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ON ALMOST EQUITABLE UNINORMS

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Uninorms, as binary operations on the unit interval, have been widely applied in information aggregation. The class of almost equitable uninorms appears when the contradictory information is aggregated. It is proved that among various uninorms of which either underlying t-norm or t-conorm is continuous, only the representable uninorms belong to the class of almost equitable uninorms. As a byproduct, a characterization for the class of representable uninorms is obtained.

Keywords: uninorm, representable uninorm, aggregation functions, negation, contradictory information

Classification: 06F05, 03E72, 03B52

1. INTRODUCTION

Uninorms constitute an important and broad class of aggregation functions in information aggregation. Since their introduction in 1996 by Yager and Rybalov [30], they have attracted a significant and varied amount of research activity, ranging from theoretical study and practical applications. The first deep study by Fodor et al. revealed the structure of uninorms in [12]. Later on it has been justified that uninorms are useful in many fields like expert systems [4, 32], fuzzy logic [14], fuzzy mathematical morphology [5], bipolar aggregation [31]. On the other hand, the theoretical study of uninorms has been even more extensive [1, 7 − 10, 13, 15 − 18, 20, 25 − 29].

Several classes of uninorms are nowadays available. So the question of how to choose the most suitable uninorm for each particular application arises. Several criteria may help in making this choice, such as satisfaction of some specific properties or the fitting of prototypical data. Another criterion is the behavior of the uninorm when receiving contradictory information: should it be tolerant, intolerant, or equitable? Some works [21 − 24] along this line have been carried out. Especially, the class of almost equitable uninorms was introduced in [24]. This paper is devoted to study of this class of uninorms. Sections 2 provides some preliminary concepts and results about uninorms. Section 3 includes the main results of this paper.

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2. PRELIMINARIES

In this section we summarize some of the essential results about t-norms, t-conorms and uninorms.

Definition 2.1. (Klement et al. [19]) A t-norm is a commutative, associative, increasing function $T : [0,1]^2 \to [0,1]$ such that $T(1,x) = x$ for all $x \in [0,1]$.

A t-norm T satisfies $T(x,y) \leq \min(x,y)$ for all $x, y \in [0,1]$. If a continuous t-norm T satisfies $T(x,x) < x$ for all $x \in]0,1[$, then it is called a continuous Archimedean t-norm. As it is well-known, each continuous Archimedean t-norm T can be represented by means of a continuous additive generator [19], i. e., a strictly decreasing continuous function $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$ such that

$$T(x,y) = t^{(-1)}(t(x) + t(y)),$$

where $t^{(-1)}\colon [0,\infty] \to [0,1]$ is the pseudo-inverse of t, and is given by

$$t^{(-1)}(u) = t^{-1}(\min(u, t(0))).$$

Moreover, if T is continuous, Archimedean and for all $x \in]0,1], 0 < y < z < 1$ implies $T(x,y) < T(x,z)$, then T is called strict. If T is continuous, Archimedean and for all $x \in]0,1[$, there exists $y \in]0,1[$ such that $T(x,y) = 0$, then T is called nilpotent. A nilpotent t-norm T has additive generator t such that $t(0) < +\infty$. This implies that T is strictly increasing on that part of the unit square where it is positive. We will use this fact in some proofs later on.

Each continuous t-norm can be represented as an ordinal sum of continuous Archimedean t-norms, i. e., there exists a uniquely determined index set K which is finite or countably infinite, and a family of uniquely determined continuous Archimedean t-norms $T_k, k \in K$ such that $T = (\langle a_k, b_k, T_k\rangle)_{k\in K}$, where $\langle a_k, b_k, T_k\rangle$ is called summand and T_k is called the corresponding t-norm in summand $\langle a_k, b_k, T_k\rangle$ [19].

Definition 2.2. (Klement et al. [19]) A t-conorm is a commutative, associative, increasing function $S : [0,1]^2 \to [0,1]$ such that $S(0,x) = x$ for all $x \in [0,1]$.

