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INVERSE PROBLEM FOR SEMILINEAR ULTRAPARABOLIC EQUATION OF HIGHER ORDER

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Abstract. We study the existence and the uniqueness of the weak solution of an inverse problem for a semilinear higher order ultraparabolic equation with Lipschitz nonlinearity. The main aim is to determine the weak solution of the equation and some functions that depend on the time variable, appearing on the right-hand side of the equation. The overdetermination conditions introduced are of integral type. In order to prove the solvability of this problem in Sobolev spaces we use the Galerkin method and the method of successive approximations.

Keywords: ultraparabolic equation; mixed problem; inverse problem; weak solution

MSC 2010: 35K70, 35R30

1. Introduction

The equation of ultraparabolic type was first introduced by A. N. Kolmogorov [5] when describing non-isotropic processes. Later on such type of equations was applied in physics, finance [7]. In the theory of partial differential equations a problem in which the solution of the equation and some of the coefficients of the equation are unknown, is called an inverse problem. Usually an inverse problem contains the same conditions as the direct problem, and overdetermination conditions related to the presence of additional unknown functions [3], [4], [6], [10], [11]. The inverse problem of recovering one or several coefficients that depend on the time and/or on spatial variables on the right-hand side for hyperbolic or parabolic equations was investigated in [1], [3], [4], [6], [11]. The main aim of this paper is to determine the solution of a semilinear higher order ultraparabolic equation and some functions that depend on the time variable, appearing on the right-hand side of the equation. In order to obtain the result we use the Galerkin method and the method of successive approximations. Note that the solvability of mixed problems for nonlinear ultraparabolic equations is studied in [8], [9], [10].

2. Formulation of the problem

Let $\Omega \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^l$ be bounded domains with boundaries $\partial\Omega \in C^{m_0}$ and $\partial D \in C^1$, respectively; $x \in \Omega$, $y \in D$, $t \in (0,T)$, $T > 0$, $Q_\tau = \Omega \times D \times (0,\tau)$, $\tau \in (0,T]$, $G = \Omega \times D$. Denote $\Sigma_T = \partial\Omega \times D \times (0,T)$, $S_T = \Omega \times \partial D \times (0,T)$, $n, l, s, m_0 \in \mathbb{N}$, $\gamma, \alpha \in \mathbb{N}^n$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. In the domain Q_T we consider the problem

$$u_t + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} + \sum_{|\alpha|=|\gamma|\leqslant m_0} (-1)^{|\gamma|} D^\gamma (a_{\alpha\gamma}(x,y,t) D^\alpha u) + c(x,y,t)u + g(x,y,t,u) = \sum_{i=1}^{s} f_i(x,y,t) q_i(t) + f_0(x,y,t); \tag{2.1}$$

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in G; \tag{2.2}$$

$$D^\alpha u|_{\Sigma_T} = 0, \quad |\alpha| \leqslant m_0 - 1, \quad u|_{S_T^1} = 0; \tag{2.3}$$

$$\int_G K_i(x,y) u(x,y,t) \,\mathrm{d}x \,\mathrm{d}y = E_i(t), \quad t \in [0,T], \ i = 1, \ldots, s, \tag{2.4}$$

where $u(x,y,t)$, $q_i(t)$, $i = 1, \ldots, s$, are unknown functions, ν is the outward unit normal vector to the surface S_T, $S_T^1 = \Big\{(x,y,t) \in S_T \colon \sum\limits_{i=1}^{l} \lambda_i(x,y,t) \cos(\nu, y_i) < 0\Big\}$. Let us assume that condition

(S) there exists $\Gamma_1 \subset \partial D \subset \mathbb{R}^{l-1}$ such that the surface $S_T^1 = \Omega \times \Gamma_1 \times (0,T)$

