) & \text{for } 0 \leqslant i < n, \\ H^{i-n}_{\mathfrak{a}}(H^n_{(\mathrm{x})}(M)) & \text{for } n \leqslant i. \end{cases}$$

Remarks 2.2 Let $\mathrm{x} = x_1, \ldots, x_n$ be an u.s.d-sequence on M.

(i) It is easy to see that any permutation of $x_1, \ldots, x_n$ is an (x)-filter regular sequence on M.

(ii) Moreover, $x_2, \ldots, x_n$ form an u.s.d-sequence on M/x_1M.

Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}M \neq M$. We denote the maximum length of all regular M-sequences in $\mathfrak{a}$ by $\operatorname{grade}(\mathfrak{a}, M)$. It is well-known that $\operatorname{grade}(\mathfrak{a}, M)$ is the least integer i such that $H^i_{\mathfrak{a}}(M) \neq 0$. Recall that the finiteness dimension $f_{\mathfrak{a}}(M)$ of M relative to $\mathfrak{a}$ is defined as follows.

$$f_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N} \mid H^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\}$$

In [7], the present author with Salarian showed that for an ideal $\mathfrak{a}$ of R, $f_{\mathfrak{a}}(M) = \ell$ if and only if there exists a sequence $x_1, \ldots, x_\ell$ in $\mathfrak{a}$ such that $x_1, \ldots, x_\ell$ is both $\mathfrak{a}$-filter regular and u.s.d-sequence on M.

Now, let $\mathrm{x} = x_1, \ldots, x_s$ be a sequence of elements of R and let $K(\mathrm{x}^t, M)$ denote the Koszul complex on $\mathrm{x}^t := x_1^t, \ldots, x_s^t$ with coefficients in M. Then, by [1, Theorem 5.2.9], one has

$$H^i_{(\mathrm{x})}(M) = \varinjlim_{t \in \mathbb{N}} H_{s-i}(K(\mathrm{x}^t, M)) \text{ for all } i \in \mathbb{N}_0.$$

Set $H^i(\mathrm{x}^t; M) = H_{-i}(\operatorname{Hom}_R(K(\mathrm{x}^t, R), M))$. Then, in view of [2, Theorem 3.5.6],

$$H^i_{(\mathrm{x})}(M) = \varinjlim_{t \in \mathbb{N}} H^i(\mathrm{x}^t; M) \text{ for all } i \in \mathbb{N}_0.$$

Hence, for all $t \in \mathbb{N}$, there exists a canonical map

$$\lambda_{\mathrm{x}^t;M} : H^i(\mathrm{x}^t; M) \longrightarrow H^i_{(\mathrm{x})}(M).$$

In the following proposition, we study the canonical map $\lambda_{\mathrm{x};M}$ in the case that x is an u.s.d-sequence on M.

Proposition 2.3 *Let* $\mathrm{x} = x_1, \ldots, x_s$ *be an u.s.d-sequence on* M *such that* $\operatorname{grade}((\mathrm{x}), M) \geqslant s - 1$. *Then the canonical map*

$$\lambda_{\mathrm{x};M} : H^{s-1}(\mathrm{x}; M) \longrightarrow H^{s-1}_{(\mathrm{x})}(M)$$

is an isomorphism.

Proof. We use induction on s, the length of the sequence. In the case when $s = 1$, since x_1 is a d-sequence on M, we have that

$$H^0_{(\mathrm{x})}(M) = H^0_{(x_1)}(M) = \cup_{t\in\mathbb{N}}(0 :_M x_1^t) = 0 :_M x_1 = H^0(\mathrm{x}; M).$$

Now, suppose inductively that $s > 1$ and the result has been proved for smaller values of s. By Remarks 2.2(i), $x_1, \ldots, x_s$ is an (x)-filter regular sequence on M and so, in view of Lemma 2.1, $H^0_{(\mathrm{x})}(M) \cong H^0_{(x_1)}(M)$. Also, since $\operatorname{grade}((\mathrm{x}), M) > 0$, by [1, Theorem 6.2.7], $H^0_{(\mathrm{x})}(M) = 0$. Hence x_1 is a non-zerodivisor on M. Moreover, since x is both (x)-filter regular sequence and u.s.d-sequence on M, by [7, Theorem A], $H^i_{(\mathrm{x})}(M)$ is finitely generated for $i < s$. Hence, in view of [3, Satz 1] and [6, Theorem (5)], there exists a positive integer ℓ such that

$$x_1^\ell H^{s-1}_{(\mathrm{x})}(M) = 0 = x_1^\ell H^{s-1}(\mathrm{x}; M).$$

Set $N := M/x_1^\ell M$ and $\mathrm{y} = x_1^\ell, \ldots, x_s$.

Then, by Remarks 2.2(ii), $x_2, \ldots, x_s$ is an u.s.d-sequence on N. Now, consider the exact sequence

$$0 \longrightarrow M \xrightarrow{x_1^\ell} M \longrightarrow N \longrightarrow 0$$

to obtain the commutative diagram

$$\begin{array}{ccccccc}
H^{s-2}_{(\mathrm{x})}(M) \cong H^{s-2}_{(\mathrm{y})}(M) & \longrightarrow H^{s-2}_{(x_2,\ldots,x_s)}(N) \cong H^{s-2}_{(\mathrm{x})}(N) & \longrightarrow & H^{s-1}_{(\mathrm{x})}(M) \cong H^{s-1}_{(\mathrm{y})}(M) & \xrightarrow{x_1^\ell} & 0 \\
 & \uparrow \lambda_{x_2,\ldots,x_s;N} & & \uparrow \lambda_{\mathrm{y};M} & & \\
 & H^{s-2}(x_2,\ldots,x_s;N) \cong H^{s-2}(\mathrm{y};N) & \longrightarrow & H^{s-1}(\mathrm{y};M) & \xrightarrow{x_1^\ell} & 0
\end{array}$$

in which the upper and lower rows are exact and by inductive hypothesis, the map $\lambda_{x_2,\ldots,x_s;N}$ is an isomorphism. Since $\operatorname{grade}((\mathrm{x}), M) \geqslant s - 1$, by [1, Theorem 6.2.7], $H^{s-2}_{(\mathrm{x})}(M) = 0$. Therefore $\lambda_{\mathrm{x};M}$ is an isomorphism and the result now follows by induction. □

