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STANDARD AND NONSTANDARD REPRESENTABILITY OF POSITIVE UNCERTAINTY ORDERINGS

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Axioms are given for positive comparative probabilities and plausibilities defined either on Boolean algebras or on arbitrary sets of events. These axioms allow to characterize binary relations representable by either standard or nonstandard measures (i. e. taking values either on the real field or on a hyperreal field). We also study relations between conditional events induced by preferences on conditional acts.

Keywords: comparative probability, comparative plausibilities, hyperreal field, representability by nonstandard measures

Classification: 60A05, 62C10, 91B08

1. INTRODUCTION

In a comparative framework it is natural to require that every possible event E is strictly more likely than the impossible event $\emptyset$. This condition must be often weakened when we require that the binary relation is representable by a standard probability measure. In fact, as it is well known, for any partition of the sure event Ω at most a countable set of its elements can have positive probability. This is actually due to the numerical framework where probability takes values, i. e. the real field, which is Archimedean. The same problem obviously arises when we look for belief functions, which are envelopes of suitable classes of probability measures [40]. On the contrary, for a plausibility function (the dual function of belief), which is the upper envelope of a suitable class of probability measures, we can have positive functions for any cardinality of the Boolean algebra.

Therefore, to have a binary relation representable by a probability measure (belief function) on an uncountable Boolean algebra, there must be possible events E equally likely to the impossible event (i. e., $E \sim \emptyset$). Hence it is unavoidable to require only that every possible event E is at least as probable as the impossible one and that the sure event is strictly more probable than the impossible one (i. e. $\emptyset \preccurlyeq E$ and $\emptyset \prec \Omega$). For this reason Savage [39] introduced the concept of null events identified through the condition that any two acts are indifferent (and so indistinguishable) when restricted to one of such events.

Nevertheless also in the case that, like for plausibility, it is possible to consider positive functions for all the events of an infinite Boolean algebra, for assuring representability

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it is necessary to impose some Archimedean axiom to the comparative structure (see Example 4.8).

Moreover, the problem of positivity occurs even when the set of events is countable or finite. In fact, the same problem can turn up also in a dynamic situation in which we start from a possibly finite set of random quantities (acts) and we consider as initial set of events only those induced by these random quantities. Even when we require all these events E_i to be not-null (i. e. $\emptyset \prec E_i$) and the comparative probability to be representable by a coherent (positive) probability assessment, it is not sure that the comparative probability can be extended to new events as a comparative probability representable by a positive probability (see Example 4.2).

In order to have $\emptyset \prec E$, for any possible event E, it is necessary to resume different representability: either through a probability with values on a hyperreal field $\mathbb{R}^* \supsetneq \mathbb{R}$ or through a conditional probability (taking values in $\mathbb{R}^*$ or in $\mathbb{R}$).

In this paper we study the representability of strictly positive binary relations by different uncertainty functions (probabilities, plausibilities and belief) whose values are taken either in the real field or in a hyperreal field, both in the finite and infinite case.

In order to characterize relations representable by a plausibility [belief], we compare conditional plausibility [belief] functions taking values in $\mathbb{R}$ with those taking values in $\mathbb{R}^*$. Similarly to what Krauss already proved in the probabilistic framework [31], we prove that every conditional plausibility [belief] can be represented by a not unique strictly positive plausibility [belief] with values in $\mathbb{R}^*$.

In the literature related to the expected utility framework, several attempts to generalize Savage axioms to deal with *possible negligible (null)* events, partial assessments and dynamic decision have been proposed, see e. g. [2, 3, 5, 22, 25, 26, 32, 41]. Dynamism in fact can be reduced to reason about classes of preferences, each relation being conditioned to a specific scenario (information). This can be formalized through conditional preference relations. In this contribution we bring to the light peculiarities of conditional binary relations with specific representability requirements. We study the representability problem of partial conditional relations by referring either to nonstandard conditional probabilities and plausibilities or to standard conditional probabilities and plausibilities.

2. NUMERICAL MODELS OF REFERENCE

We adopt the following concept of event: an event E is any fact singled-out by a (non-ambiguous) proposition, that can be either true or false. Among the events Boolean operations of disjunction $\vee$ and conjunction $\wedge$ are considered. Note that any Boolean algebra is in a one-to-one correspondence (by Stone's theorem) with a Boolean algebra of subsets of a given set, however a set is actually composed of elements (or points), and so its subdivision into subsets necessarily stops when the subdivision reaches its constituent points; on the contrary, events allow to go on in the subdivision by defining suitable new events, singled-out by further relevant propositions. Two particular events are the certain event (the top element) Ω and the impossible event (the bottom element) $\emptyset$. Moreover, by an abuse of notation, taking into account the analogy with sets, we use the symbol E^c to denote the contrary event of E and we denote by the symbol $E \subseteq F$ the assertion: the events E, F are such that $E^c \vee F = \Omega$.

Often, expecially in decision making, we need to consider arbitrary families $\mathcal{E}$ of events, so it is not required to assume that there is a given specific structure for the family $\mathcal{E}$, even if we could consider the generated free algebra $\langle\mathcal{E}\rangle$ as reference structure.

In this section we briefly recall the numerical models that could be used as reference for the representability of binary relations (see Sect. 4). We focus on conditional probability and plausibility.

We need to introduce the coherence notion characterizing functions on arbitrary domains which are partial assessments of an uncertainty measure. Moreover, in order to overcome the poorness of the reals we deal with uncertainty measures with hyperreal values.

2.1. Conditional probabilities

In what follows, $\mathcal{B}\times\mathcal{H}$ denotes a set of conditional events with $\mathcal{B}$ a Boolean algebra and $\mathcal{H}$ an additive set (i. e., closed with respect to finite disjunctions), such that $\mathcal{H}\subseteq\mathcal{B}^0$, where $\mathcal{B}^0=\mathcal{B}\setminus\{\emptyset\}$.

Definition 2.1. A function $P:\mathcal{B}\times\mathcal{H}\to[0,1]$ is a conditional probability if it satisfies the following conditions:

(i) $P(E|H)=P(E\wedge H|H)$, for every $E\in\mathcal{B}$ and $H\in\mathcal{H}$;

(ii) $P(\cdot|H)$ is a (finitely additive) probability on $\mathcal{B}$, for any $H\in\mathcal{H}$;

(iii) $P(E\wedge F|H)=P(E|H)\cdot P(F|E\wedge H)$, for any $H, E\wedge H\in\mathcal{H}$ and $E,F\in\mathcal{B}$.

In [35] condition *(ii)* is replaced by the stronger one of countable additivity.

It is well known that a conditional probability is not representable by just a probability, but a (not necessarily unique) class of charges [21] (or a class of countably additive measures when *(ii)* is replaced by countable additivity [35]).

A function $\mu:\mathcal{B}\to[-\infty,\infty]$ is said to be a charge on $\mathcal{B}$ if $\mu(\emptyset)=0$ and μ is finitely additive.

Moreover a charge is *real* if $-\infty<\mu(F)<\infty$ for any $F\in\mathcal{B}$, it is *bounded* if $\sup\{|\mu(F)|:F\in\mathcal{B}\}<\infty$, and it is *positive* if $\mu(F)\geq 0$ for any $F\in\mathcal{B}$. A charge is a *probability* if it is positive and $\mu(\Omega)=1$.

The aforementioned class of charges is unique when the conditional probability P is full [21, 31], that means that P is defined on $\mathcal{B}\times\mathcal{B}^0$.

2.2. Conditional plausibility and belief

A real-valued function g on a Boolean algebra $\mathcal{B}$ is n-alternating (alternating of order n) if for every finite collection $A_1,\ldots,A_n$ of events in $\mathcal{B}$

$$g\left(\bigwedge_{i=1}^{n}A_i\right)\leq\sum_{\emptyset\neq I\subset\{1,\ldots,n\}}(-1)^{|I|+1}g\left(\bigvee_{i\in I}A_i\right)\tag{1}$$

where $|\cdot|$ indicates the cardinality function. A real-valued function g on a Boolean algebra $\mathcal{B}$ is said *totally alternating* (∞-alternating) if it is n-alternating for each $n\geq 1$.

A plausibility function Pl on a Boolean algebra $\mathcal{B}$ is a function totally alternating such that $Pl(\emptyset) = 0$ and $Pl(\Omega) = 1$ [19, 40].

A belief function on a Boolean algebra $\mathcal{B}$ is the dual function of a plausibility function, i. e. for any $B \in \mathcal{B}$

$$Bel(B) = 1 - Pl(B^c).$$

Then belief functions are totally monotone, that is for any $n \geq 1$ inequality (1) holds by exchanging disjunctions and conjunctions and by reverting inequality.

In the literature there are many definitions of conditioning for belief and plausibility functions, we recall the following axiomatic definition (see [9, 10]):

Definition 2.2. Let $\mathcal{B}$ be a Boolean algebra and $\mathcal{H} \subseteq \mathcal{B} \backslash \{\emptyset\}$ an additive set. A function Pl defined on $\mathcal{B} \times \mathcal{H}$ is a conditional plausibility if it satisfies the following conditions

(i) $Pl(E|H) = Pl(E \wedge H|H)$;

(ii) $Pl(\cdot|H)$ is a plausibility function for any $H \in \mathcal{H}$;

(iii) For every $E \in \mathcal{B}$ and $H, K \in \mathcal{H}$

$$Pl(E \wedge H|K) = Pl(E|H \wedge K) \cdot Pl(H|K).$$

Moreover, given a conditional plausibility Pl, a conditional belief function $Bel(\cdot|\cdot)$ is defined by duality as follows: for every event $E|H \in \mathcal{B} \times \mathcal{H}$

$$Bel(E|H) = 1 - Pl(E^c|H). \tag{2}$$

We say that a conditional plausibility (belief) is *full* on $\mathcal{B}$ if it is defined on $\mathcal{B} \times \mathcal{B}^0$.

The axiomatic definition of conditional belief (see equation (2)) extends the Dempster's rule of conditioning (formula 4.2 in [18]), i. e. for all conditioning events H such that $Pl(H) > 0$

$$Bel(F|H) = 1 - \frac{Pl(F^c \wedge H)}{Pl(H)}.$$

An easy consequence of Definition 2.2, in particular of 2-alternance for any conditioning event H, is a weak form of "disintegration formula" for a conditional plausibility of an event $E|H$ with respect to a partition $H_1, ..., H_N$ of H

$$Pl(E|H) \leq \sum_{k=1}^{N} Pl(H_k|H)Pl(E|H_k) \tag{3}$$

Notice that for a conditional belief an equivalent weak disintegration formula holds.