holds. Denote $\Gamma_2 = \partial D \backslash \Gamma_1$. We shall use the following spaces: $L^\infty(\cdot)$, $L^2(\cdot)$, $W^{1,2}(\cdot)$, $C^k(\cdot)$, $W_0^{m_0,2}(\Omega)$, see [2], $V_1(Q_T) := \{w \colon Q_T \to \mathbb{R} | w,\ D^\alpha w \in L^2(Q_T),\ |\alpha| \leqslant m_0,\ D^\gamma w\big|_{\Sigma_T} = 0,\ |\gamma| \leqslant m_0 - 1\}$, $V_2(G) := L^2(D;\ W_0^{m_0,2}(\Omega))$, $V_3(Q_T) := \{w \colon Q_T \to \mathbb{R};\ w,\ D^\alpha w,\ w_{y_j} \in L^2(Q_T),\ |\alpha| \leqslant m_0,\ j = 1, \ldots, l,\ w|_{S_T^1} = 0,\ D^\gamma w\big|_{\Sigma_T} = 0,\ |\gamma| \leqslant m_0 - 1\}$, $C([0,T];\ L^2(G)) := \{w \colon [0,T] \to L^2(G);\ \|w(\cdot,\cdot,t); L^2(G)\| \in C([0,T])\}$, $L^2(0,T; V_2^*(G)) := \{w \colon (0,T) \to V_2^*(G);\ \|w(\cdot,\cdot,t); V_2^*(G)\| \in L^2(0,T)\}$. According to [2], $L^2(0,T; V_2^*(G)) + L^2(Q_T) := \{z_1 + z_2 \colon z_1 \in L^2(0,T; V_2^*(G)),\ z_2 \in L^2(Q_T)\}$ is a Banach space with the norm $\|z; L^2(0,T; V_2^*(G)) + L^2(Q_T)\| = \inf\limits_{\substack{z_1 \in L^2(0,T;V_2^*(G)),\\ z_2 \in L^2(Q_T),\, z_1+z_2=z}} \max\{\|z_1; L^2(0,T; V_2^*(G))\|; \|z_2; L^2(Q_T)\|\}$. Denote by $\langle \cdot, \cdot \rangle$ the scalar product between the spaces $V_2^*(G)$ and $V_2(G)$.

We also assume that the following hypotheses hold:

(A) $a_{\alpha\gamma} \in L^\infty(Q_T)$, $|\alpha| = |\gamma| \leqslant m_0$,

$$\sum_{|\alpha|=|\gamma|\leqslant m_0} \int_\Omega a_{\alpha\gamma}(x,y,t) D^\alpha w D^\gamma w \,\mathrm{d}x \geqslant a_0 \int_\Omega \sum_{|\alpha|=m_0} |D^\alpha w|^2 \,\mathrm{d}x$$

for almost all $(y,t) \in D \times (0,T)$ and for all $w \in W_0^{m_0,2}(\Omega)$, $a_0 > 0$;

(C) $c \in L^\infty(Q_T)$, $c(x,y,t) \geqslant c_0$ for almost all $(x,y,t) \in Q_T$, c_0 being a constant;

(E) $E_i \in W^{1,2}(0,T)$, $i = 1, \ldots, s$;

(F) $f_i \in C([0,T]; L^2(G))$, $i = 0, \ldots, s$;

(G) $g(x,y,t,\xi)$ is measurable with respect to (x,y,t) in the domain Q_T for all $\xi \in \mathbb{R}^1$ and is continuous with respect to ξ for almost all $(x,y,t) \in Q_T$; moreover, there exists a positive constant g^0 such that $|g(x,y,t,\xi) - g(x,y,t,\eta)| \leqslant g^0 |\xi - \eta|$ for almost all $(x,y,t) \in Q_T$ and for all $\xi, \eta \in \mathbb{R}^1$;

(K) $K_i \in C^1(D; C^1(\overline{\Omega}))$, $K_i\big|_{\partial\Omega\times D} = 0$, $K_i\big|_{\Omega\times\Gamma_2} = 0$ for all $i = 1, \ldots, s$;

(L) $\lambda_i \in L^\infty(0,T; C(\overline{G}))$, $\lambda_{iy_i} \in L^\infty(Q_T)$ for all $i = 1, \ldots, l$;

(U) $u_0, u_{0,y_j} \in L^2(G)$, $j = 1, \ldots, l$, $u_0\big|_{\partial\Omega\times D} = 0$, $u_0\big|_{\Omega\times\Gamma_1} = 0$.