A t-conorm S satisfies $S(x,y) \geq \max(x,y)$ for all $x, y \in [0,1]$. If a continuous t-conorm S satisfies $S(x,x) > x$ for all $x \in]0,1[$, then it is called a continuous Archimedean t-conorm. Moreover, if S is continuous, Archimedean and for all $x \in [0,1[, 0 < y < z < 1$ implies $S(x,y) < S(x,z)$, then S is called strict. If S is continuous, Archimedean and for all $x \in]0,1[$, there exists $y \in]0,1[$ such that $S(x,y) = 1$, then S is called nilpotent.

More information concerning t-norms and t-conorms can be found in [19].

Definition 2.3. (Fodor et al. [12], Yager and Rybalov [30]) A uninorm is a two-place function: $U : [0,1]^2 \to [0,1]$ which is associative, commutative, increasing and there exists some element $e \in [0,1]$, called neutral element, such that $U(e,x) = x$ for all $x \in [0,1]$.

We summarize some fundamental results from [12].

It is clear that the uninorm U becomes a t-norm when $e = 1$ and a t-conorm when $e = 0$. For any uninorm we have $U(0,1) \in \{0,1\}$. Throughout this paper, we exclusively consider uninorms with a neutral element e strictly between 0 and 1. With any uninorm U with neutral element $e \in]0,1[$, we can associate two binary operations $T_U, S_U : [0,1]^2 \to [0,1]$ defined by

$$T_U(x,y) = \frac{1}{e} \cdot U(ex, ey)$$

and

$$S_U(x,y) = \frac{1}{1-e}(U(e + (1-e)x, e + (1-e)y) - e).$$

It is easy to see that T_U is a t-norm and that S_U is a t-conorm. In other words, on $[0,e]^2$ any uninorm U is determined by the t-norm T_U, and on $[e,1]^2$ any uninorm U is determined by the t-conorm S_U; T_U is called the underlying t-norm, and S_U is called the underlying t-conorm. Let us denote the remaining part of the unit square by E, i. e., $E = [0,1]^2 \setminus ([0,e]^2 \cup [e,1]^2)$. On the set E, any uninorm U is bounded by the minimum and maximum of its arguments, i. e., for any $(x,y) \in E$ it holds that

$$\min(x,y) \leq U(x,y) \leq \max(x,y). \tag{1}$$

Now, we recall the characterizations of several classes of uninorms.

Theorem 2.4. (Fodor et al. [12]) Suppose that U is a uninorm with neutral element $e \in]0,1[$ and both functions $x \mapsto U(x,1)$ and $x \mapsto U(x,0)$ $(x \in [0,1])$ are continuous except perhaps at the point $x = e$. Then U is given by one of the following forms.

(i) If $U(0,1) = 0$, then

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{2}$$

(ii) If $U(0,1) = 1$, then

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwise.} \end{cases} \tag{3}$$

Denote $\mathcal{U}_{\min}$ the class of uninorms having the form (2) and $\mathcal{U}_{\max}$ the class of uninorms with the form (3).

Proposition 2.5. (Fodor et al. [12]) Consider $e \in]0,1[$ and strictly increasing continuous function $h : [0,1] \to [-\infty, +\infty]$ with $h(0) = -\infty, h(e) = 0$ and $h(1) = +\infty$. The binary operator U defined by

$$U(x,y) = h^{-1}(h(x) + h(y))$$

for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and either $U(0,1) = U(1,0) = 0$ or $U(0,1) = U(1,0) = 1$, is a uninorm which is continuous in $[0,1]^2 \setminus \{(0,1),(1,0)\}$.

Uninorms defined in Proposition 2.5 are called representable uninorms, function h is called an additive generator of U.

Remark 2.6. For a representable uninorm U, both the underlying t-norm T_U and underlying t-conorm S_U are strict.

Uninorms continuous in $]0,1[^2$ form another class of uninorms that contains the class of representable uninorms. They were characterized in [17, 28] as follows.

