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CALCULATING ALL ELEMENTS OF MINIMAL INDEX IN THE INFINITE PARAMETRIC FAMILY OF SIMPLEST QUARTIC FIELDS

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Abstract. It is a classical problem in algebraic number theory to decide if a number field is monogeneous, that is if it admits power integral bases. It is especially interesting to consider this question in an infinite parametric familiy of number fields. In this paper we consider the infinite parametric family of simplest quartic fields K generated by a root ξ of the polynomial $P_t(x) = x^4 - tx^3 - 6x^2 + tx + 1$, assuming that $t > 0$, $t \neq 3$ and $t^2 + 16$ has no odd square factors. In addition to generators of power integral bases we also calculate the minimal index and all elements of minimal index in all fields in this family.

Keywords: simplest quartic field; power integral base; monogeneity

MSC 2010: 11Y50, 11D25, 11R04

1. Introduction

There is an extensive literature (cf. [3]) of *monogenic fields* K, that is, algebraic number fields of degree n having a *power integral basis* $1, \vartheta, \ldots, \vartheta^{n-1}$. This is the case exactly if the *index* of ϑ, that is

$$I(\vartheta) = (\mathbb{Z}_K^+ : \mathbb{Z}[\vartheta]^+),$$

(where $\mathbb{Z}_K$ denotes the ring of integers of K) is equal to 1.

The first author developed algorithms for calculating generators of power integral bases (cf. [3]) and also succeeded in determining all possible generators of power

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integral bases in some infinite parametric families of number fields, see I. Gaál and M. Pohst [6], I. Gaál and G. Lettl [4].

Among the best known infinite parametric families of number fields are the family of simplest cubic (see [16]), simplest quartic and simplest sextic fields (see [13]). These are exactly the families of totally real cyclic number fields, having a transformation of type $z \mapsto (az+b)/(cz+d)$ as the generator of the Galois group.

In the present paper we deal with the family of *simplest quartic fields*, that is $K = \mathbb{Q}(\xi)$, where ξ is a root of the polynomial

$$P_t(x) = x^4 - tx^3 - 6x^2 + tx + 1 \tag{1}$$

where $t \in \mathbb{Z}$, $t \neq 0, \pm 3$. This family was considered by M. N. Gras [7], A. J. Lazarus [11] and many other authors. Using the integral basis constructed by H. K. Kim and J. H. Lee [10], P. Olajos [15] showed that K has power integral bases only in two exceptional cases. He used the method of I. Gaál, A. Pethő and M. Pohst [5] to solve the index form equations.

In the present paper we determine the minimal indices and all elements of minimal index in the fields belonging to the infinite parametric family of simplest quartic fields. Our basic tool is the method [5], involving extensive formal calculations using Maple [1] and the resolution of a great number of special Thue equations using Kash [2].

Note that B. Jadrević [8], [9] has determined the minimal indices and all elements of minimal index formerly in certain families of bicyclic biquadratic fields. In those fields the index form equation splits into the product of three quadratic forms, which makes the problem much easier.

2. Simplest quartic fields

For simplicity let $t > 0$, $t \neq 3$ and let ξ be a root of the polynomial (1). We also assume that $t^2 + 16$ is not divisible by an odd square since this was needed by H. K. Kim and J. H. Lee [10] to determine the integral bases. (M. N. Gras [7] showed that $t^2 + 16$ represents infinitely many square free integers. This implies that there are infinitely many parameters t with the above properties.)

Lemma 1 (H. K. Kim and J. H. Lee [10]). *Under the above assumptions, an integral basis of K is given by*

$$\left(1, \xi, \xi^2, \frac{1+\xi^3}{2}\right) \quad \text{if} \quad v_2(t) = 0,$$

$$\left(1, \xi, \frac{1+\xi^2}{2}, \frac{\xi+\xi^3}{2}\right) \quad \text{if} \quad v_2(t) = 1,$$
$$\left(1, \xi, \frac{1+\xi^2}{2}, \frac{1+\xi+\xi^2+\xi^3}{4}\right) \quad \text{if} \quad v_2(t) = 2,$$
$$\left(1, \xi, \frac{1+2\xi-\xi^2}{4}, \frac{1+\xi+\xi^2+\xi^3}{4}\right) \quad \text{if} \quad v_2(t) \geqslant 3.$$

P. Olajos [15] determined all generators of power integral bases (up to translation by elements of $\mathbb{Z}$).

Lemma 2 (P. Olajos [15]). *Under the above assumptions power, integral bases exist only for $t = 2$ and $t = 4$. All generators of power integral bases are given by*