Let $(R, \mathfrak{m})$ be a local ring and M be a non-zero finitely generated R-module of dimension $d > 0$. We say that M is a generalized Cohen-Macaulay R-module precisely when $H^i_{\mathfrak{m}}(M)$ is finitely generated for all $i \neq d$. (Such modules were called 'quasi-Cohen-Macaulay module' by Schenzel in [9, p.238].) Hence M is generalized Cohen-Macaulay if $f_{\mathfrak{m}}(M) = d$. On the other hand, a system of parameters $\mathrm{x} := x_1, \ldots, x_d$ for M is said to be standard if

$$(\mathrm{x})H^i_{\mathfrak{m}}(M/(x_1, \ldots, x_j)M) = 0,$$

for all $i, j \in \mathbb{N}_0$ with $i + j < d$ (cf. [10, Definition 3.1]). It follows from [13, Theorem 2.1 and Proposition 3.1] and [11, (3.7)] that M is generalized Cohen-Macaulay if and only if there exists a positive integer n such that every system of parameters of M in $\mathfrak{m}^n$ is standard.

Let $(R, \mathfrak{m})$ be an n-dimensional local ring and $\mathrm{x} = x_1, \ldots, x_n$ be a standard system of parameters for R. Then, by [10, Theorem A], x is an u.s.d-sequence on R. Suppose that $\operatorname{cmd} R \leqslant 1$ where $\operatorname{cmd} R$ is the Cohen-Macaulay defect of R, that is $\operatorname{cmd} R = \dim R - \operatorname{depth} R$. Hence $\operatorname{grade}((\mathrm{x}), R) \geqslant \mu((\mathrm{x})) - 1 = n - 1$ where, for an ideal $\mathfrak{a}$ $\mu(\mathfrak{a})$ is the number of elements of a minimal generating set of $\mathfrak{a}$. Recall that an ideal $\mathfrak{a}$ is an almost complete intersection if $\operatorname{grade}(\mathfrak{a}, R) \geqslant \mu(\mathfrak{a}) - 1$. Now, by using Proposition 2.3 in conjunction with Lemma 1.5 in [8], we have the following corollary.

Corollary 2.4 ([8, Theorem 3.1(i)].) *Let* $(R, \mathfrak{m})$ *be an n-dimensional local ring and* $\operatorname{cmd} R \leqslant 1$. *Let* x *be an standard system of parameters for R. Then*

$$\Omega_R^{n-1}(H_{\mathfrak{m}}^{n-1}(R)) \cong \Omega_R^{n+1}(R/(\mathrm{x})).$$

As we mentioned in the introduction, in the next theorem, we want to show that an u.s.d-sequence in $\mathfrak{a}$ which is an $\mathfrak{a}$-filter regular sequence is an excellent analogue of standard system of parameters.

Theorem 2.5 *Let* $\mathfrak{a}$ *be an almost complete intersection ideal of a local ring R and r be the number of elements of a minimal generating set of* $\mathfrak{a}$. *Suppose that* $H_{\mathfrak{a}}^i(R)$ *is finitely generated for all* $i < r$. *Then there exists a sequence* $\mathrm{x} = x_1, \ldots, x_r$ *of elements in* $\mathfrak{a}$ *which is both an* $\mathfrak{a}$*-filter regular and u.s.d-sequence on R and*

$$\Omega_R^{r-1}(H_{\mathfrak{a}}^{r-1}(R)) \cong \Omega_R^{r+1}(R/(\mathrm{x})).$$

Proof. Since r is the number of elements of a minimal generating set of $\mathfrak{a}$, by [1, Corollary 3.3.3], $H_{\mathfrak{a}}^i(R) = 0$ for all $i > r$. Hence, in view of [14, Proposition 3.1] in conjunction with [4, Remark 2.5], $H_{\mathfrak{a}}^r(R)$ is not finitely generated and so $r = f_{\mathfrak{a}}(R)$. Therefore, by [7, Theorem A], there exists a sequence $\mathrm{x} = x_1, \ldots, x_r$ of elements in $\mathfrak{a}$ which is both an $\mathfrak{a}$-filter regular and u.s.d-sequence on R. So, by Lemma 2.1,

$$H_{\mathfrak{a}}^i(R) \cong H_{(\mathrm{x})}^i(R) \text{ for all } i < r. \quad (*)$$

Since $\mathfrak{a}$ is an almost complete intersection, $\operatorname{grade}(\mathfrak{a}, R) \geqslant r - 1$. It follows from $(*)$ and [1, Theorem 6.2.7] that (x) is an almost complete intersection ideal of R. Therefore, by [8, Lemma 1.5], Proposition 2.3 and $(*)$,

$$\begin{aligned} \Omega_R^{r+1}(R/(\mathrm{x})) &\cong \Omega_R^{r-1}(H^{r-1}(\mathrm{x}; R)) \\ &\cong \Omega_R^{r-1}(H_{(\mathrm{x})}^{r-1}(R)) \\ &\cong \Omega_R^{r-1}(H_{\mathfrak{a}}^{r-1}(R)). \end{aligned}$$

□

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