We shall use Friedrichs' inequality:

$$\int_\Omega \sum_{|\alpha|=j} |D^\alpha w|^2 \,\mathrm{d}x \leqslant \gamma_{k,j} \int_\Omega \sum_{|\alpha|=k} |D^\alpha w|^2 \,\mathrm{d}x,$$

$j = 0, 1, \ldots, k$, $w \in W_0^{k,2}(\Omega)$, where the constant $\gamma_{k,j}$ depends on Ω, k, j. Denote $\Gamma_k = \sum_{j=1}^k \gamma_{k,j}$.

3. Main results

First we assume that $q_i(t) = q_i^*(t)$, $i = 1, \ldots, s$ in (2.1), where $q_i^* \in L^2(0,T)$ are known functions, and we introduce the operator

$$L[u,v] := \int_0^T \langle u_t, v\rangle \,\mathrm{d}t + \int_{Q_T} \Big[\sum_{i=1}^l \lambda_i(x,y,t) u_{y_i} v + \sum_{|\alpha|=|\gamma|\leqslant m_0} a_{\alpha\gamma}(x,y,t) D^\alpha u D^\gamma v + c(x,y,t)uv + g(x,y,t,u)v\Big] \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t.$$

Definition 3.1. A function $u^*(x,y,t)$ is a weak solution to the problem (2.1)–(2.3) if $u^* \in V_3(Q_T) \cap C([0,T]; L^2(G))$, $u_t^* \in L^2(0,T; V_2^*(G)) + L^2(Q_T)$ and if it satisfies the equality $L[u^*, v] = \int_{Q_T} \Big(\sum\limits_{i=1}^s f_i(x,y,t) q_i^*(t) + f_0(x,y,t)\Big) v \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t$ for all functions $v \in V_1(Q_T)$ and the condition (2.2) holds.

Theorem 3.1. *Suppose that the hypotheses* (A), (C), (G), (L), (F), (U), (S) *hold. Then the problem* (2.1)–(2.3) *has at most one weak solution. Moreover, if we add the following assumptions:*

1) $a_{\alpha\gamma},\ D^{\alpha}a_{\alpha\gamma},\ c_{y_k}\in L^{\infty}(Q_T)$, $|\alpha|=|\gamma|\leqslant m_0$, $k=1,\dots,l$, $q_j^*\in L^2(0,T)$, $f_{i,y_k}\in L^2(Q_T)$, $i=0,\dots,s$, $k=1,\dots,l$, $j=1,\dots,s$;
2) $|g_{y_i}(x,y,t,\xi)|\leqslant g^1$, $i=1,\dots,l$ *for almost all* $(x,y,t)\in Q_T$ *and for all* $\xi\in\mathbb{R}^1$, *where* g^1 *is a positive constant;*
3) $f_i|_{S_T^1}=0$, $i=0,1,\dots,s$,

then a weak solution to the problem (2.1)–(2.3) *exists.*

The proof is carried out according to the scheme of proof of Theorem 2 in [9], where we use the Galerkin method, and we build the sequence $\{u^{*,N}\}_{N=1}^{\infty}$ that converges in $V_3(Q_T)$ weakly to the solution u^* of the problem (2.1)–(2.3) as $N\to\infty$, and the sequence $\{u_t^{*,N}\}_{N=1}^{\infty}$ converges to u_t^* in $L^2(0,T;V_2^*(G))+L^2(Q_T)$ weakly.

□

Definition 3.2. A set of functions $(u(x,y,t),q_1(t),q_2(t),\dots,q_s(t))$ is a weak solution to the problem (2.1)–(2.4) if $u\in V_3(Q_T)\cap C([0,T];L^2(G))$, $u_t\in L^2(0,T;V_2^*(G))+L^2(Q_T)$, $q_i\in L^2(0,T)$, $i=1,\dots,s$, and it satisfies the equality

$$L[u,v]=\int_{Q_T}\Bigl(\sum_{i=1}^{s}f_i(x,y,t)q_i(t)+f_0(x,y,t)\Bigr)v\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \tag{3.1}$$

for all functions $v\in V_1(Q_T)$ and the conditions (2.2) and (2.4) hold.