- ▷ $t = 2$, $\alpha = x\xi + y(1+\xi^2)/2 + z(\xi+\xi^3)/2$ *where* $(x, y, z) = (4, 2, -1), (-13, -9, 4), (-2, 1, 0), (1, 1, 0), (-8, -3, 2), (-12, -4, 3), (0, -4, 1), (6, 5, -2), (-1, 1, 0), (0, 1, 0)$;
- ▷ $t = 4$, $\alpha = x\xi + y(1+\xi^2)/2 + z(1+\xi+\xi^2+\xi^3)/4$ *where* $(x, y, z) = (3, 2, -1), (-2, -2, 1), (4, 8, -3), (-6, -7, 3), (0, 3, -1), (1, 3, -1)$.

3. Elements of minimal index in the family of simplest quartic fields

If K admits a power integral basis, then its generator has index 1. Otherwise we call m the *minimal index* of K if m is the least positive integer such that $\alpha \in \mathbb{Z}_K$ exists with

$$I(\alpha) = m.$$

If there are no power integral bases then it is important to determine the minimal index of the field K and all elements of minimal index. It is easily seen from Lemma 1 that ξ has index 2, 4, 8, 16 according as $v_2(t) = 0, 1, 2, \geqslant 3$, respectively. We shall see that in some cases there are elements of smaller index, as well. Moreover, we determine all elements of minimal index.

In the following theorem the coordinates of the elements are given in the integral bases of Lemma 1. We display only the last three coordinates of the elements (and omit the first) since the index is translation invariant.

Our main result is the following theorem.

Theorem 3. *Assume that $t > 0$, $t \neq 3$ and $t^2 + 16$ has no odd square factors. Except the parameters $t = 2, 4, 8, 12, 16, 20, 24, 28, 32$, for different values of $v_2(t)$ the minimal indices m of the field K and all elements of minimal index are listed below:*

$$v_2(t) = 0,\ m = 2$$
$$(1,0,0), (6,t,-2), \left(\frac{5+t}{2}, \frac{-1+t}{2}, -1\right), \left(\frac{5-t}{2}, \frac{1+t}{2}, -1\right);$$
$$v_2(t) = 1,\ m = 4$$
$$(1,0,0), (7,2t,-2), \left(\frac{6+t}{2}, -1+t, -1\right), \left(\frac{6-t}{2}, 1+t, -1\right);$$
$$v_2(t) = 2,\ m = 8$$
$$(1,0,0), (7,2+2t,-4), \left(\frac{6+t}{2}, t, -2\right), \left(\frac{6-t}{2}, 2+t, -2\right);$$
$$v_2(t) \geqslant 3,\ m = 16$$
$$(1,0,0), (9+2t, 4-4t, -4), \left(\frac{10+t}{2}, -4-2t, -2\right), \left(\frac{6+3t}{2}, -2t, -2\right).$$

For $t = 2, 4, 8, 12, 16, 20, 24, 28$ the minimal indices and all elements of minimal index are listed below:

$t = 2,\ m = 1$

$(4,2,-1), (-13,-9,4), (-2,1,0), (1,1,0), (-8,-3,2), (-12,-4,3),$
$(0,-4,1), (6,5,-2), (-1,1,0), (0,1,0);$

$t = 4,\ m = 1$

$(3,2,-1), (-2,-2,1), (4,8,-3), (-6,-7,3), (0,3,-1), (1,3,-1);$

$t = 8,\ m = 4$

$(-7,8,1), (17,-28,-3), (-8,8,1), (5,-10,-1), (20,-26,-3), (-4,10,1);$

$t = 12,\ m = 3$

$(10,6,-1), (8,6,-1), (-4,-20,3), (-2,7,-1), (16,19,-3), (4,-7,1);$

$t = 16,\ m = 8$

$(-14,16,1), (27,-52,-3), (-13,16,1), (6,-18,-1), (-34,50,3), (-7,18,1);$

$t = 20,\ m = 5$

$(14,10,-1), (0,-32,3), (12,10,-1), (6,-11,1), (-8,11,-1), (-20,-31,3);$

$t = 24,\ m = 12$

$(-19,24,1), (37,-76,-3), (-20,24,1), (-9,26,1), (8,-26,-1), (-48,74,3);$

$t = 28,\ m = 7$

$(-18,-14,1), (-16,-14,1), (-4,44,-3), (-24,-43,3), (-13,14,1), (10,-15,1).$

For $t = 32$, in addition to those given for $v_2(t) \geqslant 3$ there are further elements of index 16 having coefficients

$$t = 32,\ m = 16$$
$$(-26, 32, 1), (-25, 32, 1), (47, -100, -3), (11, -34, -1), (-10, 34, 1),$$
$$(62, -98, -3).$$

4. Proof of Theorem 3

In this section we list the results that we need to prove Theorem 3 and then we describe its proof.