Among the definitions of conditional belief present in the literature (see for instance [1, 28, 40, 42]) we recall the conditional belief obtained through the product rule, i. e.

$$Bel_P(E \wedge F|H) = Bel_P(E|F \wedge H)Bel_P(F|H)$$

for any $E \in \mathcal{B}$ and $F \wedge H, H \in \mathcal{H}$. For this conditional belief a weak form of disintegration formula holds: given $E|H$ and a partition $H_1, \ldots, H_N$ of H

$$Bel_P(E|H) \geq \sum_{k=1}^{N} Bel_P(H_k|H)Bel_P(E|H_k). \tag{4}$$

On the other hand, the dual conditional function of a conditional belief defined through product rule, satisfies neither axiom (iii) of Definition 2.2 nor inequality (3).

A comparison of these conditioning rules from a comparative point of view is carried out in [13].

Obviously, as it occurs for conditional probability, a conditional plausibility is not representable by only one (unconditional) plausibility. When the conditional plausibility Pl is full on a finite Boolean algebra $\mathcal{B}$, there is a unique suitable class of plausibilities on $\mathcal{B}$ giving out a chain representation of Pl, as proved in the following result:

Theorem 2.3. Let $\mathcal{B}$ be a finite Boolean algebra and $\mathcal{C}_{\mathcal{B}}$ be its set of atoms. For a function $g : \mathcal{B} \times \mathcal{B}^0 \to [0,1]$ the following statements are equivalent:

a) g is a full conditional plausibility;

b) there is a class of plausibilities $\{Pl_0, \ldots Pl_k\}$ $(k \leq |\mathcal{C}_{\mathcal{B}}|-1)$ on $\mathcal{B}$ such that supports S_i of Pl_i are a partition of the set $\mathcal{B}^0$ and for any $i = 0, \ldots, k$ and any pair of events $E, H \subseteq \bigcup_{j=i}^{k} S_j$

$$Pl_i(E \wedge H) = g(E|H)Pl_i(H). \tag{5}$$

Proof. *a)* $\to$ *b)* Suppose that g is a full conditional plausibility, the function $Pl_0(\cdot) = g(\cdot|\Omega)$ is a plausibility with support $S_0 = \{H \in \mathcal{B}^0 : g(H|\Omega) > 0\}$.
Consider $H_0^0 = \vee\{H : H \in \mathcal{B}^0 \setminus S_0\}$ by formula (3) $pl(H_0^0) = 0$. If H_0^0 is different from $\emptyset$, let $Pl_1(\cdot) = g(\cdot|H_0^0)$ build S_1 analogously to S_0 in the previous step so that $H_1^0 = \vee\{H : H \in \mathcal{B}^0 \setminus (S_0 \bigcup S_1)\}$. Proceed in this way till H_k^0 is $\emptyset$ (that means that for any $H \in \mathcal{B}^0$ there is a unique i such that $H \in S_i$). Thus, the sets S_i are a partition of $\mathcal{B}^0$. Moreover, take $E, H \subseteq \bigvee_{j=i}^{k} S_j$ so that for $i > 0$

$$Pl_i(E \wedge H) = g(E \wedge H|H_0^{i-1}) = g(E|H)g(H|H_0^{i-1}) = g(E|H)Pl_i(H);$$

while for $i = 0$

$$g(E \wedge H|\Omega) = Pl_0(E \wedge H) = g(E|H)Pl_0(H) = g(E|H)g(H|\Omega).$$

b)$\to$ *a)* Consider a class of plausibilities $\{Pl_0, \ldots Pl_k\}$ as in *b)*, we need to check that g is a full conditional plausibility. Axiom *(i)* of Definition 2.2 follows easily. Let $H \in \mathcal{B}^0$, then there is a unique $i \in \{0, \ldots, k\}$ such that $Pl_i(H) > 0$ and from equation (5) we have that $g(E|H) = \frac{Pl_i(E \wedge H)}{Pl_i(H)}$ for any $E \in \mathcal{B}$, so axiom *(ii)* of Definition 2.2 holds.

Concerning axiom *(iii)*: for any $F \wedge H, H \in \mathcal{B}^0$ and $E \in \mathcal{B}$ consider the indexes i and j in $\{0, \ldots, k\}$ such that $Pl_i(F \wedge H) > 0$ and $Pl_j(H) > 0$, then $i \geq j$. If $i > j$, then $Pl_j(E \wedge H) = 0$ and from equation (5) $g(E \wedge F|H) = g(F|H) = 0$ so the thesis follows. Otherwise $(i = j)$,

$$g(E \wedge F|H) = \frac{Pl_i(E \wedge F \wedge H)}{Pl_i(H)} = \frac{Pl_i(E \wedge F \wedge H)}{Pl_i(F \wedge H)} \frac{Pl_i(F \wedge H)}{Pl_i(H)} = g(E|F \wedge H)g(F|H).$$

□

The class of plausibilities in condition *b)* of Theorem 2.3 is said to agree with g. The fullness requirement for g is fundamental for the unicity of the agreeing class. The existence of at least an agreeing class is a consequence of the following Theorem 2.4, which shows, in particular, that every conditional plausibility on $\mathcal{B}\times\mathcal{H}$ can be extended (not in a unique way) to a full conditional plausibility on $\mathcal{B}$. Actually, the result in Theorem 2.4 is more general and the aforementioned part has been already proved in [14]; for completeness we give here all the proof.

Theorem 2.4. *Let $\mathcal{H}$ be an additive set, $\mathcal{B}$ and $\mathcal{A}$ two finite Boolean algebras with $\mathcal{H}\subset\mathcal{B}\subset\mathcal{A}$. If $Pl:\mathcal{B}\times\mathcal{H}\to[0,1]$ is a conditional plausibility, then there exists a full conditional plausibility $Pl^\sharp$ on $\mathcal{A}$ such that its restriction $Pl^\sharp_{|\mathcal{B}\times\mathcal{H}}=Pl$.*

Proof. Let $H_0^0=\bigvee_{H\in\mathcal{H}}H$. Firstly we prove that Pl is extendible as full conditional plausibility on $\mathcal{B}$.

The function $Pl(\cdot|H_0^0)$ univocally defines the extension $Pl'(\cdot|\cdot)$ to any $E|H\in\mathcal{B}\times\mathcal{B}^0$ such that $Pl(H|H^0)>0$ trough *(iii)* of Definition 2.2.

Let $\mathcal{H}_0^1=\{H\in\mathcal{B}^0: Pl(H|H_0^0)=0\}$ and $H_0^1=\bigvee_{H\in\mathcal{H}_0^1}H$ belonging to $\mathcal{B}^0$.

If $H_0^1\in\mathcal{H}$, Pl' is univocally defined for any $E|H\in\mathcal{B}\times\mathcal{B}^0$ such that $Pl(H|H_0^1)>0$, so we can proceed as before.

If $H_0^1\notin\mathcal{H}$ check whether the set $\mathcal{K}=\{H\in\mathcal{H}: Pl(H|H_0^1)>0\}$ is not empty. If it is not empty, consider in $\mathcal{H}$ the event $K_1=\bigvee_{H\in\mathcal{K}}H\subseteq H_0^1$ and define $Pl'(E|H_0^1)=Pl(E|K_1)$ for any $E\in\mathcal{B}$. Note that $Pl'(K_1|H_0^1)=1$, $Pl'(K_1^c|H_0^1)=0$ and $Pl'(\cdot|H_0^1)$ is a plausibility since $Pl(\cdot|K_1)$ is.

Otherwise if $\mathcal{K}$ is empty define $Pl'(E|H_o^1)=1$ for any $E\in\mathcal{B}$ such that $E\wedge H_0^1\neq\emptyset$. It is easy to check that even in this case $Pl'(\cdot|H_0^1)$ is a plausibility.

Now, define $\mathcal{H}_0^2=\{H\in\mathcal{B}^0: Pl(H|H_0^1)=0\}$ and proceed as before.

It is easy to check that Pl' satisfies the axioms *(iii)* of Definition 2.2 and so it is a conditional plausibility (see the proof of Theorem 2.3).

Now, we need to prove that the full conditional plausibility Pl' on $\mathcal{B}$ is extendible as full conditional plausibility on any other finite Boolean algebra $\mathcal{A}\supseteq\mathcal{B}$. For any $A\in\mathcal{A}$ consider in $\mathcal{B}$ the event

$$A^*=\bigvee_{C\in\mathcal{B}:C\wedge A\neq\emptyset}C,$$

then for any $A,B\in\mathcal{A}$ one has $(A\wedge B)^*=A^*\wedge B^*$. Define $Pl''(A)=Pl(A^*)$ so that by construction $Pl''(A|B)=Pl'(A^*|B)$ for any $A|B\in\mathcal{A}\times\mathcal{B}^0$, so Pl'' is a conditional plausibility on $\mathcal{A}\times\mathcal{B}^0$. As shown in the first part of the proof the function Pl'' admits an extension $Pl^\sharp$ on $\mathcal{A}\times\mathcal{A}^0$ as a full conditional plausibility. □

2.3. Coherent conditional probability and plausibility assessments

Any uncertainty function is defined on a domain with a specific Boolean structure thus, in order to remove any restriction on the domain, we go back to the concept of coherence, originally introduced by de Finetti for (finitely additive) probabilities [17]. Although different equivalent definitions can be given for specific functions, for the aim of this paper we recall the following one:

Definition 2.5. An assessment φ on a set $\mathcal{E} = \{E_i|H_i\}_{i\in I}$ of conditional events is a *coherent* conditional probability [plausibility] if there exists a conditional probability [plausibility] $\varphi' : \mathcal{B} \times \mathcal{H} \to [0,1]$ (with $\mathcal{B}$ the algebra generated by $\{E_i, H_i\}_{i\in I}$ and $\mathcal{H}$ the additive set generated by $\{H_i\}_{i\in I}$) whose restriction to $\mathcal{E}$ coincides with φ.

In [8] a characterization of coherent conditional probability, defined on a finite set of conditional events $\mathcal{E}$ ($\mathcal{E}$ does not generally coincides with $\mathcal{B} \times \mathcal{H}$), is provided in terms of a class of probabilities. Moreover, in [11] it is shown that coherence, with respect to a conditional probability, can be checked by proving the coherence on any finite subset. For the sake of completeness we recall here the latter result:

Theorem 2.6. Let $\mathcal{E} = \{E_i|H_i \,:\, i \in I\}$ be an arbitrary family of conditional events and $\mathcal{B}_0 = \langle\{E_i, H_i \,:\, i \in I\}\rangle$. For a real function $P : \mathcal{E} \to [0,1]$ the following statements are equivalent:

(a) P is a *coherent* conditional probability on $\mathcal{E}$;

(b) there exists (at least) a class of charges $\{\mu_\alpha\}$, such that μ_0 is defined on $\mathcal{B}_0$ and μ_α ($\alpha \geq 1$) is defined on $\mathcal{B}_\alpha = \langle\{E_i, H_i \,:\, i \in I \text{ and } \mu_{\alpha-1}(H_i) = 0\}\rangle$; moreover for any conditional event $E|H \in \mathcal{E}$ one has that $P(E|H)$ is a solution of all the equations

$$\mu_\beta(E \wedge H) = x\mu_\beta(H) \tag{6}$$

with $H \in \mathcal{B}_\beta$, moreover there exists a unique α such that $0 < \mu_\alpha(H) < \infty$ and $P(E|H)$ is the unique solution of the corresponding equation (6);

(c) for any finite subset $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ of $\mathcal{E}$, denoting by $\mathcal{A}_o$ the set of atoms A_r generated by the events $E_1, H_1, \ldots, E_n, H_n$, there exists (at least) a *class* of coherent probabilities $\{P_0, P_1, \ldots P_k\}$, each probability P_α being defined on a suitable subset $\mathcal{A}_\alpha \subseteq \mathcal{A}_0$, such that for any $E_i|H_i \in \mathcal{E}$ there is a unique P_α with

$$\sum_{\substack{r \\ A_r\subseteq H_i}} P_\alpha(A_r) > 0 \qquad \text{and} \qquad P(E_i|H_i) = \frac{\sum_{\substack{r \\ A_r\subseteq E_i\wedge H_i}} P_\alpha(A_r)}{\sum_{\substack{r \\ A_r\subseteq H_i}} P_\alpha(A_r)}\ ;$$

moreover $\mathcal{A}_{\alpha'} \subset \mathcal{A}_{\alpha"}$ for $\alpha' > \alpha"$ and $P_{\alpha"}(A_r) = 0$ iff $A_r \in \mathcal{A}_{\alpha'}$.