Theorem 2.7. Suppose that U is a uninorm continuous in $]0,1[^2$ with neutral element $e \in]0,1[$. Then, one of the following cases is satisfied:

(i) There exist $u \in [0,e[, \lambda \in [0,u]$, two continuous t-norms T_1, T_2 and a representable uninorm R such that U can be represented as

$$U(x,y) = \begin{cases} \lambda T_1(\frac{x}{\lambda}, \frac{y}{\lambda}) & \text{if } x,y \in [0,\lambda], \\ \lambda + (u-\lambda) T_2(\frac{x-\lambda}{u-\lambda}, \frac{y-\lambda}{u-\lambda}) & \text{if } x,y \in [\lambda,u], \\ u + (1-u) R(\frac{x-u}{1-u}, \frac{y-u}{1-u}) & \text{if } x,y \in]u,1[, \\ 1 & \text{if } \min(x,y) \in]\lambda,1], \ \max(x,y)=1, \\ \min(x,y) \ \text{ or } \ 1 & \text{if } (x,y) \in \{(\lambda,1),(1,\lambda)\}, \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{4}$$

(ii) There exist $\gamma \in]e,1], \delta \in [\gamma,1]$, two continuous t-conorms S_1, S_2 and a representable uninorm R such that U can be represented as

$$U(x,y) = \begin{cases} \gamma R(\frac{x}{\gamma}, \frac{y}{\gamma}) & \text{if } x,y \in]0,\gamma], \\ \gamma + (\delta-\gamma) S_1(\frac{x-\gamma}{\delta-\gamma}, \frac{y-\gamma}{\delta-\gamma}) & \text{if } x,y \in [\gamma,\delta], \\ \delta + (1-\delta) S_2(\frac{x-\delta}{1-\delta}, \frac{y-\delta}{1-\delta}) & \text{if } x,y \in [\delta,1], \\ 0 & \text{if } \max(x,y) \in [0,\delta[, \ \min(x,y)=0, \\ \max(x,y) \ \text{ or } \ 0 & \text{if } (x,y) \in \{(\delta,0),(0,\delta)\}, \\ \max(x,y) & \text{otherwise.} \end{cases} \tag{5}$$

Denote $\mathcal{CU}^{\min}$ the class of uninorms with the form (4) and $\mathcal{CU}^{\max}$ the class of uninorms with the form (5).

Remark 2.8. Any uninorm U in $\mathcal{CU}^{\min}$ with $u = 0$ or U in $\mathcal{CU}^{\max}$ with $\gamma = 1$ is a representable uninorm. Both the underlying t-norm T_U and underlying t-conorm S_U of U in $\mathcal{CU}^{\min}$ (or in $\mathcal{CU}^{\max}$) are continuous.

Definition 2.9. (De Baets [2]) A uninorm $U : [0,1]^2 \to [0,1]$ is said to be idempotent whenever $U(x,x) = x$ for all $x \in [0,1]$.

Theorem 2.10. (Ruiz-Aquilera and Torrens [27]) Consider $e \in]0,1[$. The following items are equivalent:

(i) U is idempotent uninorm with neutral element e.

(ii) There exists a decreasing, Id-symmetrical function $g : [0, 1] \to [0, 1]$ with fixed point e such that, for all $(x, y) \in E = [0, 1]^2 \setminus ([0, e]^2 \cup [e, 1]^2)$

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } (y = g(x) \text{ and } x < g^2(x)), \\ \max(x,y) & \text{if } y > g(x) \text{ or } (y = g(x) \text{ and } x > g^2(x)), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g^2(x). \end{cases}$$

Moreover, U is commutative on the set of points $(x, g(x))$ such that $x = g^2(x)$.

Definition 2.11. A uninorm U is called a uninorm with continuous underlying operators if both the underlying t-norm T_U and underlying t-conorm S_U are continuous.

Recently, a characterization of the class of uninorms with strict underlying t-norm and t-conorm was presented in [10, 13, 15].

Theorem 2.12. Let U be a uninorm with neutral element $e \in]0, 1[$ such that T_U is strict and S_U is strict. Then one of the following seven statements holds:

(i) $U \in \mathcal{U}_{\min}$.

(ii)

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 1 & \text{if } x = 1 \text{ or } y = 1, \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{6}$$

(iii)

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 1 & \text{if } x = 1, y \neq 0 \text{ or } x \neq 0, y = 1, \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{7}$$

(iv) $U \in \mathcal{U}_{\max}$.