4.1. The corresponding family of Thue equations. In our calculation we shall use the result giving all solutions $p, q \in \mathbb{Z}$ of the infinite parametric family of Thue equations

$$F_t(p, q) = p^4 - tp^3q - 6p^2q^2 + tpq^3 + q^4 = w \tag{2}$$

for given $w \in \mathbb{Z}$.

These equations were considered by G. Lettl and A. Pethő [12] and by G. Lettl, A. Pethő, P. Voutier [13]. Note that if (p, q) is a solution, then so also is $(-p, -q)$ but we list only one of them. G. Lettl and A. Pethő [12] showed:

Lemma 4.
For $w = 1$ all solutions are the following:
For any $t > 0$, $t \neq 3$: $(p, q) = (1, 0), (0, 1)$.
For $t = 4$: $(p, q) = (2, 3), (3, -2)$.
For $w = -1$ all solutions are the following:
For $t = 1$: $(p, q) = (1, 2), (2, -1)$.
For $w = 4$ all solutions are the following:
For $t = 1$: $(p, q) = (3, 1), (1, -3)$.
For $w = -4$ all solutions are the following:
For any $t > 0$, $t \neq 3$: $(p, q) = (1, 1), (1, -1)$.
For $t = 4$: $(p, q) = (5, 1), (1, -5)$.

Note that congruence consideration mod 8 shows that equation (2) is not solvable for $w = \pm 2$. Further,

$$\text{if} \quad F_t(p, q) = -4c \quad \text{then} \quad F_t\Big(\frac{p-q}{2}, \frac{p+q}{2}\Big) = c$$

(in this case indeed p and q have the same parity), and

$$\text{if} \quad F_t(p,q) = c \quad \text{then} \quad F_t(p-q, p+q) = 4c.$$

Using Lemma 4 and the above notes we can figure out the solutions of (2) for all w being a power of 2 or its negative, and just this is what we need in our calculation.

4.2. Index form equations in arbitrary quartic fields. In this section we detail the method of I. Gaál, A. Pethő and M. Pohst [5] (see also I. Gaál [3]) which will play an essential role in our arguments.

Let K be a quartic field generated by a root ξ with minimal polynomial $f(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \in \mathbb{Z}[x]$. We represent any $\alpha \in \mathbb{Z}_K$ in the form

$$\alpha = \frac{1}{d}(a + x\xi + y\xi^2 + z\xi^3) \tag{3}$$

with coefficients $a, x, y, z \in \mathbb{Z}$ and with a common denominator $d \in \mathbb{Z}$. Let $n = I(\xi)$, let

$$F(u,v) = u^3 - a_2u^2v + (a_1a_3 - 4a_4)uv^2 + (4a_2a_4 - a_3^2 - a_1^2a_4)v^3$$

be a binary cubic form over $\mathbb{Z}$ and

$$\begin{aligned} Q_1(x,y,z) &= x^2 - xya_1 + y^2a_2 + xz(a_1^2 - 2a_2) + yz(a_3 - a_1a_2) \\ &\quad + z^2(-a_1a_3 + a_2^2 + a_4), \\ Q_2(x,y,z) &= y^2 - xz - a_1yz + z^2a_2 \end{aligned}$$

ternary quadratic forms over $\mathbb{Z}$.

Lemma 5. *If α of (3) satisfies*

$$I(\alpha) = m, \tag{4}$$

then there is a solution $(u,v) \in \mathbb{Z}^2$ of

$$F(u,v) = \pm\frac{d^6m}{n} \tag{5}$$

such that

$$\begin{aligned} Q_1(x,y,z) &= u, \\ Q_2(x,y,z) &= v. \end{aligned} \tag{6}$$

In [5] an algorithm is also given for the resolution of the system of equations (6) which we shall apply in the sequel.