Any class $\{P_\alpha\}$ singled-out by condition *(c)* is said *to agree* with the coherent conditional probability P restricted to the family $\mathcal{F}$.

Concerning coherence, another fundamental result is the following, essentially due to de Finetti [17] even originally referred to unconditional events and to an *equivalent* form of coherence in terms of a betting scheme (see also [27, 36, 44]).

Theorem 2.7. Let $\mathcal{K}$ be any family of conditional events, and take an *arbitrary* family $\mathcal{E} \subseteq \mathcal{K}$. Let P be an assessment on $\mathcal{E}$; then there exists a (possibly not unique) coherent conditional probability extension of P to $\mathcal{K}$ *if and only if* P is coherent conditional probability on $\mathcal{E}$.

From Theorems 2.3 and 2.4 the following characterization of coherent conditional plausibility functions in terms of a class of plausibilities arises:

Theorem 2.8. Let $\mathcal{F} = \{E_1|H_1, \dots, E_m|H_m\}$, $\mathcal{B}_0 = \{E_1, H_1, \dots, E_m, H_m\}$ and $H_0^0 = \bigvee_{j=1}^m H_j$. For $Pl : \mathcal{F} \to [0,1]$ the following statements are equivalent:

(a) Pl is a coherent conditional plausibility;

(b) there exists (at least) a class $\mathcal{L} = \{Pl_0, \dots Pl_k\}$ $(k < m)$ of coherent plausibilities with Pl_0 on $\mathcal{B}_0$ and Pl_α $(\alpha = 1, \dots, k)$ on $\mathcal{B}_\alpha = \{E_i, H_i \in \mathcal{B}_{\alpha-1} : Pl_{\alpha-1}(H_i) = 0\}$ such that for any conditional event $E|H \in \mathcal{F}$ one has that $Pl(E|H)$ is a solution of all the equations

$$Pl_\beta(E \wedge H) = x Pl_\beta(H) \tag{7}$$

with $H \in \mathcal{B}_\beta$, moreover there exists a unique α such that $Pl_\alpha(H) > 0$ and $Pl(E|H)$ is the unique solution of the corresponding equation (7);

(c) all the following systems (S^α), with $\alpha = 0, \dots, k \leq m$, admit a solution $\mathbf{x}^\alpha = (\mathbf{x}_1^\alpha, \dots, \mathbf{x}_{j_\alpha}^\alpha)$ where j_α is the cardinality of $\langle \mathcal{B}_\alpha \rangle$ and (for $\alpha = 1, \dots, k$) $\mathcal{B}_\alpha = \{E_i, H_i \in \mathcal{B}_{\alpha-1} : \sum_{F_k \in \langle \mathcal{B}_{\alpha-1} \rangle : F_k \wedge H_i \neq \emptyset} \mathbf{x^{\alpha-1}}_k = 0\}$:

$$(S^\alpha) = \begin{cases} \sum\limits_{F_k \wedge H_i \neq \emptyset} x_k^\alpha \cdot Pl(E_i|H_i) = \sum\limits_{F_k \wedge E_i \wedge H_i \neq \emptyset} x_k^\alpha, & \forall H_i \subseteq K_0^\alpha \\ \sum\limits_{F_k \subseteq K_0^\alpha} x_k^\alpha = 1 & \\ x_k^\alpha \geq 0, & \forall F_k \subseteq K_0^\alpha \end{cases}$$

with $K_0^\alpha = \bigvee \{F_k \in \mathcal{B}_\alpha : \mathbf{x^{\alpha-1}}_k = 0\}$.

In particular, conditions *(b)* and *(c)* stress that this conditional assessment can be written in terms of a suitable class of unconditional plausibilities Pl_α or basic assignments m_α (with $m_\alpha(F_j) = \mathbf{x}^\alpha{}_j$).

Note that every class $\mathcal{L}$ (condition *(b)* of Theorem 2.8) is said to agree with conditional plausibility Pl.

From Theorem 2.4 it follows a result analogous to Theorem 2.7 for coherent conditional probabilities:

Corollary 2.9. Let $\mathcal{K}$ be any family of conditional events, and take an arbitrary family $\mathcal{E} \subseteq \mathcal{K}$. Let Pl be an assessment on $\mathcal{E}$; then there exists a (possibly not unique) coherent conditional plausibility extending Pl on $\mathcal{K}$ if and only if Pl is coherent conditional plausibility on $\mathcal{E}$.

2.4. Nonstandard measures

Let us preliminarily state some well-known notions about hyperreal fields $\mathbb{R}^*$, which are completely ordered non-Archimedean fields, extending the real one $\mathbb{R}$ [38]. Let us denote by $\mathcal{I}$ the set of *infinitesimals* of $\mathbb{R}^*$, that is the set of $\xi \in \mathbb{R}^*$ such that, for every natural number $n > 0$ one has $0 < |\xi| < 1/n$. Let $\mathcal{F}$ be the set of *finite values*, i. e. the set of $\xi \in \mathbb{R}^*$ such that for some $n > 0, |\xi| < n$. It results that $\mathbb{R} \subseteq \mathcal{F} \subset \mathbb{R}^*$, with $\mathcal{F}$ an integral domain, $\mathcal{I}$ a maximal ideal in $\mathcal{F}$ and $\mathcal{F}/\mathcal{I} \cong \mathbb{R}$. As usual, for every $\xi \in \mathcal{F}$, $Re[\xi] \in \mathbb{R}$ denotes the *real part* of ξ: for every $\xi, \eta \in \mathcal{F}$ we have $Re[\xi+\eta] = Re[\xi]+Re[\eta]$, $Re[\xi \cdot \eta] = Re[\xi]\, Re[\eta]$, while $Re[\xi] = 0$ for any $\xi \in \mathcal{I}$.

We finally recall that hyperreal fields $\mathbb{R}^*$ satisfy the transfer principle which states that *valid first order logic statements about* $\mathbb{R}$ *are also valid in* $\mathbb{R}^*$.

As already mentioned, a conditional probability is characterized by a class of probabilities [31]. To avoid to refer to a class of measures, Krauss [31] suggested to use as a bridge a finitely additive strictly positive probability on $\mathcal{B}$ taking values on a hyperreal field $\mathbb{R}^*$ (nonstandard probability).

Starting from a positive nonstandard probability p^*, the conditional nonstandard probability P^* on $\mathcal{B} \times \mathcal{B}^0$ taking values on $\mathbb{R}^*$ is defined by

$$P^*(E|H) = \frac{p^*(E \wedge H)}{p^*(H)}.$$

Note that P^* satisfies the axioms in Definition 2.1 since they involve only finite numbers of events.

From our point of view, countable additivity for nonstandard probabilities is not so meaningful. This is due to the fact that in $\mathbb{R}^*$ a bounded nondecreasing sequence does not have necessarily a unique least upper bound, and this creates operational troubles especially for extension procedures. Thus, in the following we refer only to (conditional) finitely additive probabilities.

A standard conditional probability is obtained by considering the real part of P^*, i. e. the function P defined for any $E|H \in \mathcal{B} \times \mathcal{B}^0$ as

$$P(E|H) = Re[P^*(E|H)].$$

Nevertheless it can be interesting, especially in a qualitative framework, to consider the two different structures induced by standard and nonstandard probability.

In [25] a comparison of different probability spaces is carried out also by considering lexicographic probability spaces: in particular it has been proved that in the finite case lexicographic probability spaces are equivalent to nonstandard probability spaces, where equivalence means that the order induced by the expected values on real valued random variables is the same. However, this equivalence breaks down in the infinite case although considering countably additivity in spite of simple finite additivity. On the other hand, in the finite case, Example 5.3 of [25] shows that nonstandard probability spaces are not equivalent (in the above sense) to de Finetti probabilities with real values. Obviously, the equivalence is preserved by considering the standard part as done in [31].

As it arises from the proof of the above Theorems 2.3 and 2.4 (see also [9]) a conditional plausibility on $\mathcal{B} \times \mathcal{H}$ taking values on the real field cannot be described by just one plausibility, but by a class of plausibilities even when the Boolean algebra is finite. This class is univocally defined when the conditional plausibility is full: the class $\mathcal{L} = \{Pl_0, \dots Pl_k\}$ of plausibilities on $\mathcal{B}$ is such that

- $Pl_0(\cdot) = Pl(\cdot|\Omega)$,
- $Pl_\alpha(\cdot) = Pl(\cdot|H_0^\alpha)$ with $H_0^\alpha = \{H \in \mathcal{B}^0 \,:\, Pl_{(\alpha-1)}(H) = 0\}$ for $\alpha = 1, \ldots, k$.

An example showing how to compute the induced class of plausibilities by starting from a conditional plausibility can be found in [13].

To avoid to refer to a class of plausibilities, following Krauss approach [31] to conditional probability, we can define a nonstandard plausibility as a function

$$pl^* : \mathcal{B} \to [0,1]^*$$

with range in a hyperreal interval $[0,1]^*$, which is totally alternating and such that $pl^*(\emptyset) = 0, pl^*(\Omega) = 1$, for any $B \in \mathcal{B}^0$ $pl^*(B) > 0$. The function $bel^* : \mathcal{B} \to [0,1]^*$ defined as

$$bel^*(E) = 1 - pl^*(E^c)$$

is a function totally monotone.

Starting from a positive nonstandard plausibility pl^* on $\mathcal{B}$, the nonstandard conditional plausibility Pl^* on $\mathcal{B} \times \mathcal{B}^0$ taking values on $\mathbb{R}^*$ is defined by

$$Pl^*(E|H) = \frac{pl^*(E \wedge H)}{pl^*(H)}. \tag{8}$$

It is immediate to see that Pl^* satisfies the axioms in Definition 2.2, by taking into account that they involve only finite numbers of events.