The equation (2.1) and the conditions (2.4) imply the equality

$$\sum_{i=1}^{s}q_i(t)\int_G K_j(x,y)f_i(x,y,t)\,\mathrm{d}x\,\mathrm{d}y=F_j(t),\quad t\in[0,T],\ j=1,\dots,s, \tag{3.2}$$

where

$$\begin{aligned}F_j(t):=E_j'(t)-\int_G\Bigl(&K_j(x,y)f_0(x,y,t)+\sum_{i=1}^{l}(\lambda_i(x,y,t)K_j(x,y))_{y_i}u\\&-\sum_{|\alpha|=|\gamma|\leqslant m_0}D^{\gamma}K_j(x,y)a_{\alpha\gamma}(x,y,t)D^{\alpha}u-K_j(x,y)c(x,y,t)u\\&-K_j(x,y)g(x,y,t,u)\Bigr)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Denote $B(t) := [b_{ij}(t)]_{s\times s}$, where $b_{ij}(t) = \int_G K_i(x,y)f_j(x,y,t)\,\mathrm{d}x\,\mathrm{d}y$, $\Delta(t) := \det B(t)$, $A_{ij}(t)$—the algebraical complements of the elements of $B(t)$. Let $\Delta(t)\neq 0$,

$$\alpha_{ij}(x,y,t) := A_{ji}(t)(\Delta(t))^{-1}\Bigl(-K_j(x,y)c(x,y,t) + \sum_{i=1}^{l}(\lambda_i(x,y,t)K_j(x,y))_{y_i}\Bigr),$$
$$\beta_{ij\alpha\gamma}(x,y,t) := -A_{ji}(t)(\Delta(t))^{-1}D^{\gamma}K_j(x,y)a_{\alpha\gamma}(x,y,t),$$
$$\widetilde{E}_{ij}(t) = A_{ji}(t)(\Delta(t))^{-1}\Bigl(E_j'(t) - \int_G K_j(x,y)f_0(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\Bigr).$$

Then from (3.2) we obtain

$$\begin{aligned}(3.3)\quad q_i(t) = \sum_{j=1}^{s}\Bigl(\widetilde{E}_{ij}(t) - \int_G\Bigl(\alpha_{ij}(x,y,t)u + \sum_{|\alpha|=|\gamma|\leqslant m_0}\beta_{ij\alpha\gamma}(x,y,t)D^{\alpha}u\Bigr)\,\mathrm{d}x\,\mathrm{d}y \\ + \int_G A_{ji}(t)(\Delta(t))^{-1}K_j(x,y)g(x,y,t,u)\,\mathrm{d}x\,\mathrm{d}y\Bigr),\quad t\in[0,T],\ i=1,\dots,s.\end{aligned}$$

Theorem 3.2. *Let the conditions of Theorem* 3.1 *and hypotheses* (K), (E) *hold. The set of functions* $(u(x,y,t),q_1(t),q_2(t),\dots,q_s(t))$ *is a weak solution to the problem* (2.1)–(2.4) *if and only if this set satisfies* (2.2), (3.2) *and* (3.1) *for all* $v\in V_1(Q_T)$.

The proof is carried out with the use of Lemma 2.2 in [4]. □

Denote: $\lambda^1 = \max_i \operatorname{ess\,sup}_{Q_T}|\lambda_{iy_i}(x,y,t)|$, $f^1 = \max_i\max_{[0,T]}|f_i(x,y,t)|^2$, $\zeta_1 = l\lambda^1 - 2c_0 + 2g^0 + 1 + 1/T_1$, where $0 < T_1 \leqslant T$, $M_1 := f^1\mathrm{e}^{\zeta_1T_1}/\min\{1,2a_0\}$,

$$\begin{aligned}M_2 := 3s\max\Bigl\{\sup_{[0,T_1]}\sum_{i,j=1}^{s}\Bigl(\int_G(\alpha_{ij}(x,y,t))^2\,\mathrm{d}x\,\mathrm{d}y \\ + (A_{ji}(t)(\Delta(t))^{-1}g^0)^2\int_G(K_j(x,y))^2\,\mathrm{d}x\,\mathrm{d}y\Bigr); \\ m_0^3\Gamma_{m_0}\max_{\alpha,\beta}\sup_{[0,T_1]}\sum_{i,j=1}^{s}\Bigl(\int_G(\beta_{ij\alpha\gamma}(x,y,t))^2\,\mathrm{d}x\,\mathrm{d}y\Bigr)\Bigr\}.\end{aligned}$$