4.3. Index form equations in simplest quartic fields. Using the coefficients of the polynomial (1), in Lemma 5 we substitute

$$a_1 = -t, \quad a_2 = -6, \quad a_3 = t, \quad a_4 = 1;$$

then we become

$$F(u,v) = (u+2v)(u^2+4uv-v^2(t^2+12)), \tag{7}$$

$$Q_1(x,y,z) = x^2+txy-6y^2+(t^2+12)xz-5tyz+(t^2+37)z^2, \tag{8}$$

$$Q_2(x,y,z) = y^2-xz+tyz-6z^2. \tag{9}$$

We deal with all four cases ($v_2(t) = 0, 1, 2, \geqslant 3$) in Lemma 1 parallelly. According to Lemma 5, for a given m, in order to determine the elements of index m we first have to solve the equation

$$F(u,v) = \frac{g^6 m}{I(\xi)}$$

with

$$F(u,v) = (u+2v)(u^2+4uv-v^2(t^2+12)).$$

Here $g = 2, 2, 4, 4$ and $I(\xi) = 2, 4, 8, 16$ according as $v_2(t) = 0, 1, 2, \geqslant 3$. We proceed by taking $m = 1, 2, \ldots, I(\xi)$ until we find solutions.

We write $g^6 m/I(\xi)$ in the form $a \cdot 2^l$ with an odd $a \in \{1,3,5,7,9,11,13,15\}$ and $4 \leqslant l \leqslant 12$. We confer

$$\begin{aligned} u+2v &= \pm a_1 \cdot 2^i, \\ u^2+4uv-v^2(t^2+12) &= \pm a_2 \cdot 2^{l-i} \end{aligned} \tag{10}$$

with odd numbers a_1, a_2 satisfying $a_1 a_2 = a$ and $i = 0, 1, \ldots, l$. We obtain

$$a_1^2 \cdot 2^{2i} \pm a_2 \cdot 2^{l-i} = v^2(t^2+16). \tag{11}$$

The left hand side $v^2(t^2+16)$ is either zero or positive.

Case I. In case $v^2(t^2+16) = 0$ we get $v = 0$ for arbitrary t. (11) implies

$$\frac{a_1^2}{a_2} = 2^{l-3i},$$

which is only possible for $a_1 = a_2 = 1$ and $(i, l) = (2, 6), (3, 9), (4, 12)$. By $v = 0$ equation (10) implies $u = \pm 2^i$, therefore (following the method of [5])

$$Q_0(x, y, z) = vQ_1(x, y, z) - uQ_2(x, y, z) = 0$$

whence

$$2^i(y^2 - xz + yzt - 6z^2) = 0.$$

A nontrivial solution of this quadratic equation is $(x_0, y_0, z_0) = (-6, 0, 1)$. Using an idea of [14] Chapter 7 we parametrize all solutions x, y, z with rational parameters p, q, r in the form

$$x = -6p + r, \quad y = q, \quad z = r. \tag{12}$$

Substituting it into $Q_0(x, y, z) = 0$ we obtain

$$r(2^i p - 2^i qt) = 2^i q^2.$$

Multiplying all equations by $2^i p - 2^i qt$ we obtain

$$\begin{aligned} kx &= 2^i p^2 - 2^i pqt - 6 \cdot 2^i q^2, \\ ky &= 2^i pq - 2^i tq^2, \\ kz &= 2^i q^2 \end{aligned} \tag{13}$$

with $k \in \mathbb{Q}$. Arrange the coefficients of p^2, pq, q^2 on the left hand sides of the equations above into a 3×3 matrix $C = (c_{ij})$. Multiplying all equations by the square of the common denominators of p, q and dividing them by the gcd of the elements of the matrix $C = (c_{ij})$ we can replace k, p, q by integer parameters (cf. [14], [5]) and k can be shown to divide

$$\frac{\det(C)}{(\gcd(c_{ij}))^2} = 2^i$$

(cf. [5], [3]). Substituting (13) into $Q_1(x, y, z) = \pm u$ we obtain

$$2^{2i} \cdot F_t(p, q) = \pm 2^i \cdot k^2 \tag{14}$$

with

$$F_t(p, q) = p^4 - tp^3 q - 6p^2 q^2 + tpq^3 + q^4$$

where $i = 2, 3, 4$ and $k \mid 2^i$.

This equation is not solvable for $k = 1$. Using Lemma 4 and the remarks after it, solutions exist only for $i = 2$ and $i = 4$. The corresponding values are $l = 6, 12$, respectively. Using the formula

$$F(u, v) = \pm\frac{g^6 m}{I(\xi)} = 2^l$$

we can figure out which $v_2(t), I(\xi)$ and m may possibly correspond to i. The case $m = 1$ was dealt with by P. Olajos [15]. Using Lemma 4 we obtain the solutions (p, q) of equation (14), then (x, y, z) is obtained by (13).