As proved in the following Proposition 2.10, a standard conditional plausibility and belief (in the sense of Definition 2.2) can be obtained by considering the real part of Pl^*, i.e. the function φ defined for any $E|H \in \mathcal{B} \times \mathcal{B}^0$ as

$$\varphi(E|H) = Re\,[Pl^*(E|H)] \tag{9}$$

and its dual function

$$\psi(E|H) = 1 - \varphi(E^c|H). \tag{10}$$

Proposition 2.10. Let pl^* be a nonstandard plausibilty on $\mathcal{B}$. Then the function $\varphi : \mathcal{B} \times \mathcal{B}^0 \to [0,1]$ defined by (9), through (8), is a full conditional plausibility.

Proof. For any $E, H \in \mathcal{B}^0$ one has $0 \leq \varphi(E|H) \leq 1$ and $\varphi(E|H) = 1$ if $E \supseteq H$.

Consider any $E_1, \ldots, E_n \in \mathcal{B}$ and $H \in \mathcal{B}^0$, for any non-empty $I \subset \{1, \ldots, n\}$

$$\varphi(\bigwedge_{i=1}^{n} E_i|H) = Re\left[\frac{pl^*((\wedge_{i=1}^n E_i) \wedge H)}{pl^*(H)}\right] \leq Re\left[\sum_I (-1)^{|I|+1} \frac{pl^*((\vee_{i\in I} E_i) \wedge H)}{pl^*(H)}\right] =$$

$$\sum_I Re\left[(-1)^{|I|+1} \frac{pl^*((\vee_{i\in I} E_i) \wedge H)}{pl^*(H)}\right] = \sum_I (-1)^{|I|+1} \varphi((\vee_{i\in I} E_i)|H).$$

In order to check the axiom *(iii)* of Definition 2.2, let $F \wedge H, H \in \mathcal{B}^0$ and $E \in \mathcal{B}$

$$\varphi(E \wedge F|H) = Re\left[\frac{pl^*(E \wedge F \wedge H)}{pl^*(H)}\right] = Re\left[\frac{pl^*(E \wedge F \wedge H)}{pl^*(F \wedge H)}\right] Re\left[\frac{pl^*(F \wedge H)}{pl^*(H)}\right]$$
$$= \varphi(E|F \wedge H)\varphi(F|H).$$

□

The above result proves that every nonstandard plausibility on $\mathcal{B}$ induces a full conditional plausibility. Conversely, the following Theorem 2.11 shows that any conditional plausibility can be seen as the real part of a (not unique) plausibility taking values in a hyperreal field.

Theorem 2.11. For any full conditional plausibility $Pl : \mathcal{B} \times \mathcal{B}^0 \to [0, 1]$ there is (at least) a strictly positive plausibility $pl^* : \mathcal{B} \to [0, 1]^*$ such that for any $E|H \in \mathcal{B} \times \mathcal{B}^0$

$$Pl(E|H) = Re\left[\frac{pl^*(E \wedge H)}{pl^*(H)}\right]. \tag{11}$$

Proof. For such a proof we can mimic that of Theorem 3.4 in [31] given by Krauss for full conditional probabilities. We will stress here just the slight differences due to the peculiarities of plausibility functions, by taking for granted the main algebraic constructions built in the aforementioned proof, starting from the initial ultrapower **H** of the reals based on finite sequences of the reals (for details refer to aforementioned paper). Hence in the sequel we will take for given the existence in **H** of elements $\xi_0, \ldots, \xi_k$ such that $\xi_i > n\xi_{i+1} > 0$ for all $n > 0$, and $\xi_0 + \ldots + \xi_k = 1$. The main issue is to construct another ultrapower **F**, that extends **H** and that will turn out to be an ordered non-Archimedean extension field of the reals. For each finite subalgebra $\mathcal{A}$ of $\mathcal{B}$, by Theorem 2.3 we can have a chain representation for Pl restricted to $\mathcal{A} \times \mathcal{A}^0$ in terms of a class of $\{Pl_0, \ldots, Pl_k\}$. Since, as already stated, exist $\xi_0, \ldots, \xi_k \in \mathbf{H}$ such that $\xi_i > n\xi_{i+1} > 0$ for all $n > 0$, and $\xi_0 + \ldots + \xi_k = 1$, we can define a **H**-valued plausibility $pl^*_{\mathcal{A}}$ on $\mathcal{A}$ by

$$pl^*_{\mathcal{A}}(E) = \sum_{i=0}^{k} \xi_i Pl_i(E) \tag{12}$$

for any $E \in \mathcal{A}$. Obviously $pl^*_{\mathcal{A}}$ is strictly positive since by statement *b)* of Theorem 2.3 for any $E \in \mathcal{A}^0$ there exists a unique $j \le k$ such that $Pl_j(E) > 0$; $pl^*_{\mathcal{A}}(\Omega) = \sum \xi_i = 1$, and for any finite set of events $pl^*_{\mathcal{A}}$ is totally monotone. We have to show now that the real parts of such **H**-valued plausibility functions $pl^*_{\mathcal{A}}$ characterize Pl restricted to $\mathcal{A} \times \mathcal{A}^0$, i.e. that

$$Pl(E|H) = Re\left[\frac{pl^*_{\mathcal{A}}(E \wedge H)}{pl^*_{\mathcal{A}}(H)}\right], \tag{13}$$

holds for any $E \in \mathcal{A}$ and $H \in \mathcal{A}^0$. This easily stems from the choice (12) and from the fact that the class $\{Pl_i\}_{i=0}^k$ is a chain representation for Pl restricted to $\mathcal{A} \times \mathcal{A}^0$. In fact, as already recalled, there exist unique j and l, with $j \le l \le k$, such that

$$\frac{pl^*_{\mathcal{A}}(E \wedge H)}{pl^*_{\mathcal{A}}(H)} = \frac{\xi_l Pl_l(E \wedge H)}{\xi_j Pl_j(H)} = \frac{\xi_l}{\xi_j} \frac{Pl_l(E \wedge H)}{Pl_j(H)}. \tag{14}$$

If $j = l$ then the latter fraction is actually the real value $\frac{Pl_j(E \wedge H)}{Pl_j(H)}$ and (13) holds since (5). Otherwise, if $j < l$ we have that $Pl_j(E \wedge H) = 0$ so that $Pl(E|H) = 0$ again by (5). Moreover, we have by construction $\frac{\xi_l}{\xi_j} < \frac{1}{n}$ for all $n > 0$ and thus $\frac{\xi_l}{\xi_j} \in I$ so that $Re[\frac{\xi_l}{\xi_j}] = 0$. Hence (13) trivially holds since

$$Re\left[\frac{\xi_l}{\xi_j}\frac{Pl_l(E \wedge H)}{Pl_j(H)}\right] = Re\left[\frac{\xi_l}{\xi_j}\right] Re\left[\frac{Pl_l(E \wedge H)}{Pl_j(H)}\right] = 0 = Pl(E|H). \tag{15}$$

Now we need to prove the thesis for any pair of events $E \in \mathcal{B}$ and $H \in \mathcal{B}^0$. Thus for each finite subalgebra $\mathcal{A}$ of $\mathcal{B}$, let $\Gamma_{\mathcal{A}}$ be the set of all strictly positive plausibilities $pl^*_{\mathcal{A}}$ with values in $\mathbf{H}$ whose domain includes $\mathcal{A}$ and such that condition (13) holds for any $E \in \mathcal{A}$ and $H \in \mathcal{A}^0$. We have just seen that, for each finite subalgebra $\mathcal{A}$ of $\mathcal{B}$, $\Gamma_{\mathcal{A}} \neq \emptyset$, and we have that the family $\{\Gamma_{\mathcal{A}} : \mathcal{A} \subset \mathcal{B}$ finite subalgebras$\}$ has the finite intersection property, since for any finite collection $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of finite subalgebras of $\mathcal{B}$, there exist a finite subalgebra $\mathcal{G}$ containing $\bigcup_{i=1}^n \mathcal{A}_i$, hence $\emptyset \neq \Gamma_{\mathcal{G}} \subseteq \bigcap_{i=1}^n \Gamma_{\mathcal{A}_i}$. The finite intersection property of the family $\{\Gamma_{\mathcal{A}} : \mathcal{A} \subset \mathcal{B}$ finite subalgebras$\}$ guaranties (see e. g. Th. 4.1 in [33]) that it is contained in an ultrafilter $\mathfrak{U}$ on $\Phi = \bigcup\{\Gamma_{\mathcal{A}} : \mathcal{A} \subset \mathcal{B}$ finite subalgebras$\}$. Let $\mathbf{F}$ be the ultrapower $\mathbf{H}^\Phi/\mathfrak{U}$, then such $\mathbf{F}$ is an ordered extension field of $\mathbf{H}$, and thus of the reals.

For each $E \in \mathcal{B}$ and $pl^*_{\mathcal{A}} \in \Phi$ define

$$F_E(pl^*_{\mathcal{A}}) = pl^*_{\mathcal{A}}(E) \quad \text{if } E \in \mathcal{A} \tag{16}$$

$$F_E(pl^*_{\mathcal{A}}) = 0 \quad \text{otherwise.} \tag{17}$$

Then $F_E \in \mathbf{H}^\Phi$ and $F^*_E = F_E/\mathfrak{U}$. Thus $F^*_{(\cdot)}$ is the searched strictly positive plausibility on $\mathcal{B}$ with values in $\mathbf{F}$ and such that $Pl(E|H) = Re\left[\frac{F^*_{(E \wedge H)}}{F^*_H}\right]$. □

3. COMPARATIVE PROBABILITY AND COMPARATIVE PLAUSIBILITY

We indicate by $\preccurlyeq$ a binary relation between the events of a set $\mathcal{E}$. As usual we denote by $\prec$ and $\sim$ the asymmetrical and symmetrical parts of $\preccurlyeq$, respectively. Depending on the choice of the framework of reference, the relation $\preccurlyeq$ expresses the idea of "no more probable than" (a comparative probability) or "no more plausible than" (a comparative plausibility). For a comparative probability on a Boolean algebra $\mathcal{B}$ the following axioms have been introduced since the seminal articles [15, 16, 29]

(1) for any $E \in \mathcal{B}$ we have $\emptyset \preccurlyeq E$ and the not-triviality requirement $\emptyset \prec \Omega$;

(2) $\preccurlyeq$ is a weak order;

(P) if $A, B, C, A \vee C, B \vee C \in \mathcal{B}$ are such that $A \wedge C = B \wedge C = \emptyset$ then

$$A \preccurlyeq B \iff A \vee C \preccurlyeq B \vee C\,.$$

Actually, de Finetti in [15] instead of **(1)** required the following positivity axiom **(1')**, which expresses that every possible event is more probable than the impossible one:

(1') $\emptyset \prec E$.