Let a number T_1 satisfy the inequalities

$$(3.4)\qquad \zeta_1 > 0,\quad |M_1M_2T_1| < 1.$$

Theorem 3.3. *Let* $\Delta(t) \neq 0$ *for all* $t \in [0,T]$ *and let the hypotheses* (A), (C), (F), (L), (U), (G), (E), (K), (S) *hold. Then the problem* (2.1)–(2.4) *has at most one weak solution. If, besides,* $a_{\alpha\gamma y_k}$, $D^\alpha a_{\alpha\gamma}$, $c_{y_k} \in L^\infty(Q_T)$, $f_{i,y_k} \in L^2(Q_T)$, $|\alpha| = |\gamma| \leqslant m_0$, $i = 0,\dots,s$, $k = 1,\dots,l$, *and* $f_i|_{S_T^1} = 0$, $i = 0,\dots,s$ *then a weak solution to the problem* (2.1)–(2.4) *exists.*

P r o o f. The proof is divided into three parts.

Part I. Let $T = T_1$. Similarly to [1], we construct the approximation of the solution to the problem (2.1)–(2.4) in such way: $q_i^1(t) := 0$, $i = 1,\dots,s$,

$$(3.5)\qquad q_i^m(t) = \sum_{j=1}^{s}\Bigg(\widetilde{E}_{ij}(t) - \int_G\Big(\alpha_{ij}(x,y,t)u^{m-1} + \sum_{|\alpha|=|\gamma|\leqslant m_0}\beta_{ij\alpha\gamma}(x,y,t)D^\alpha u^{m-1}\Big)\,\mathrm{d}x\,\mathrm{d}y + \int_G \frac{A_{ji}(t)}{\Delta(t)}K_j(x,y)g(x,y,t,u^{m-1})\,\mathrm{d}x\,\mathrm{d}y\Bigg),$$
$$t \in [0,T_1],\ i = 1,\dots,s,\ m \geqslant 2,$$

u^m satisfies the equality

$$(3.6)\qquad L[u^m, v] = \int_{Q_{T_1}}\Big(\sum_{i=1}^{s} f_i(x,y,t)q_i^m(t) + f_0(x,y,t)\Big)v\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \quad m \geqslant 1$$

for all $v \in V_1(Q_{T_1})$ and the condition

$$(3.7)\qquad u^m(x,y,0) = u_0(x,y), \quad (x,y) \in G.$$

It follows from (3.5) that $q_i^m \in L^2(0,T_1)$, $m \geqslant 2$, $i = 1,\dots,s$. According to Theorem 3.1 for each $m \in \mathbb{N}$ there exists a unique function $u^m \in V_3(Q_{T_1}) \cap C([0,T_1]; L^2(G))$, $u_t^m \in L^2(0,T_1; V_2^*(G)) + L^2(Q_{T_1})$, which satisfies (3.6), (3.7).

Now we show that $\{(u^m(x,y,t), q_1^m(t), q_2^m(t),\dots,q_s^m(t))\}_{m=1}^\infty$ converges to the weak solution of the problem (2.1)–(2.4). Denote $r_i^m(t) := q_i^m(t) - q_i^{m-1}(t)$, $z^m := z^m(x,y,t) = u^m(x,y,t) - u^{m-1}(x,y,t)$, $s^m(t) := \int_G \big[|z^m|^2 + \sum\limits_{|\alpha|=m}|D^\alpha z^m|^2\big]\,\mathrm{d}x\,\mathrm{d}y$, $i = 1,\dots,s$, $m \geqslant 2$. From (3.7) we get $z^m(x,y,0) = 0$, $(x,y) \in G$, $m \geqslant 2$. Moreover, using (3.6), Lemma 2 in [9] and considering $L[u^m, z^m\mathrm{e}^{-\zeta_1 t}] - L[u^{m-1}, z^m\mathrm{e}^{-\zeta_1 t}]$ we obtain the equality

$$
\begin{aligned}
(3.8)\qquad & \frac{1}{2}\int_G |z^m(x,y,\tau)|^2 \mathrm{e}^{-\zeta_1\tau}\,\mathrm{d}x\,\mathrm{d}y + \sum_{i=1}^{l} \lambda_i(x,y,t) z^m_{y_i} z^m \\
& + \int_{Q_\tau} \Bigg[\frac{\zeta_1}{2}|z^m|^2 + \sum_{|\alpha|=|\gamma|\leqslant m_0} a_{\alpha\gamma}(x,y,t) D^\alpha z^m D^\gamma z^m \\
& + c(x,y,t)(z^m)^2 + (g(x,y,t,u^m) - g(x,y,t,u^{m-1}))z^m\Bigg] \mathrm{e}^{-\zeta_1 t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
& = \int_{Q_\tau} \sum_{i=1}^{s} f_i(x,y,t) r_i^m(t) z^m \mathrm{e}^{-\zeta_1 t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \quad \tau \in (0,T_1],\ m \geqslant 2.
\end{aligned}
$$