Finally, we obtain the following solutions (and their negatives):

For any t with $v_2(t) = 0$ or $v_2(t) = 1$

$$(x, y, z) = (2, 0, 0),\ (12, 2t, -2),\ (5 \pm t, \mp 1 + t, -1),\ (-5 \pm t, \mp 1 - t, 1).$$

For any t with $v_2(t) = 2$ or $v_2(t) \geqslant 3$

$$(x, y, z) = (4, 0, 0),\ (24, 4t, -4),\ (10 \pm 2t, \mp 2 + 2t, -2),\ (-10 \pm 2t, \mp 2 - 2t, 2).$$

The corresponding coordinates in the integral basis are listed in Theorem 3.

Case II: Using equation (11) and considering the possible values of a_1, a_2, i, l in case $v^2(t^2 + 16) > 0$ we obtain specific values for t. The possible triples (t, u, v) are listed below.

t	2	2	4	4	4	4	4	4	4	4	4	4	7	7	7	7
u	0	−4	20	−28	16	−32	12	20	8	−24	20	−44	1	3	20	12
v	1	1	2	2	4	4	2	2	4	4	6	6	−1	−1	−2	2
t	8	8	8	8	8	8	8	8	8	8	8	8	8	8		
u	10	−14	12	20	22	34	14	−18	−2	−10	34	−54	4	−12		
v	1	1	2	2	3	3	1	1	3	3	5	5	2	2		
t	12	12	12	12	16	16	16	16	16	16	16	16	20	20	20	20
u	20	−28	14	−18	10	−14	28	−36	18	−22	2	−6	14	−18	36	−44
v	2	2	1	1	1	1	2	2	1	1	1	1	1	1	2	2
t	24	24	24	24	24	24	28	28	32	32	32	32	32	32	48	48
u	2	−6	44	−52	18	−22	52	−60	30	−34	2	−6	60	−68	46	−50
v	1	1	2	2	1	1	2	2	1	1	1	1	2	2	1	1
t	64	64	80	80	96	96	112	112	128	128	144	144	240	240	256	256
u	62	−66	78	−82	94	−98	110	−114	126	−130	142	−146	238	−242	254	−258
v	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

For each triple (t, u, v) we have to solve the system of equations (6). We demonstrate our procedure for $(t, u, v) = (12, 20, 2)$. We have

$$Q_1(x, y, z) = x^2 + 12xy - 6y^2 + 156xz - 60yz + 181z^2 = \pm 20, \tag{15}$$

$$Q_2(x, y, z) = y^2 - xz + 12yz - 6z^2 = \pm 2, \tag{16}$$

$$Q_0(x, y, z) = 2x^2 + 24xy - 32y^2 + 332xz - 360yz + 482z^2 = 0.$$

The last equation has the nontrivial solution $(x_0, y_0, z_0) = (15, 11, -1)$. We set

$$x = 15r + p, \quad y = 11r + q, \quad z = -r \tag{17}$$

(with rational parameters) whence $Q_0(x, y, z) = 0$ implies

$$(8p - 16q)r = 2p^2 + 24pq - 32q^2.$$

Multiplying equation (17) by $8p - 16q$ and using the above equation we arrive at

$$\begin{aligned} kx &= -38p^2 - 344pq + 480q^2, \\ ky &= -22p^2 - 272pq + 368q^2, \\ kz &= 2p^2 + 24pq - 32q^2 \end{aligned} \tag{18}$$

with $k \in \mathbb{Q}$. Multiplying all equations by the square of the common denominators of p, q and dividing them by the gcd of the elements of the above coefficient matrix $C = (c_{ij})$ on the right hand side of (18) we can replace k, p, q by integer parameters (cf. [5]). The number k divides $\det(C)/(\gcd(c_{ij}))^2 = 3 \cdot 2^7$. Substituting equations (18) into the equations (15) and (16) we obtain

$$F_2(p, q) = 8p^4 + 128p^3q + 128p^2q^2 - 3072pq^3 + 3328q^4 = \pm 2 \cdot k^2.$$

We could solve all these equations by using the program package Kash [2]. The total CPU time on an average laptop took a couple of hours. The solutions (x, y, z) are listed below. Note that for $(t, -u, -v)$ we get the solutions $(-x, -y, -z)$.

(p, q)	(x, y, z),
$(1, 0)$	$(19, 11, -1)$,
$(2, 1)$	$(15, 11, -1)$,
$(10, -1)$	$(-5, -37, 3)$.

The corresponding coordinates in the integral basis are listed in Theorem 3.

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