We will say a comparative probability (or more generally a relation) to be *positive* if it satisfies **(1')** for any $E \in \mathcal{B}^0$. It is immediate to see that for a binary relation defined on a Boolean algebra $\mathcal{B}$ axioms **(1)** [**(1')**], **(2)**, **(P)** are necessary for the existence of a [strictly positive] probability P representing $\preccurlyeq$, i. e. for every $A, B \in \mathcal{B}$

$$A \preccurlyeq B \iff P(A) \le P(B)\,.$$

Previous axioms are instead not sufficient for [strict] representability if the cardinality of $\mathcal{B}$ is greater than 2^4 (see [30]).

We recall now the proper condition introduced in [7] whenever the comparisons are made among events of an arbitrary set $\mathcal{E}$. Before to do it, let us denote with $C_{\mathcal{E}}$ the set of atoms generated by $\mathcal{E}$ and with $I_E : \mathcal{C}_{\mathcal{E}} \to \{0,1\}$ the usual indicator function

$$I_E(c) = \begin{cases} 1 & \text{if } \{c\} \wedge E \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \quad . \tag{18}$$

We can recall now the coherence condition:

Definition 3.1. A binary relation $\preccurlyeq$ on $\mathcal{E}$ is said to be a *coherent comparative probability* if

(c) for any finite sub-family $\mathcal{F} = \bigcup_{i=1,\dots,n}\{E_i, F_i\}$ with $E_i, F_i \in \mathcal{E}$ such that $E_i \preccurlyeq F_i$ and for every $\lambda_i > 0$

$$\sup_{c \in \mathcal{C}_{\mathcal{F}}} \sum_{i=1}^{n} \lambda_i (I_{F_i}(c) - I_{E_i}(c)) \le 0 \quad \Longrightarrow \quad E_i \sim F_i \quad i = 1, \dots, n. \tag{19}$$

Coherence implies that for any event $E \in \mathcal{E}$ we have $\emptyset \preccurlyeq E$, reflexivity and transitivity of $\preccurlyeq$ and axiom **(P)**. On the contrary it does not imply neither **(1')** nor the non-triviality $\emptyset \prec \Omega$ conditions.

If the set of events is a finite Boolean algebra, then coherence is equivalent to the well known condition introduced by Koopman [29].

As proved in [7], coherence is a necessary and sufficient condition for representability of a non-trivial comparative probability $\preccurlyeq$ in any finite subset.

To define *comparative plausibilities* axiom **(P)** must be replaced by one of the following ones

(pl) for every $E, F, H \in \mathcal{B}$ with $E \subset F$ and $E \vee H = \Omega$

$$E \prec F \quad \Longrightarrow (E \wedge H) \prec (F \wedge H)$$

or

(pl') for every $E, F, H \in \mathcal{B}$ with $E \subseteq F$ and $F \wedge H = \emptyset$

$$E \sim F \quad \Longrightarrow (E \vee H) \sim (F \vee H).$$

Axiom **(pl)** was essentially introduced in [45] while **(pl')** in [4]. For a weak order $\preccurlyeq$ the equivalence between **(pl)** and **(pl')** has been proved in [13].

The effectiveness of condition **(pl)** (or equivalently **(pl')**) is based on the requirement that the set of events is a Boolean algebra. If the set of events $\mathcal{E}$ is arbitrary there is no efficient algorithm to check that $\preccurlyeq$ is extendible as a (positive) comparative plausibility, into the Boolean algebra $\mathcal{A} = \langle \mathcal{E} \rangle$ generated by $\mathcal{E}$. For that we introduce a coherence condition similar to **(c)**.

Given a finite set of events $\mathcal{E}$, for every element E of $\mathcal{E}$ it is hence possible to introduce the *covering* function $J_E : \langle \mathcal{E} \rangle \rightarrow \{0,1\}$ with

$$J_E(A_k) = \begin{cases} 1 & \text{if } A_k \wedge E \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Definition 3.2. A binary relation $\preccurlyeq$ is a *coherent comparative plausibility* if the following condition holds

(cpl) for every finite sub-family $\mathcal{F} = \bigcup_{i=1,\ldots,n} \{E_i, F_i\}$ with $E_i, F_i \in \mathcal{E}$ such that $E_i \preccurlyeq F_i$ and for every $\lambda_i > 0$

$$\sup_{A_k \in \mathcal{A}} \sum_{i=1}^{n} \lambda_i (J_{F_i}(A_k) - J_{E_i}(A_k)) \leq 0 \quad \Longrightarrow \quad E_i \sim F_i \quad i = 1, \ldots, n. \tag{21}$$

A similar condition for comparative belief has been already introduced in [34, 37]. Condition **(cpl)** can have a "betting" interpretation: consider a hypothetical bet, related to $\{E_i, F_i\}$, of stake λ_i where the gambler plays in favor of F_i and against E_i. The gain is given by $\lambda_i(k_{F_i} - k_{E_i})$ with k_{F_i} the number of outputs compatible with F_i, and similarly for k_{E_i}. Coherence requires that for any compound betting if the gambler always plays in favor of the most plausible events and against the less plausible ones, there should be at least one possible win (positive gain), if the couples $\{F_i, E_i\}$ are not all equally plausible.

4. REPRESENTABILITY

In this section we study representability of positive comparative probabilities and plausibilities, both by standard and nonstandard probabilities and plausibilities.

4.1. Comparative probabilities

Theorem 4.1. Let $\mathcal{E}$ be a finite set of events and let $\mathcal{C}_{\mathcal{E}}$ be the set of atoms generated by $\mathcal{E}$. For a binary relation $\preccurlyeq$ on $\mathcal{E}$ the following statements are equivalent:

i) $(\mathcal{E}, \preccurlyeq) \cup (\emptyset \prec C_k)_{C_k \in \mathcal{C}_{\mathcal{E}}}$ is a coherent comparative probability;

ii) there exists a strictly positive probability measure $p : \langle \mathcal{E} \rangle \rightarrow [0,1]$ which represents $\preccurlyeq$;

iii) there exists a strictly positive nonstandard probability function $p^* : \langle \mathcal{E} \rangle \to [0,1]^*$ which represents $\preccurlyeq$.

Proof. Equivalence between *i)* and *ii)* is the usual representation theorem of coherent comparative probabilities (see e. g. [7]). Equivalence between *ii)* and *iii)* directly derives from the transfer principle between $[0,1]$ and $[0,1]^*$: in fact in a finite setting representability by a (standard or nonstandard) probability can be easily reduced to a first-order statement, since it corresponds to the solution of a linear system (for the real case see [7]). An alternative proof is derivable since this statement for the real valued case can be regarded as a valid formula in the first-order probabilistic logic, for which a sound and complete axiomatization named AX_{MEAS} has been introduced by Fagin et al. in [24] (see in particular Th.2.2 and Lemma 2.3 therein).

Strict positivity of p derives from the fact that the represented comparative probability is positive also on atoms. □

A coherent comparative probability on $\mathcal{E}$ can be extended to a coherent comparative probability on $\mathcal{E}' \supset \mathcal{E}$, but a positive coherent comparative probability is not necessarily extendible to a positive one. In particular positivity requirement could not hold for some atom as shown in the following example:

Example 4.2. Let $\mathcal{E} = \{A, B, C\}$ with $A \wedge B \wedge C = \emptyset$. Consequently the Boolean algebra generated by $\mathcal{E}$ has 7 atoms and it is immediate to see that the binary relation

$$\emptyset \prec A \prec B \prec C, \quad A \vee B \sim A \vee B \vee C \tag{22}$$

is representable by a probability but that necessarily it implies $A^c \wedge B^c \wedge C \sim \emptyset$.

Nevertheless it is important to note that, given a positive coherent comparative probability on an atomic countable Boolean algebra, there is a positive coherent extension on any atomic countable Boolean super-algebra, representable by a positive probability:

Theorem 4.3. Let $\mathcal{A}, \mathcal{B}$ be two atomic countable Boolean algebras such that $\mathcal{B} \subseteq \mathcal{A}$, given a positive comparative probability $\preccurlyeq$ on $\mathcal{B}$, which is representable by a positive finitely [countably] additive probability, there is at least a positive comparative probability $\preccurlyeq'$ in $\mathcal{A}$ extending $\preccurlyeq$, which is representable by a positive finitely [countably] additive probability.

Proof. By hypothesis $\preccurlyeq$ is representable by a positive probability p on $\mathcal{B}$. Let $\mathcal{C}_{\mathcal{B}}$ be the set of atoms of $\mathcal{B}$, then $p(C) > 0$ for any $C \in \mathcal{C}_{\mathcal{B}}$. Moreover, let $\mathcal{C}_{\mathcal{A}}$ be the set of atoms of $\mathcal{A}$, then for any $K \in \mathcal{C}_{\mathcal{A}}$ there is a unique $C \in \mathcal{C}_{\mathcal{B}}$ such that $K \subseteq C$. Given any $C \in \mathcal{C}_{\mathcal{B}}$ consider the set $\mathcal{K}_C = \{K \in \mathcal{C}_{\mathcal{A}} : K \subseteq C\}$, it follows that the sets $\mathcal{K}_C$ are a partition of $\mathcal{C}_{\mathcal{A}}$. Furthermore, let us consider a function p' on $\mathcal{C}_{\mathcal{A}}$ defined in such a way that $p'(K) = \frac{p(C)}{|\mathcal{K}_C|}$ for any $K \in \mathcal{K}_C$ with $\mathcal{K}_C$ finite and $p'(K_n) = \frac{p(C)}{2^n}$ for any $K_n \in \mathcal{K}_C$ with $\mathcal{K}_C$ countable (but not finite).

The strict positivity of p' on $\mathcal{C}_{\mathcal{A}}$ follows from strict positivity of p on $\mathcal{C}_{\mathcal{B}}$.

Moreover,

$$p(C) = \sum_{K \in \mathcal{K}_C} p'(K)$$

even when C is obtained as a countable (but not finite) disjunction of atoms in $\mathcal{C}_\mathcal{A}$ and

$$\sum_{K \in \mathcal{C}_\mathcal{A}} p'(K) = \sum_{C \in \mathcal{C}_\mathcal{B}} \sum_{K \in \mathcal{K}_C} p'(K) = \sum_{C \in \mathcal{C}_\mathcal{B}} p(C)$$

that is not necessarily 1 when $\mathcal{B}$ is not finite and p is not countably additive.

Then, for any $A \in \mathcal{A}$ the function p' on $\mathcal{C}_\mathcal{A}$ can be extended on $\mathcal{A}$ as follows: consider for any $A \in \mathcal{A}$ the event in $\mathcal{B}$

$$B = \bigvee_{C \in \mathcal{C}_\mathcal{B} : C \subseteq A} C$$

and let

$$K_A^B = \bigvee_{K \subseteq A \wedge B^c, K \in \mathcal{C}_\mathcal{A}} K,$$

then $A = B \vee K_A^B$.