After using the inequality $|ab| \leqslant \frac{1}{2}\delta a^2 + \frac{1}{2\delta}b^2$, $a, b \in \mathbb{R}$, with $\delta = T_1$ and hypotheses (A)–(F) in (3.8) we obtain the estimates

$$
(3.9)\qquad \int_0^\tau s^m(t)\,\mathrm{d}t \leqslant T_1 M_1 \int_0^\tau \sum_{i=1}^{s} |r_i^m(t)|^2\,\mathrm{d}t, \quad \tau \in (0,T_1],\ m \geqslant 2,
$$

$$
(3.10)\quad \int_G |z^m(x,y,\tau)|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant T_1 f^1 \mathrm{e}^{\zeta_1 T_1} \int_0^\tau \sum_{i=1}^{s} |r_i^m(t)|^2\,\mathrm{d}t, \quad \tau \in (0,T_1],\ m \geqslant 2.
$$

Now we estimate $|r_i^m(t)|$, $m \geqslant 3$, using (3.5), Hölder's and Friedrichs' inequalities:

$$
(3.11)\qquad \int_0^\tau \sum_{i=1}^{s} |r_i^m(t)|^2\,\mathrm{d}t \leqslant M_2 \int_0^\tau s^{m-1}(t)\,\mathrm{d}t, \quad \tau \in (0,T_1],\ m \geqslant 3.
$$

Moreover, (3.9) and (3.11) imply the inequalities $\int_0^\tau \sum\limits_{i=1}^{s} |r_i^{m+1}(t)|^2\,\mathrm{d}t \leqslant M_2 \times \int_0^\tau s^m(t)\,\mathrm{d}t \leqslant T_1 M_3 \int_0^\tau \sum\limits_{i=1}^{s} |r_i^m(t)|^2\,\mathrm{d}t$, $m \geqslant 2$, $\tau \in (0,T_1]$, $M_3 := M_1 M_2$. Therefore

$$
(3.12)\quad \int_0^\tau \sum_{i=1}^{s} |r_i^m(t)|^2\,\mathrm{d}t \leqslant (T_1 M_3)^{m-1} \int_0^\tau \sum_{i=1}^{s} |r_i^1(t)|^2\,\mathrm{d}t, \quad \tau \in (0,T_1],\ m \geqslant 3.
$$

Let $k \in \mathbb{N}$. Taking into account (3.4) and (3.12), we obtain the inequalities

$$
\int_0^\tau |q_i^{m+k}(t) - q_i^m(t)|^2\,\mathrm{d}t \leqslant \sum_{j=m+1}^{m+k} \int_0^\tau |r_i^j(t)|^2\,\mathrm{d}t \leqslant \frac{(T_1 M_3)^m}{1 - T_1 M_3} \int_0^\tau \sum_{i=1}^{s} |r_i^1(t)|^2\,\mathrm{d}t,
$$

$\tau \in (0,T_1]$, $m \geqslant 3$. Then for all $i = 1,\ldots,s$ and for each $\varepsilon > 0$ there exists $\widetilde{m}$ such that for all $k \in \mathbb{N}$ and $m > \widetilde{m}$ the inequality $\|q_i^{m+k}(t) - q_i^m(t); L^2(0,T_1)\| \leqslant \varepsilon$ holds.