(v)

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 0 & \text{if } x = 0 \text{ or } y = 0, \\ \max(x,y) & \text{otherwise.} \end{cases} \tag{8}$$

(vi)

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 0 & \text{if } x = 0, y \neq 1 \text{ or } x \neq 1, y = 0, \\ \max(x,y) & \text{otherwise.} \end{cases} \tag{9}$$

(vii) U is representable.

Definition 2.13. (Fodor and Roubens [11]) A function $N : [0,1] \to [0,1]$ is said to be a negation if it is decreasing and satisfies $N(0) = 1, N(1) = 0$. Moreover, if N is continuous and strictly decreasing, then it is called a strict negation. If a strict negation N is involutive, i. e., $N(N(x)) = x$ for all $x \in [0,1]$, then it is called a strong negation.

In [22, 24], the Non-Contradiction and Excluded-Middle logical principles are discussed for aggregation operators when receiving contradictory information, where contradiction information is represented by means of couples $(x, N(x))$ and N is a strong negation. In order to describe the equitable behavior of uninorms when receiving contradictory information, the class of almost equitable uninorms was introduced in [24].

Definition 2.14. A uninorm U with neutral element $e \in]0,1[$ is said to be almost equitable with respect to a strict negation N if $U(x, N(x)) = e$ for all $x \in]0,1[$.

Remark 2.15. In Definition 2.14, N is a strict negation. Since the strict negation is often used to represent the contradictory information [11], we give the more general definition.

3. THE MAIN RESULTS

In this section, we discuss the class of almost equitable uninorms.

Proposition 3.1. Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$ and N be a strict negation with the fixed point e. If U is almost equitable with respect to N then N is a strong negation.

Proof. Suppose that N is not a strong negation. Then there exists $x \in]0,1[$ such that $N(N(x)) \neq x$. Without loss of generality, assume that $x \in]0,e[$ and $N(N(x)) > x$. Since U is almost equitable with respect to N, we have $U(x, N(x)) = e$ and $U(N(x), N(N(x))) = e$. Hence, $U(z, N(x)) = e$ for all $z \in [x, N(N(x))]$ by the monotonicity of U. Taking $z \in]x, N(N(x))[$, we have

$$U(x, U(z, N(x))) = U(x, e) = x$$

and

$$U(U(x, N(x)), z) = U(e, z) = z,$$

a contradiction with the associativity and the commutativity of U. Hence, the result holds. □

Proposition 3.2. Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$ and N be a strict negation. If U is almost equitable with respect to N, then e is the only *fixed point* of N, i. e., $N(e) = e$.

Proof. The proof is easy by taking $x = e$ in Definition 2.14. □

Proposition 3.3. Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$ and N be a strict negation. If U is locally internal on $[0,e[\times]e,1]\cup]e,1] \times [0,e[$ (i. e., $U(x,y) \in \{x,y\}$ for any (x,y) in this region), then U is not almost equitable with respect to N.

P r o o f. If $N(e) \neq e$ then U is not almost equitable with respect to N by Proposition 3.2. Now, assume that e is the fixed point of N, i. e., $N(e) = e$. Since U is locally internal, we have $U(x, N(x)) \in \{x, N(x)\}$ for all $x \in [0,1]$. Hence, since N is strictly decreasing and $N(e) = e$, there exists $x \in [0,1[, x \neq e$ such that $U(x, N(x)) \neq e$. □

Corollary 3.4. Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$ and N be a strict negation. If U is an idempotent uninorm then U is not almost equitable with respect to N.

Corollary 3.5. Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$ and N be a strict negation. If $U \in \mathcal{U}_{\min}$ or $U \in \mathcal{U}_{\max}$, then U is not almost equitable with respect to N.

Proposition 3.6. Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$ and N be a strict negation. If U is continuous in $]0,1[^2$, then U is almost equitable with respect to N if and only if U is a representable uninorm with additive generator $h : [0,1] \to [-\infty, +\infty]$ and $N = N_U$ is a strong negation, where $N_U(x) = h^{-1}(-h(x))$ for all $x \in [0,1]$.

P r o o f. If U is a representable uninorm and $N = N_U$ then $U(x, N(x)) = e$ for all $x \in]0,1[$ by Proposition 6 in [3].