Define $p'(A) = p(B) + \sum_{K \subseteq A \wedge B^c, K \in \mathcal{C}_\mathcal{A}} p'(K)$. Notice that when $A \in \mathcal{B}$ (so $B = A$) it follows $p'(A) = p(A)$.

We need to prove that p' is a finitely additive probability.

For any set of pairwise incompatible events $A_1, \ldots, A_n \in \mathcal{A}$, and $A = \vee_{i=1}^n A_i$, there are the corresponding events $B_1, \ldots, B_n, B \in \mathcal{B}$ contained in $A_1, \ldots, A_n, A$, respectively, and the events $K_{A_i}^{B_i} = \bigcup_{K \subseteq A_i \wedge B_i^c, K \in \mathcal{C}_\mathcal{A}} K$ for $i = 1, \ldots, n$ and $K_A^B = \bigcup_{K \subseteq A \wedge B^c, K \in \mathcal{C}_\mathcal{A}} K$ with $\vee_{i=1}^n B_i \subseteq B$, $K_A^B \subseteq \vee_{i=1}^n K_{A_i}^{B_i}$.

Then, $B = \vee_{i=1}^n (B_i \vee (K_{A_i}^{B_i} \wedge B))$ and $K_A^B = \vee_{i=1}^n (K_A^B \wedge K_{A_i}^{B_i})$, so

$$p'(A) = p'(\vee_{i=1}^n A_i) = p'(B) + p'(K_A^B)$$

$$= p'(\vee_{i=1}^n B_i) + p'(\vee_{i=1}^n (K_{A_i}^{B_i} \wedge B)) + p'(\vee_{i=1}^n (K_{A_i}^{B_i} \wedge K_A^B))$$

$$= \sum_{i=1}^n p'(B_i) + \sum_{i=1}^n p'(K_{A_i}^{B_i} \wedge B) + \sum_{i=1}^n p'(K_{A_i}^{B_i} \wedge K_A^B) = \sum_{i=1}^n p'(A_i).$$

The probability p' on $\mathcal{A}$ induces a positive comparative probability $\preccurlyeq'$ that extends $\preccurlyeq$ on $\mathcal{B}$.

Moreover, if p is countably additive, then p' is countably additive by construction and so it induces a positive comparative probability representable by a countably additive probability on $\mathcal{A}$. □

The proof of the above result aims to build a probability extending the given one by preserving the positivity: actually the existence of a generic (i. e. not strictly positive) extension is well known since we are dealing with countable algebras.

An example related to Theorem 4.3 is the following

Example 4.4. Let $\mathcal{B}$ be a Boolean algebra generated by the set $\{C_1, \ldots, C_4\}$ of atoms with C_i = "the number i is drawn", for $i = 1, 2, 3$ and C_4 = "a number greater or equal to 4 is drawn". We define $\preccurlyeq$ on $\mathcal{B}$ as induced by the probability on $\mathcal{B}$ such that $p(C_i) = 1/2^{1+i}$ for $i = 1, 2, 3$ and $p(C_4) = 9/16$.

If we extend the binary relation on the atomic Boolean algebra $\mathcal{A}$ of finite and co-finite subsets of the natural numbers, we could take that one generated by $p'(\{n\}) = 1/2^{1+n}$ for any $n \in \mathbb{N}$ and $p'(K) = 1 - p'(K^c)$ for any $K \subset \mathbb{N}$ co-finite .

Note that such p' is a finitely additive but not countably additive probability: in fact the sum over all the atoms $\{n\} \in \mathbb{N}$ is 1/2.

It has been already stressed that coherence is equivalent to representability by real valued probabilities only for finite settings. Such equivalence for $\preccurlyeq$ defined on sets of events $\mathcal{E}$ with arbitrary cardinality can be maintained only for nonstandard representability. In fact, the following result holds:

Theorem 4.5. Let $\mathcal{E}$ be a set of events and $\preccurlyeq$ a binary relation on $\mathcal{E}$, then the following are equivalent:

i) for every finite subset $\mathcal{F} \subseteq \mathcal{E}$ the binary relation $(\mathcal{F}, \preccurlyeq_{|\mathcal{F}}) \cup (\emptyset \prec C_k)_{C_k \in \mathcal{C}_\mathcal{F}}$ is a coherent comparative probability;

ii) there exists an extension of $\preccurlyeq$ to $\langle\mathcal{E}\rangle$ which is a positive coherent comparative probability;

iii) there exists a strictly positive nonstandard probability function $p^* : \langle\mathcal{E}\rangle \to [0,1]^*$ which represents $\preccurlyeq$.

Proof. Statements *ii)* implies *i)* trivially. The reverse implication is stated in [7] (Theorem 4). Theorem 4.1 shows that *i)* is equivalent to have for any finite $\mathcal{F} \subset \mathcal{E}$ a strictly positive p, defined on the Boolean algebra generated by $\mathcal{F} \cup \mathcal{C}_\mathcal{F}$, which represents $\preccurlyeq$ restricted to $\mathcal{F} \cup \mathcal{C}_\mathcal{F}$. As already proved in [33][Th.5.1], there exists a strictly positive nonstandard probability p^* defined on the whole Boolean algebra $\langle\mathcal{E}\rangle$ that represents $\preccurlyeq$. The explicit proof of strict positivity of p^* derives directly from representability of a positive comparative probability (see again last rows of the proof of Th.5.1 in [33]). □

An example of positive comparative probability, coherent on any finite set that is not representable by a strictly real valued positive probability is the well known following one introduced in [6]:

Example 4.6. Let $\mathcal{A}$ be the Boolean algebra of finite and co-finite subsets of $\mathbb{N}$ and $\preccurlyeq$ induced by cardinalities, i. e.:

$$A \preccurlyeq B \Leftrightarrow \begin{cases} |A| \leq |B| & \text{if A is finite} \\ |B^c| \leq |A^c| & \text{if B is co-finite.} \end{cases} \tag{23}$$

It is representable through the nonstandard probability generated by $p^*(n) = \epsilon$, with ϵ any infinitesimal of $[0,1]^*$, if $n \in \mathbb{N}$ and $p^*(B) = 1 - |B^c|\epsilon$ if B is a co-finite.

While it can be only weakly represented by a real valued probability since $n \sim m$ for all $n, m \in \mathbb{N}$ implies inevitably $p(n) = p(m) = 0$ and consequently $p(A) = p(B) = 0$ for finite A and B even if with different cardinalities.

4.2. Comparative plausibilities

First of all we prove the following Theorem 4.7, which states that the coherence condition (**cpl**) characterizes the binary relations $\preccurlyeq$, defined on arbitrary set of events $\mathcal{E}$, extendible as a (positive) comparative plausibility on the Boolean algebra $\langle\mathcal{E}\rangle$:

Theorem 4.7. Let $\mathcal{E}$ be a set of events and consider, for every finite subfamily $\mathcal{F}$ of $\mathcal{E}$, the Boolean algebra $\langle\mathcal{F}\rangle$ generated by $\mathcal{F}$. For a binary relation $\preccurlyeq$ on $\mathcal{E}$ the following statements are equivalent:

i) there exists a comparative plausibility $\preccurlyeq'$ on $\langle\mathcal{E}\rangle$ extending $\preccurlyeq$;

ii) $\preccurlyeq$ on $\mathcal{E}$ is a coherent comparative plausibility;

iii) for every finite subfamily $\mathcal{F}$ of $\mathcal{E}$ there exists a plausibility function $Pl : \langle\mathcal{F}\rangle \to [0,1]$ which represents $\preccurlyeq_{|\mathcal{F}}$.

Proof. Equivalence between *i)* and *iii)* has been proved in [13], so we only need to prove equivalence between *ii)* and *iii)*. Condition *ii)* is equivalent to the solvability, for every finite $\mathcal{F} \subset \mathcal{E}$ of the following system $S_{\mathcal{F}}$: with unknowns $x_r = m(A_r) \geq 0$ for $A_r \in \langle\mathcal{F}\rangle$,

$$(S_{\mathcal{F}}) = \begin{cases} \sum\limits_{A_r \wedge E_i \neq \emptyset} x_r \leq \sum\limits_{A_r \wedge F_i \neq \emptyset} x_r & \text{for every } E_i, F_i \in \mathcal{F} \text{ with } E_i \preccurlyeq F_i \\ \sum\limits_{A_r \wedge E_j \neq \emptyset} x_r < \sum\limits_{A_r \wedge F_j \neq \emptyset} x_r & \text{for every } E_j, F_j \in \mathcal{F} \text{ with } E_j \prec F_j \\ x_r \geq 0 & \text{for every} A_r \in \langle\mathcal{F}\rangle \\ \sum\limits_{A_r \in \langle\mathcal{F}\rangle} x_r = 1. \end{cases} \tag{24}$$

By using a classical alternative theorem (see for instance [23]) it is easy to see that systems $(S_{\mathcal{F}})$ have a solution if and only if **(cpl)** holds. □

Note that condition *i)* (or equivalently *ii)*) is not sufficient for the existence of a real valued plausibility representing $\preccurlyeq$ on the whole $\mathcal{E}$, as the the following example shows:

Example 4.8. Let $\preccurlyeq$ be as in Example 4.6 which obviously satisfies **(1')**, **(2)** and **(pl)** and so, by previous theorem, it is a coherent comparative plausibility. Therefore, in any finite subalgebra $\mathcal{F}$ of $\mathcal{A}$ there is a plausibility $Pl_{\mathcal{F}}$ representing $\preccurlyeq$ restricted to $\mathcal{F}$. On the contrary, there is not a plausibility Pl representing $\preccurlyeq$ because, if by absurd it will be, then for any $h, k \in \mathbb{N}$ we would have

$$0 < Pl(\{k\}^c) = Pl(\{h\}^c) < Pl(\mathbb{N}) = 1. \tag{25}$$

Then, by passing to the associated belief function Bel, we would have

$$1 > 1 - Bel(\{k\}) = 1 - Bel(\{h\}) > 0 \tag{26}$$

so that we can set $Bel(\{k\}) = Bel(\{h\}) = \delta \in \mathbb{R}$. Then for any natural $m \geq \frac{1}{\delta}$, we have

$$Pl\left(\bigvee_{i=1}^{m+1}\{i\}\right) \geq Bel\left(\bigvee_{i=1}^{m+1}\{i\}\right) \geq \frac{m+1}{m} > 1 \tag{27}$$

that is an absurd.

The following Theorems 4.9 – 4.10 are the analogous in the plausibility framework of Theorems 4.1-4.5 proved in the probability framework.