Thus the sequence $\{q_i^m\}_{m=1}^{\infty}$ is fundamental in $L^2(0,T_1)$. Then from (3.9) and (3.10) we obtain that $\{u^m\}_{m=1}^{\infty}$ is fundamental in $V_1(Q_{T_1})\cap C([0,T_1];L^2(G))$, therefore for $m\to\infty$

$$
\begin{aligned}
&u^m \to u \quad \text{in} \quad V_1(Q_{T_1})\cap C([0,T_1];L^2(G)),\\
&q_i^m \to q_i \quad \text{in} \quad L^2(0,T_1),\ i=1,\dots,s.
\end{aligned} \tag{3.13}
$$

Moreover, the following estimates were obtained for $u^{m,N}$ (here $u^{m,N}$ are approximations of u^m in the Galerkin method), see [9], page 4, (16) and (18):

$$
\begin{aligned}
&\int_G \sum_{i=1}^{l} |u_{y_i}^{m,N}(x,y,\tau)|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C_1\int_0^{T_1}\sum_{i=1}^{s}|q_i^m(t)|^2\,\mathrm{d}t + C_2, \quad \tau\in[0,T_1],\\
&\|u_t^{m,N}; L^2(0,T_1;V_3^*(G)) + L^2(Q_{T_1})\| \leqslant C_3,
\end{aligned} \tag{3.14}
$$

where the constants C_1, C_2, C_3 do not depend on N. The boundedness of the right-hand side of (3.14) follows from (3.13). Passing to the limit as $N\to\infty$ and taking into account the estimate $\|v;L^2(Q_{T_1})\|^2 \leqslant \lim\limits_{N\to\infty}\|v^N;L^2(Q_{T_1})\|^2$, see [2], page 20, we obtain $\int_G \sum\limits_{i=1}^{l} |u_{y_i}^m(x,y,\tau)|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C_4$, $\tau\in[0,T_1]$, $\|u_t^m; L^2(0,T_1;V_3^*(G)) + L^2(Q_{T_1})\| \leqslant C_5$, where the constants C_4, C_5 do not depend on m. Consequently, we can choose a subsequence from $\{u^m\}_{m=1}^{\infty}$ such that

$$
\begin{aligned}
&u_{y_i}^{m_k} \to u_{y_i} \quad \text{in } L^2(Q_{T_1}) \text{ weakly as } m_k\to\infty,\ i=1,\dots,l,\\
&u_t^{m_k} \to u_t \quad \text{in } L^2(0,T_1;V_3^*(G)) + L^2(Q_{T_1}) \text{ weakly as } m_k\to\infty.
\end{aligned} \tag{3.15}
$$

Taking into account (3.13), (3.15), from (3.6), (3.5) and Theorem 3.2 we conclude that $(u,q_1,q_2,\dots,q_s)$ is a weak solution to the problem (2.1)–(2.4) in Q_{T_1}.

Part II. Let $(u^{(1)},q_1^{(1)},\dots,q_s^{(1)})$, $(u^{(2)},q_1^{(2)},\dots,q_s^{(2)})$ be two weak solutions to the problem (2.1)–(2.4) in Q_{T_1}. Then their difference $(\widetilde{u},\widetilde{q}_1^{(1)},\dots,\widetilde{q}_s^{(1)})$, where $\widetilde{u} = u^{(1)}-u^{(2)}$, $\widetilde{q}_i = q_i^{(1)} - q_i^{(2)}$, satisfies the equality $L[u^{(1)},\widetilde{u}\mathrm{e}^{-\zeta_1 t}] - L[u^{(2)},\widetilde{u}\mathrm{e}^{-\zeta_1 t}] = \int_{Q_{T_1}}\sum\limits_{i=1}^{s} f_i(x,y,t)\widetilde{q}_i(t)v\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$ for all functions $v\in V_1(Q_{T_1})$ and the condition $\widetilde{u}(x,y,0)\equiv 0$ holds. Further, using hypotheses (A)–(F) we find

$$
\int_{Q_{T_1}}\Big[|\widetilde{u}\,|^2 + \sum_{|\alpha|=m_0}|D^\alpha\widetilde{u}\,|^2\Big]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \leqslant T_1M_1\int_0^{T_1}\sum_{i=1}^{s}|\widetilde{q}_i(t)|^2\,\mathrm{d}t, \quad m\geqslant 2. \tag{3.16}
$$

Moreover, (3.2), Hölder's and Friedrichs' inequalities imply the estimate

$$
\int_0^{T_1}\sum_{i=1}^{s}|\widetilde{q}_i(t)|^2\,\mathrm{d}t \leqslant M_2\int_{Q_{T_1}}\Big[|\widetilde{u}\,|^2 + \sum_{|\alpha|=m_0}|D^\alpha\widetilde{u}\,|^2\Big]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
$$