Conversely, assume that U is almost equitable with respect to N. By Theorem 2.7, unless U is a representable uninorm, there always exists $l \in]0,e[$ such that $U(x,y) \in \{x,y\}$ for all $(x,y) \in [0,l] \times [l,1] \cup [l,1] \times [0,l]$. Hence, similarly as in Proposition 3.3, U is not almost equitable with respect to N. If U is a representable uninorm then there exists an additive generator $h : [0,1] \to [-\infty, +\infty]$ such that $h(e) = 0$ and $U(x,y) = h^{-1}(h(x) + h(y))$ for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$. We have $U(x, N(x)) = h^{-1}(h(x) + h(N(x))) = e$ and $h(x) + h(N(x)) = h(e) = 0$, i. e., $N(x) = h^{-1}(-h(x))$. By Proposition 7 in [8], $N = N_U$ is a strong negation. □

Now, we discuss the class of uninorms with continuous underlying operators. First we need the following lemmas.

Lemma 3.7. Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$. Then the following statements hold:

(i) If T_U is nilpotent then $U(x,y) \in \{x\} \cup]e,y]$ for all $(x,y) \in [0,e[\times]e,1]$.

(ii) If S_U is nilpotent then $U(x,y) \in \{y\} \cup [x,e[$ for all $(x,y) \in [0,e[\times]e,1]$.

P r o o f. We only prove (i). Taking $(x, y) \in [0, e[\times]e, 1]$, we have $U(x, y) \geq U(x, e) = x$ by the monotonicity of U. Suppose that $U(x, y) = a \in]x, e]$. Since T_U is nilpotent there exists the largest $x_1 \in]0, e]$ such that $U(x_1, x) = 0$. Then by the associativity of U, we have

$$U(U(x_1, x), y) = U(0, y) = U(U(0, x), y) = U(0, U(x, y)) = U(0, a) = 0.$$

Moreover, since the nilpotent t-norm T_U can be represented by means of an additive generator, T_U is strictly increasing on that part of the unit square where it is positive. Hence, we have

$$U(x_1, U(x, y)) = U(x_1, a) > 0,$$

a contradiction with the associativity of U. □

Remark 3.8. (i) The results of Lemma 3.7 can also be induced from Lemma 3.13 in [20] where a general result is proved by adopting the concepts of web geometry.

(ii) If T_U (or S_U) is not nilpotent then the results of Lemma 3.7 may not hold. Some examples are presented here.

Example 3.9. Let U be a representable uninorm with neutral element $e \in]0, 1[$. Then there exists $(x, y) \in [0, e[\times]e, 1]$ such that $U(x, y) \in]x, e]$ by Proposition 2.5.

Example 3.10. ([6]) Let U be a binary operator defined by

$$U(x, y) = \begin{cases} \max(x, y) & \text{if } \min(x, y) \geq \frac{1}{2}, \\ \min(x, y) & \text{if } \frac{1}{8} < \min(x, y) < \frac{1}{2} \text{ or } \max(x, y) = \frac{1}{2}, \\ \frac{1}{8} & \text{if } 0 < \min(x, y) \leq \frac{1}{8} \text{ and } \max(x, y) > \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to prove that U is a uninorm with neutral element $e = \frac{1}{2}$. However, $U(\frac{1}{16}, \frac{3}{4}) = \frac{1}{8} \in]\frac{1}{16}, \frac{1}{2}]$.

Lemma 3.11. (Hliněná et al. [6]) Let $U : [0, 1]^2 \to [0, 1]$ be a uninorm with neutral element $e \in]0, 1[$. Then, for every $x \in]0, 1[$, $U(1, x) = x$ or $U(1, x) = x' > x$. Furthermore, if $U(1, x) = x' > x$ then $U(1, z) = x'$ for all $z \in [x, x']$.

Lemma 3.12. Let $U : [0, 1]^2 \to [0, 1]$ be a uninorm with neutral element $e \in]0, 1[$. Then the following statements hold:

(i) If T_U is continuous then $U(1, y) \in \{y, 1\}$ for all $y \in [0, 1]$.

(ii) If S_U is continuous then $U(0, y) \in \{0, y\}$ for all $y \in [0, 1]$.

P r o o f. We only prove (i). If $U(1, y) \geq e$ then by the associativity and the monotonicity of U, we have

$$U(1, y) = U(U(1, 1), y) = U(1, U(1, y)) \geq U(1, e) = 1.$$

Hence, $U(1,y)=1$.