Theorem 4.9. Let $\mathcal{E}$ be a finite set of events and let $\mathcal{C}_{\mathcal{E}}$ be the set of atoms generated by $\mathcal{E}$. For a binary relation $\preccurlyeq$ on $\mathcal{E}$ the following statements are equivalent:

i) there exists a positive comparative plausibility $\preccurlyeq'$ on $\langle\mathcal{E}\rangle$ extending $\preccurlyeq$;

ii) $(\mathcal{E}, \preccurlyeq) \cup (\emptyset \prec C_k)_{C_k \in \mathcal{C}_{\mathcal{E}}}$ is a coherent comparative plausibility;

iii) there exists a plausibility function $Pl : \langle\mathcal{E}\rangle \to [0,1]$ strictly positive which represents $\preccurlyeq_{|\mathcal{E}}$;

iv) there exists a strictly positive nonstandard plausibility function $pl^* : \langle\mathcal{E}\rangle \to [0,1]^*$ which represents $\preccurlyeq_{|\mathcal{E}}$.

P r o o f. Equivalence between *i)* and *iii)* has been proved in [13]. Equivalence between *iii)* and *iv)* holds for the transfer principle between $[0,1]$ and $[0,1]^*$ since the finiteness of $\mathcal{E}$, for the same motivations given in the proof of Theorem 4.1. Strict positivity of Pl derives from the fact that the represented comparative plausibility is positive on atoms and so in any event. The proof of the equivalence between *ii)* and *iii)* has been in fact provided in the previous Theorem 4.7. □

Theorem 4.10. Let $\mathcal{E}$ be a set of events and $\preccurlyeq$ a binary relation on $\mathcal{E}$, then the following conditions are equivalent:

i) there exists a positive comparative plausibility $\preccurlyeq'$ on $\langle\mathcal{E}\rangle$ extending $\preccurlyeq$;

ii) for *every finite* subset $\mathcal{F} \subseteq \mathcal{E}$, the relation $(\mathcal{F}, \preccurlyeq_{|\mathcal{F}}) \cup (\emptyset \prec C_k)_{C_k \in \mathcal{C}_{\mathcal{F}}}$ is a coherent comparative plausibility;

iii) there exists a strictly positive *nonstandard* plausibility $pl^* : \langle\mathcal{E}\rangle \to [0,1]^*$ which represents $\preccurlyeq$.

P r o o f. By Theorem 4.9 conditions *i)* and *ii)* are equivalent to the representability of the restriction of $\preccurlyeq$ through a plausibility on any finite subalgebra of $\langle\mathcal{E}\rangle$.

Therefore, to prove the result it is sufficient to prove that the last assertion is equivalent to *iii)*: we follow, as for the finitely additive probabilities, the line of proof of Theorem 5.1 in [33], with the exception of the characteristic property required to pl^*

that must be totaly alternating on any finite set of events. For that we can construct the ultrapower $\langle \mathbb{R}^*, +, \cdot, \leq \rangle$ of $\langle \mathbb{R}, +, \cdot, \leq \rangle$[1] based on the ultrafilter $\mathfrak{U}$ on

$$\mathbf{Y} = \{\Delta | \Delta \text{ is a nonempty finite subset of } S\} \tag{28}$$

where $S = \{\alpha | \alpha \text{ is a finite subalgebra of } \langle \mathcal{E} \rangle\}$. In fact, for each $\alpha \in S$ let $\widehat{\alpha} = \{\Delta | \Delta \in \mathbf{Y} \text{ and } \alpha \in \Delta\}$. Let $\mathfrak{F} = \{\widehat{\alpha} \mid \alpha \in S\}$. If $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_n$ are in $\mathfrak{F}$ then $\widehat{\alpha}_1 \cap \ldots \cap \widehat{\alpha}_n \neq \emptyset$ since $\{\alpha_1, \ldots, \alpha_n\} \in \widehat{\alpha}_i$, for $i = 1, \ldots, n$. Thus $\mathfrak{F}$ has the finite intersection property and consequently (see again Th. 4.1 in [33]) there exists an ultrafilter $\mathfrak{U}$ on $\mathbf{Y}$ such that $\mathfrak{U} \supseteq \mathfrak{F}$.

Therefore, for each $\alpha \in S$ let Pl_α be the plausibility function representation for $(\alpha, \preccurlyeq_\alpha)$. Let β be the finite subalgebra of $\langle \mathcal{E} \rangle$ generated by a finite union $\bigcup \Delta$ of elements of $\mathbf{Y}$, and define the function $F_E : \mathbf{Y} \to [0, 1]$ as:

$$F_E(\Delta) = Pl_\beta(E) \quad \text{if } E \in \beta \tag{29}$$

$$F_E(\Delta) = 0 \quad \text{otherwise.} \tag{30}$$

It is possible now to define the function $pl^* : \langle \mathcal{E} \rangle \to \mathbb{R}^*$ as

$$pl^*(E) = F_E. \tag{31}$$

By following [33] it is easy to prove that pl^* is strictly positive and represents $\preccurlyeq$.

We give a direct proof of the finite alternating property for pl^*:

$$\forall E_1, \ldots, E_n \in \langle \mathcal{E} \rangle \quad pl^*(\wedge_1^n E_i) \leq \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} pl^*(\vee_{i \in I} E_i). \tag{32}$$

Due to the representability of $\preccurlyeq$ through a plausibility Pl_α in each finite subalgebra $\alpha \subset \langle \mathcal{E} \rangle$, we have that the analogous alternating property

$$Pl_\beta(\wedge_1^n E_i) \leq \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} Pl_\beta(\vee_{i \in I} E_i). \tag{33}$$

holds in the finite subalgebra β generated by the union $\cup \Delta$ of the finite subalgebras Δ containing the events $E_1, \ldots, E_n$. Hence, by definition (31)

$$F_{\wedge_1^n E_i}(\Delta) \leq \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} F_{\vee_{i \in I} E_i}(\Delta) \tag{34}$$

and the set of finite subalgebras Δ for which it holds is an element of the ultrafilter $\mathfrak{U}$ and we can conclude that (32) holds in the ultrapower. The final implication *iii)* $\Rightarrow$ *i)* comes directly from the validity of axioms **(1')**, **(2)** and **(pl)** (or equivalently **(pl')**) whenever $\preccurlyeq$ is represented by a plausibility function pl^*. In fact these axioms are the qualitative counterpart of the non-negativeness, the monotonicity and the n-alternating properties, and all of them hold by definition of pl^*. For example, about **(pl')**, if we

[1] With a little abuse of notation we use the same inequality symbol $\leq$ both between real and hyperreal values

have $E \subset F$ and $F \wedge H = \emptyset$ with $E \sim F$, then $F \vee H = F \vee (E \vee H)$, hence, by the monotonicity and the 2-alternating properties of pl^*, we have

$$\begin{aligned} pl^*(E \vee H) \leq pl^*(F \vee H) &= pl^*(F \vee (E \vee H)) \\ &\leq pl^*(F) + pl^*(E \vee H) - pl^*(F \wedge (E \vee H)) \\ &= pl^*(F) + pl^*(E \vee H) - pl^*(E) \\ &= pl^*(E \vee H) \end{aligned} \tag{35}$$

the last equation being valid by the hypothesis $E \sim F$ and the representation of $\preccurlyeq$ by pl^*. Thus we have $pl^*(E \vee H) = pl^*(F \vee H)$ that implies $(E \vee H) \sim (F \vee H)$. □

From Example 4.8, taking into account positivity of the analized relation, it follows that condition *i)* (or equivalently *ii)*) is not sufficient for the existence of a strictly positive real valued plausibility representing $\preccurlyeq$ on the whole $\mathcal{E}$.

5. REFERENCE DEPENDENT RELATIONS

As already stated in the Introduction, we want to deal explicitly with a dynamic context as in situations where the Decision Maker has to express preferences conditioned to different information scenarios, i. e. to different events H varying in a arbitrary set of alternatives $\mathbb{H}$. Let us focus on relations among conditional events.

Let $\mathcal{L} = \{E_i|H_i\}_{i \in I}$ be a set of conditional events with the requirement that if $E_i|H_i \in \mathcal{L}$ then $\emptyset|H_i \in \mathcal{L}$.

Denote with $\mathcal{H} = \{H_i : E_i|H_i \in \mathcal{L}\}$ the set of conditioning events and with $\mathcal{E} = \{E_i : E_i|H_i \in \mathcal{L}\}$ the set of the conditioned ones.

Let $\mathcal{A}$ be the Boolean algebra generated by $\mathcal{E} \cup \mathcal{H}$. In the following $\preccurlyeq = \bigcup_{H \in \mathcal{H}} \{\preccurlyeq_H\}$ will be a partial binary relation defined for the couples of conditional events $E|H, F|H$ in $\mathcal{L}$ conditioned to the same event $H \in \mathcal{H}$.

In such a context it is natural to search for representability of $\preccurlyeq$ by conditional measures, taking in particular consideration negligible events, even as conditioning ones. Anyhow, since we have seen that a strictly positive nonstandard representability is permitted also with the presence of negligible events, representability can be guaranteed by rationality of a simpler unconditional relation derived from $\preccurlyeq$. This can be obtained as the following *projection* of $\preccurlyeq$:

Definition 5.1. Given $\mathcal{L} = \{E_i|H_i\}_{i \in I}$ and a binary relation $\preccurlyeq$ on $\mathcal{L}$, let $\mathcal{L}^* = \{E \wedge H : E|H \in \mathcal{L}\}$. We can define a partial binary relation $\preccurlyeq^*$ in $\mathcal{L}^*$ defined through

$$E \wedge H \preccurlyeq^* F \wedge H \Leftrightarrow E|H \preccurlyeq F|H. \tag{36}$$

Let us show how such projection $\preccurlyeq^*$ suffices to guarantee a representability in $\mathbb{R}^*$ of the original conditional preference relation:

Theorem 5.2. Let $\preccurlyeq$ be a binary relation on $\mathcal{L} = \{E_i|H_i\}_{i \in I}$ and $\mathcal{A} = \langle \{E_i|H_i\}_{i \in I} \rangle$. For any finite set $\mathcal{F} \subset \mathcal{L}$, let $\preccurlyeq^*$ and $\mathcal{F}^*$ be defined as in in Definition 5.1. Then the following statements are equivalent:

i) for any finite subset $\mathcal{F} \subseteq \mathcal{L}$, the relation $(\mathcal{F}^*, \preccurlyeq^*_{|^*_{\mathcal{F}}}) \cup \{\emptyset \prec C_k\}_{C_k \in \mathcal{C}_{\mathcal{F}^*}}$ is a coherent comparative probability;

ii) there exists a strictly positive nonstandard conditional probability $p^* : \mathcal{A} \times \mathcal{A}^0 \to [0,1]^*$ that represents $\preccurlyeq$ in $\mathcal{L}$.