Applying here (3.16) we find

$$\int_0^{T_1} \sum_{i=1}^{s} |\widetilde{q}_i(t)|^2 \,\mathrm{d}t \leqslant M_3 T_1 \int_0^{T_1} \sum_{i=1}^{s} |\widetilde{q}_i(t)|^2 \,\mathrm{d}t.$$

According to (3.4), we obtain $\int_0^{T_1} \sum\limits_{i=1}^{s} |\widetilde{q}_i(t)|^2 \,\mathrm{d}t \leqslant 0$, therefore $\widetilde{q}_i \equiv 0$, $i = 1, \dots, s$, and $q_i^{(1)} = q_i^{(2)}$, $i = 1, \dots, s$. Then (3.16) implies $\int_{Q_{T_1}} |\widetilde{u}|^2 \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \leqslant 0$, so, $u^{(1)} = u^{(2)}$ in Q_{T_1}.

Part III. If $T_1 < T$, then we divide $[0, T]$ into intervals $[0, T_1], [T_1, 2T_1], \dots, [(N-1)T_1, NT_1]$, where $NT_1 = T$, and the number T_1 satisfies (3.4). The unique solvability of (2.1)–(2.4) is proved in Q_{T_1}. Denote the solution by $(u_1(x, y, t), q_{1,1}(t), q_{2,1}(t), \dots, q_{s,1}(t))$.

Let $t \in [T_1; 2T_1]$. Consider the problem (2.1), (2.3), (2.4) with the condition $u(x, y, T_1) = u_1(x, y, T_1)$, $(x, y) \in G$. Let us change variables $t = \tau + T_1$, $\tau \in [0; T_1]$ in this problem. Denote $q_i^{(1)}(\tau) = q_i(\tau + T)$, $i = 1, \dots, s$, $U(x, y, \tau) = u(x, y, \tau + T_1)$. We obtain a problem similar to (2.1), (2.3), (2.4) as $\tau \in [0; T_1]$ with the condition $U(x, y, 0) = u_1(x, y, T_1)$, $(x, y) \in G$ for the set $(U(x, y, \tau), q_1^{(1)}(\tau), q_2^{(1)}(\tau), \dots, q_s^{(1)}(\tau))$. It is obvious that all new coefficients and initial data of the problem satisfy the same conditions as the functions appearing in the problem (2.1)–(2.4). According to I, II there exists a unique weak solution in Q_{T_1} to the problem. Therefore problem (2.1), (2.3), (2.4) admits one and only one solution in $Q_{T_1, 2T_1}$ with $u(x, y, T_1) = u_1(x, y, T_1)$, $(x, y) \in G$. Denote the solution by $(u_2(x, y, t), q_{1,2}(t), q_{2,2}(t), \dots, q_{s,2}(t))$. Following a similar reasoning on the intervals $[2T_1; 3T_1], \dots, [(N-1)T_1; NT_1]$, we prove the existence and uniqueness of weak solutions $(u_k(x, y, t), q_{1,k}(t), q_{2,k}(t), \dots, q_{s,k}(t))$, $k = 3, \dots, N$, in $Q_{(k-1)T_1, kT_1} := G \times ((k-1)T_1, kT_1)$ for the problem (2.1), (2.3), (2.4) with $u(x, y, (k-1)T_1) = u_{k-1}(x, y, (k-1)T_1)$, $(x, y) \in G$. Evidently, the set of functions $(u(x, y, t), q_1(t), q_2(t), \dots, q_s(t))$, where $u(x, y, t) = u_j(x, y, t)$ if $(x, y, t) \in Q_{(j-1)T_1, jT_1}$, (here $Q_{0,T_1} := Q_{T_1}$), $q_i(t) = q_{i,j}(t)$ if $t \in [(j-1)T_1, jT_1]$, $i = 1, \dots, s$, $j = 1, \dots, N$, is a weak solution for the problem (2.1)–(2.4) in Q_T.

The uniqueness of the weak solution for the problem (2.1)–(2.4) in Q_T is proved by computations similar to those used in parts II, III. □

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