On the other hand, from Lemma 3.11 it follows $U(1,y)\geq y$. In order to prove the result, we have only to check the case $U(1,y)\in[y,e[$, i.e., $y\in[0,e[$. If we denote $U(1,y)=z$ then from Lemma 3.11, it follows $U(1,z)=z$. The continuity of T_U implies that there exists $u\in[0,e]$ such that $U(z,u)=y$. Then by the associativity of U, we have

$$U(1,y)=U(1,U(z,u))=U(U(1,z),u)=U(z,u)=y.$$

Summarizing the above results, it is clear that $U(1,y)\in\{y,1\}$ for all $y\in[0,1]$. □

Remark 3.13. (i) The results of Lemma 3.12 are consistent with Propositions 2, 3 in [18].

(ii) The results of Lemma 3.12 may not hold when T_U (or S_U) is not continuous. Uninorm U in Example 3.10 can be taken as a counterexample.

Proposition 3.14. Let $U:[0,1]^2\to[0,1]$ be a uninorm with continuous underlying operators and neutral element $e\in]0,1[$ and N be a strict negation. U is almost equitable with respect to N if and only if U is a representable uninorm with additive generator $h:[0,1]\to[-\infty,+\infty]$ and $N=N_U$ is a strong negation, where $N_U(x)=h^{-1}(-h(x))$ for all $x\in[0,1]$.

Proof. If U is a representable uninorm and $N=N_U$ then $U(x,N(x))=e$ for all $x\in]0,1[$ by Proposition 6 in [3].

Conversely, let U be a uninorm with continuous underlying operators and be almost equitable with respect to N. Then $U(x,N(x))=e$ for all $x\in]0,1[$. We can prove the result following two steps.

Step 1: T_U and S_U are Archimedean, i.e., there exists no idempotent element of U in $]0,1[$ except the neutral element e. On the contrary, suppose that $x\in]0,1[$ is an idempotent element of U. Then we have

$$U(U(x,x),N(x))=U(x,N(x))=e$$

and

$$U(x,U(x,N(x)))=U(x,e)=x.$$

Hence, $x=e$.

Step 2: T_U and S_U are strict. On the contrary, suppose that T_U is nilpotent. Then there exist $x_1,x_2\in]0,e[$ such that $x_1<x_2, U(x_1,x_2)=0$. Since U is almost equitable with respect to N, we have $U(x_1,N(x_1))=e, U(x_2,N(x_2))=e, U(N(x_1),N(x_2))\geq e$. Furthermore, by Lemma 3.7, we have

$$U(U(x_1,x_2),U(N(x_1),N(x_2)))=U(0,U(N(x_1),N(x_2)))\in\{0\}\cup]e,U(N(x_1),N(x_2))]$$

and

$$U(U(x_1,N(x_1)),U(x_2,N(x_2)))=U(e,e)=e,$$

a contradiction with the associativity and the commutativity of U. Hence, T_U is strict. By the similar proof, we can prove that S_U is strict.

Hence, both T_U and S_U are strict and U is a representable uninorm by Theorem 2.12 and Proposition 3.3. The result holds by the proof of Proposition 3.6. □

The result of Proposition 3.14 can be strengthened.

Theorem 3.15. Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$. Then the following statements hold.

(i) U is a representable uninorm if and only if T_U is continuous and there exists a strict negation $N : [0,1] \to [0,1]$ such that $U(x, N(x)) = e$ for all $x \in]0,1[$.

(ii) U is a representable uninorm if and only if S_U is continuous and there exists a strict negation $N : [0,1] \to [0,1]$ such that $U(x, N(x)) = e$ for all $x \in]0,1[$.

Proof. We only prove (i). Let U be a representable uninorm. Then T_U is continuous and N_U is a strong negation by Proposition 7 in [12]. Taking $N = N_U$. By Proposition 6 in [3], $U(x, N(x)) = e$ for all $x \in]0,1[$.