P r o o f. *i)* implies *ii)* since from Theorem 4.5 there exists a $p^* : \mathcal{A} \to [0,1]^*$ strictly positive that represents $\preccurlyeq^*$ in $\mathcal{L}^*$. Hence

$$E|H \preccurlyeq F|H \Leftrightarrow p^*(E \wedge H) \leq p^*(F \wedge H). \tag{37}$$

Since p^* is strictly positive in $\mathcal{A}$, then $p^*(H) > 0$ and hence we have

$$p^*(E|H) = \frac{p^*(E \wedge H)}{p^*(H)} \leq p^*(F|H) = \frac{p^*(F \wedge H)}{p^*(H)}. \tag{38}$$

The proof of the implication *ii)* $\Rightarrow$ *i)* goes straightforward: in fact if $p^*(\cdot|\cdot)$ represents $\preccurlyeq$ and is strictly positive, then

$$p^*(E|H) \leq p^*(F|H) \Leftrightarrow p^*(E \wedge H) \leq p^*(F \wedge H) \tag{39}$$

and hence $\preccurlyeq^*$ must be coherent, this means, from previous Th.4.3, that *i)* must hold. □

A similar result holds for comparative plausibility:

Theorem 5.3. Let $\preccurlyeq$ be a binary relation on $\mathcal{L} = \{E_i|H_i\}_{i \in I}$, for any finite set $\mathcal{F} \subset \mathcal{L}$, let $\preccurlyeq^*$ and $\mathcal{F}^*$ be defined as in in Definition 5.1. Then the following statements are equivalent:

i) for any finite subset $\mathcal{F} \subseteq \mathcal{L}$, the relation $(\mathcal{F}^*, \preccurlyeq^*_{|^*_{\mathcal{F}}}) \cup \{\emptyset \prec C_k\}_{C_k \in \mathcal{C}_{\mathcal{F}^*}}$, is a coherent comparative plausibility;

ii) for any finite $\mathcal{F} \subseteq \mathcal{L}$, the relation $\preccurlyeq^*_{|^*_{\mathcal{F}}}$ admits an extension on $\langle \mathcal{F}^* \rangle$ which is positive comparative plausibility;

iii) there exists a strictly positive nonstandard conditional plausibility $Pl^* : \mathcal{A} \times \mathcal{A}^0 \to [0,1]^*$ that represents $\preccurlyeq$ in $\mathcal{L}$;

The proof is very similar to that of Theorem 5.2.

Relations induced on $\mathcal{A} \times \mathcal{A}^0$ from a nonstandard conditional probability $P^*(\cdot|\cdot)$ are not the same of those induced by the corresponding $P(\cdot|\cdot)$, even if we limit ourselves to the $\preccurlyeq_H$, i. e. to compare events conditioned to the same reference events H.

We show how, for real valued probabilities, we can preserve the feature of distinguishing the different layers of admissibility among different scenarios. The following definition generalizes the coherence condition for a dynamical setting:

Definition 5.4. The partial binary relation $\preccurlyeq$ on $\mathcal{L} = \{E_i|H_i\}_{i\in I}$ is a *conditionally coherent comparative probability* if the following condition holds:

(ccp) for all $E_i|H_i \preccurlyeq F_i|H_i$ there exists $\delta_i \in [0,1]$, with $\delta_i > 0$ whenever $E_i|H_i \prec F_i|H_i$, such that for every $n \in \mathbb{N}$, $\lambda_i > 0$ and $E_i|H_i \preccurlyeq F_i|H_i$, $i = 1,\ldots,n$, we have

$$\sup_{c\in\mathcal{C}_{\mathcal{L}}} \sum_{i=1}^{n} \lambda_i(I_{F_i}(c) - I_{E_i}(c) - \delta_i)I_{H_i}(c) \geq 0. \tag{40}$$

Note that if there is a single conditioning event, i.e. $\mathcal{H} = \{H\}$, then Definition 5.4 coincides with the so called *strong coherence* condition (sc) in [7, 12].

The following theorem shows that the previous rationality requirement is what is needed to have the representability of the preference relation through conditional probabilities:

Theorem 5.5. Let $\preccurlyeq$ be a partial binary relation on $\mathcal{L}$. The following statements are equivalent:

i) $\preccurlyeq$ is a conditionally coherent comparative probability;

ii) there exists a coherent standard conditional probability $P : \mathcal{L} \to [0,1]$ that represents $\preccurlyeq$.

P r o o f. This proof is a particular case of the more general one already proved in [12]. In fact, for more general comparative conditional assessments where comparisons can be made also among different conditioning events, the following coherence condition has been proved to be equivalent to the representability through a conditional probability: for all $E_i|H_i \preccurlyeq F_i|K_i$ there exists $\alpha_i, \beta_i \in [0,1]$, with $\alpha_i \leq \beta_i$ and $\alpha_i < \beta_i$ whenever $E_i|H_i \prec F_i|K_i$, such that for every $n \in \mathbb{N}$ and $\lambda_i, \lambda'_i \geq 0$ for every $E_i|H_i \preccurlyeq F_i|K_i$ we have

$$\sup_{c\in\mathcal{C}_{H^0}} \sum_{i=1}^{n} [\lambda_i(I_{F_i\wedge K_i}(c) - \beta_i I_{K_i}(c)) + \lambda'_i(\alpha_i I_{H_i}(c) - I_{E_i\wedge H_i}(c))] \geq 0 \tag{41}$$

with $H^0 = \left(\bigvee_{\lambda_i>0} K_i\right) \vee \left(\bigvee_{\lambda'_i>0} H_i\right)$ disjunction of the conditioning events whose corresponding λ_i or λ'_i is positive.

Now we are dealing with a relation $\preccurlyeq$ that compares only events conditioned to the same event H. Hence if (41) holds then **(ccp)** is obtained by taking $\lambda_i = \lambda'_i$ and $\delta_i = \beta_i - \alpha_i$. Vice versa, suppose **(ccp)** holds. We will show that every single therm of the summation in (41) can be obtained and the non-negativity of the supremum maintained. The single therm in **(ccp)** is of the form

$$\lambda_i I_{F_i\wedge H_i}(c) - \lambda_i I_{E_i\wedge H_i}(c) - \lambda_i\delta_i I_{H_i}(c) \tag{42}$$

and has a supremum greater or equal to 0 ($n = 1$) for some specific $\delta_i \geq 0$ and for any λ_i. Moreover, since $\emptyset|H_i \preccurlyeq F_i|H_i$ is always coherent, there will exists a $\gamma_i \geq 0$ such that, for any $\lambda'_i > \lambda_i$, and in particular for $\lambda'_i \geq \gamma_i + \delta_i\lambda_i$, the quantity

$$(\lambda'_i - \lambda_i)I_{F_i\wedge H_i}(c) - (\lambda'_i - \lambda_i)\gamma_i I_{H_i}(c) \tag{43}$$

has a non-negative supremum over H_i. By adding (43) to (42) non-negativity of the supremum is maintained and a therm of those of (41) is obtained by taking $\alpha_i = \delta_i$ and $\beta_i = \gamma_i + \delta_i \frac{\lambda_i}{\lambda'_i}$. □

Once again similar considerations can be done about comparative plausibility, taking into account the following property of conditional pl-coherence:

Definition 5.6. The binary relation $\preccurlyeq$ on $\mathcal{L}$ is a *conditionally coherent comparative plausibility* if the following condition holds:

(ccpl) for all $E_i|H_i \preccurlyeq F_i|H_i$ there exists $\delta_i \in [0,1]$, with $\delta_i > 0$ whenever $E_i|H_i \prec F_i|H_i$, such that for every $n \in \mathbb{N}$, $\lambda_i > 0$ and $E_i|H_i \preccurlyeq F_i|H_i$, $i = 1, \dots, n$, we have

$$\sup_{A_k \in \langle \mathcal{L} \rangle} \sum_{i=1}^{n} \lambda_i (J_{F_i}(A_k) - J_{E_i}(A_k) - \delta_i) J_{H_i}(A_k) \geq 0 \tag{44}$$

with $J_E : \langle \mathcal{L} \rangle \to \{0,1\}$ defined as in (20).

Then, also the following analogous representation theorem holds

Theorem 5.7. Let $\preccurlyeq$ be a partial binary relation on $\mathcal{L}$. The following statements are equivalent:

i) $\preccurlyeq$ is a conditionally coherent comparative plausibility;

ii) there exists a coherent standard conditional plausibility $Pl : \mathcal{L} \to [0,1]$ that represents $\preccurlyeq$.

The proof follows the same line of the proof of Theorem 5.5.

Remark 5.8. Note that if $\mathcal{E} = \mathcal{A}$ is a *finite* Boolean algebra and $\mathbb{H} = \mathcal{A}^0$, then $\preccurlyeq$ defined on $\mathcal{A} \times \mathcal{A}^0$ and positive has a projection $\preccurlyeq^*$ as in Definition 5.1 coherent if and only if it is conditionally coherent. In fact, if $\preccurlyeq$ has a coherent projection $\preccurlyeq^*$ and since it contains $\preccurlyeq_\Omega$, from Theorem 4.1 $\preccurlyeq^*$ is strictly positive, coherent and complete on $\mathcal{A}$ if and only if it is representable through a strictly positive standard (or equivalently nonstandard) probability p (p^*). Hence for any couple $E_i|H_i \preccurlyeq F_i|H_i$ it holds $\frac{p^*(E_i \wedge H_i)}{p^*(Hi)} \leq \frac{p^*(F_i \wedge H_i)}{p^*(Hi)}$ as well as $\frac{p(E_i \wedge H_i)}{p(Hi)} \leq \frac{p(F_i \wedge H_i)}{p(Hi)}$. So we have the representability through a conditional probability that, by the previous Theorem 5.5, is equivalent to the conditional coherence.

On the other hand, in the *infinite* case we have that a positive $\preccurlyeq$ can admit a representation through a nonstandard probability P^* in $[0,1]^*$, and hence it is coherent, but it can happen that it is not representable through a standard probability P in $[0,1]$. In fact, P comes from P^* as $Re[P^*]$ and it could only *almost represent* and not represent $\preccurlyeq$. On the other hand, as already stated in the motivations, if we want to represent $\preccurlyeq$ on $\mathcal{A} \times \mathcal{A}^0$ through a standard conditional probability $P(\cdot|\cdot)$, it is impossible to require all $E_i|H_i$ being more probable than $\emptyset$, since this surely could not hold for $H_i = \Omega$, as already shown in Example 4.2. Consequently a binary relation $\preccurlyeq$ representable by a standard conditional probability P is necessarily less fine of another relation representable by a nonstandard conditional probability, except of course the finite case as proved earlier.

6. CONCLUSIONS

We have considered strictly positive binary relations on sets of (conditional) events and provided necessary and sufficient conditions for their representability by positive standard or nonstandard (conditional) measures. We stress that all conditions considered in this paper do not ensure uniqueness of the representing measures. In the case that the binary relation is induced by a preference relation among acts, the not unicity of the uncertainty measure representing $\preccurlyeq$ reflects on the original preference relation, obtaining representability only by a family of conditional expected utilities (in the case of probability) or of the Choquet expected utility (in the case of belief or plausibility). This is true even if the set of acts is complete and the induced set of events is a σ-algebra. To have uniqueness we need a condition similar to that of *fineness* and *tightness* used by Savage. On the other hand, as stated by Narens in his seminal paper "as more elements are included into the qualitative structure the more unique the representation become".

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