Conversely, suppose that T_U is continuous and U is almost equitable with respect to N. By the similar proof of Proposition 3.14, we obtain that T_U is strict. From Propositions 3.1 and 3.2 we know that N is a strong negation with fixed point e. For all $(x, z_1), (x, z_2) \in E$, if $U(x, z_1) = U(x, z_2) = e$ then $z_1 = z_2$. Indeed,

$$z_1 = U(e, z_1) = U(U(x, z_2), z_1) = U(U(x, z_1), z_2) = U(e, z_2) = z_2$$

by the commutativity and the associativity of U.

For all $x, y \in]0,1[$, we have $U(U(x,y), N(U(x,y))) = e$, $U(x, N(x)) = e$ and $U(y, N(y)) = e$. Furthermore,

$$U(U(x,y), U(N(x), N(y))) = U(U(x, N(x)), U(y, N(y))) = U(e, e) = e$$

by the associativity and the commutativity of U. Hence, for all $x, y \in]0,1[$,

$$N(U(x,y)) = U(N(x), N(y)).$$

So, S_U is $N-$dual to T_U and is strict. Therefore, U is a representable uninorm by Theorem 2.12 and Propositions 3.3, 3.14. □

Corollary 3.16. Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in]0,1[$. Then the following statements hold.

(i) U is a representable uninorm if and only if T_U is continuous and there exists a strong negation $N : [0,1] \to [0,1]$ such that $U(x, N(x)) = e$ for all $x \in]0,1[$.

(ii) U is a representable uninorm if and only if S_U is continuous and there exists a strong negation $N : [0,1] \to [0,1]$ such that $U(x, N(x)) = e$ for all $x \in]0,1[$.

4. CONCLUSIONS

In this paper, the class of almost equitable uninorms has been discussed. We have proved that among several well known classes of uninorms, only the class of representable uninorms has nonempty intersection with the class of almost equitable uninorms.

In our future work, we plan to deal with the problem: whether the above result is still true for more general cases of uninorms?

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REFERENCES

[1] H. Bustince, J. Montero, and R. Mesiar: Migrativity of aggregation functions. Fuzzy Sets Syst. *160* (2009), 766–777. DOI:10.1016/j.fss.2008.09.018

[2] B. De Baets: Idempotent uninorms. Eur. J. Oper. Res. *118* (1998), 631–642. DOI:10.1016/s0377-2217(98)00325-7

[3] B. De Baets and J. Fodor: Residual operators of uninorms. Soft Comput. *3* (1999), 89–100. DOI:10.1007/s005000050057

[4] B.De Baets, J. Fodor: Van Melle's combining function in MYCIN is a representable uninorm: An alternative proof. Fuzzy Sets Syst. *104* (1999), 133–136. DOI:10.1016/s0165-0114(98)00265-6

[5] B. De Baets, N. Kwasnikowska, E. Kerre: Fuzzy morphology based on uninorms. In: Seventh IFSA World Congress, Prague 1997, pp. 215–220.

[6] D. Hliněná, M. Kalina, and P. Král': A class of implications related to Yager's f−implications. Information Sciences *260* (2014), 171–184.

[7] P. Drygaś: Discussion of the structure of uninorms. Kybernetika *41* (2005), 213–226.

[8] P. Drygaś: On the structure of continuous uninorms. Kybernetika *43* (2007), 183–196.

[9] P. Drygaś: On properties of uninorms with underlying t-norm and t-conorm given as ordinal sums. Fuzzy Sets Syst. *161* (2010), 149–157. DOI:10.1016/j.fss.2009.09.017

[10] P. Drygaś, D. Ruiz-Aguilera, and J. Torrens: A characterization of uninorms locally internal in $A(e)$ with continuous underlying operators. Fuzzy Sets Syst. (submitted).

[11] J. Fodor and M. Roubens: Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer Academic Publishers, Dordrecht 1994. DOI:10.1007/978-94-017-1648-2

[12] J. Fodor, R. R. Yager, and A. Rybalov: Structure of uninorms. Int. J. Uncertainty, Fuzziness, Knowledge-Based Syst. *5* (1997), 411–427. DOI:10.1142/s0218488597000312

[13] J. Fodor and B. De Baets: A single-point characterization of representable uninorms. Fuzzy Sets Syst. *202* (2012), 89–99. DOI:10.1016/j.fss.2011.12.001

[14] D. M. Gabbay and G. Metcalfe: Fuzzy logics based on $[0,1)$-continuous uninorms. Arch. Math. Log. *46* (2007), 425–449. DOI:10.1007/s00153-007-0047-1

[15] G. Li and H.-W. Liu: Distributivity and conditional distributivity of a uninorm with continuous underlying operators over a continuous t-conorm. Fuzzy Sets and Systems (2015). DOI:10.1016/j.fss.2015.01.019

[16] G. Li, H.-W. Liu, and J. Fodor: Single-point characterization of uninorms with nilpotent underlying t-norm and t-conorm. Int. J. Uncertainty, Fuzziness, Knowledge-Based Syst. *22* (2014), 591–604. DOI:10.1142/s0218488514500299

[17] S. Hu and Z. Li: The structure of continuous uninorms. Fuzzy Sets Syst. *124* (2001), 43–52. DOI:10.1016/s0165-0114(00)00044-0

[18] M. Mas, M. Monserrat, D. Ruiz-Aguilera, and J. Torrens: Migrative uninorms and nullnorms over t-norms and t-conorms. Fuzzy Sets and Syst. *261* (2015), 20–32. DOI:10.1016/j.fss.2014.05.012

[19] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000. DOI:10.1007/978-94-015-9540-7

[20] M. Petrík and R. Mesiar: On the structure of special classes of uninorms. Fuzzy Sets Syst. *240* (2014), 22–38. DOI:10.1016/j.fss.2013.09.013

[21] A. Pradera: Uninorms and non-contradiction. In: Lecture Notes in Computer Sciences, vol. 5285, (V. Torra and Y. Narukawa, eds.), Springer 2008, pp. 50–61. DOI:10.1007/978-3-540-88269-5_6

[22] A. Pradera and E. Trillas: Aggregation, non-contradiction and excluded-middle. Mathware Soft Comput. *XIII* (2006), 189–201.

[23] A. Pradera and E. Trillas: Aggregation operators from the ancient NC and EM point of view. Kybernetika *42* (2006), 243–260.

[24] A. Pradera, G. Beliakov, and H. Bustince: Aggregation functions and contradictory information. Fuzzy Sets Syst. *191* (2012), 41–61. DOI:10.1016/j.fss.2011.10.007

[25] F. Qin and B. Zhao: The distributive equations for idempotent uninorms and nullnorms. Fuzzy Sets Syst. *155* (2005), 446–458. DOI:10.1016/j.fss.2005.04.010

[26] D. Ruiz and J. Torrens: Distributivity and conditional distributivity of a uninorm and a continuous t-conorm. IEEE Trans. Fuzzy Syst. *14* (2006), 180–190. DOI:10.1109/tfuzz.2005.864087

[27] D. Ruiz-Aguilera, J. Torrens, B. De Baets, and J. C. Fodor: Some remarks on the characterization of idempotent uninorms. In: IPMU 2010, LNAI 6178 (E. Hüllermeier, R. Kruse and F. Hoffmann, eds.), Springer-Verlag, Berlin – Heidelberg 2010, pp. 425–434. DOI:10.1007/978-3-642-14049-5_44

[28] D. Ruiz-Aguilera and J. Torrens: S- and R-implications from uninorms continuous in $]0,1[^2$ and their distributivity over uninorms. Fuzzy Sets Syst. *160* (2009), 832–852. DOI:10.1016/j.fss.2008.05.015

[29] A. Xie and H. Liu: On the distributivity of uninorms over nullnorms. Fuzzy Sets Syst. *211* (2013), 62–72. DOI:10.1016/j.fss.2012.05.008

[30] R. Yager and A. Rybalov: Uninorm aggregation operators. Fuzzy Sets Syst. *80* (1996), 111–120. DOI:10.1016/0165-0114(95)00133-6

[31] R. Yager and A. Rybalov: Bipolar aggregation using the Uninorms. Fuzzy Optim. Decis. Making *10* (2011), 59–70. DOI:10.1007/s10700-010-9096-8

[32] R. Yager: Uninorms in fuzzy systems modeling. Fuzzy Sets Syst. *122* (2001), 167–175. DOI:10.1016/s0165-0114(00)00027-0

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