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SUM-OF-SQUARES BASED OBSERVER DESIGN FOR POLYNOMIAL SYSTEMS WITH A KNOWN FIXED TIME DELAY

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An observer for a system with polynomial nonlinearities is designed. The system is assumed to exhibit a time delay whose value is supposed to be constant and known. The design is carried out using the sum-of-squares method. The key point is defining a suitable Lyapunov–Krasovskii functional. The resulting observer is in form of a polynomial in the observable variables. The results are illustrated by two examples.

Keywords: sum-of-squares polynomial, observer, polynomial system

Classification: 93B51

1. INTRODUCTION

There are numerous examples of systems in biology, technology, sociology etc. that exhibit a time delay. Moreover, a time delay occurs in control systems with communication over networks or systems with quantization of input or output signals. This is why problems related to analysis and synthesis of control of time delay systems are intensively studied nowadays.

Linear matrix inequalities (LMI) are a standard tool for handling these problems [11]. Two main approaches are used: the first one is based on the Lyapunov–Krasovskii functional while the second one uses the Lyapunov–Razumikhin functional. Many results allow to deal with nonlinearities by estimating the nonlinearity by the Lipschitz inequality. If the delay can vary with time, its derivative must usually be bounded in order to use the Lyapunov–Krasovskii functional while the Razumikhin functional allows even discontinuous time delay. On the other hand, the Lyapunov–Krasovskii functional yields less conservative results, see [8]. Both approaches use a weighting function in form of a function of the current state. In addition, the Lyapunov–Krasovskii functional contains a set of time integrals. How conservative and/or general the result is depends on a suitable choice of these integral terms. However, [26] points out that a too large number of optimization variables leads to practical problems with implementation of the proposed scheme. There exist numerous results using this approach. As an example, let

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us mention [9, 19, 25] which present various results on stabilization of linear time-delay systems using the Lyapunov–Krasovskii functional.

A natural generalization of the LMIs seem to be the so-called sum-of-squares polynomials (s.o.s. polynomials). Analysis of dynamic polynomial systems was studied in [24] and others. Controller synthesis is more challenging than in the case of LMIs as, unlike the case of linear systems, a simple conversion allowing to recover the convex structure cannot be carried out. Nevertheless, control design using s.o.s. polynomials is presented in [2, 12, 27], for bilinear systems in [23], the use of Hamilton-Jacobi equations is described in [14], see also theses [16, 33]. Lam [18] proposes a controller for a polynomial time delay system using the Lyapunov–Krasovskii functional with quadratic weighting matrices. The quantized feedback is also studied there.

Input-to-state (ISS) stability in connections with time delay systems is investigated in [6] using LMI and in [7] using nonlinear matrix inequalities (matrix inequalities depending also on the state of the system). More about ISS can be found among other papers in [21, 32]. The similar notion of integral Input-to-state stability is studied e. g. in [15].

Observers for time delay systems have been proposed by [22] using algebraic considerations, [10] introduces an observer using delayed values of the output injection. An observer with a cascade structure is designed in [31]. A partial differential equation is solved in [17] whose solution is used to construct a Luenberger-type observer for time delay systems.

Observers derived using LMIs are often applicable to nonlinear systems as well as long as the nonlinearity can be estimated using the Lipschitz inequality. For instance, [30, 35] rely on the Lyapunov–Krasovskii functional or [4] where the Razumikhin functional is used. [36] derives an observer using the Lyapunov–Krasovskii functional for the case when the delay is known and fixed which is the same assumption as in this paper. An observer for a discrete-time nonlinear system is proposed in [5], the design is based on the Lyapunov–Krasovskii functional. [34] introduces a combination of an observer for a time delay system (proposed by the Lyapunov–Krasovskii functional again) with a reset element improving the settling time and overshoot performance. A separation principle for a class of nonlinear time delay systems is presented in [3]. The connection between the observer problem for a system without the time delay and the input-to-state stability (ISS) was studied in [1]. A s.o.s. based observer design for a polynomial system without presence of time delays is designed in [13].

An observer for nonlinear time-delay systems with polynomial nonlinearities is proposed in this paper using the s.o.s. methodology using a suitable Lyapunov–Krasovskii functional. The time delay is assumed to be known. The design procedure is inspired by some results concerning synthesis of control systems for time-delay polynomial systems. This involves above all the choice of the Lyapunov–Krasovskii functional. Some of the results also appeared in [28], however, in this paper, they are extended. Effects of imprecise knowledge of time delays are studied in [29].

After this introductory section, basic facts about sum of squares polynomials and input to state stability are repeated. Next, the observer problem for polynomial time delay systems is described. The observer is designed in the following section. Verification is carried out using simple examples, one presenting a comparison with existing results.

2. SUM OF SQUARES POLYNOMIALS AND INPUT TO STATE STABILITY

This part serves as a brief survey of definitions needed in the following text. A more detailed explanation can be found in the referenced sources, e. g. [16, 33].

Definition 2.1. A polynomial $p(x_1, \ldots, x_N)$ is said to be a sum-of-squares polynomial (s.o.s. polynomial) if there exist m polynomials $p_1(x_1, \ldots, x_N), \ldots, p_m(x_1, \ldots, x_N)$ such that

$$p = p_1^2 + \cdots + p_m^2. \tag{1}$$

The problem of finding polynomials $p_1, \ldots p_m$ for a given polynomial p such that (1) holds can be converted into a convex optimization problem. Efficient and user-friendly software for this conversion is available, [20] was used to obtain results in this paper.

S.o.s. polynomials establish a convenient platform for investigation of stability of polynomial systems. They can be considered as a generalization of the linear matrix inequalities to these systems. In brief, stability analysis boils down to the question whether there exists a function in a form of a s.o.s. polynomial such that its derivative along trajectories multiplied by -1 is again a s.o.s. polynomial. Such a function is the Lyapunov function for the investigated system. Contrary to the linear case where LMIs are used, synthesis of a controller is a more challenging task. While both problems are in its raw form nonconvex, convexity in the linear case is easily reestablished. This is done by a suitable transformation of the set of LMI. This cannot be carried out in the case of polynomial systems as this transformation would be nonlinear. Hence, iterative algorithms were proposed. On the other hand, observer problem seems to possess convexity property so that this action is not necessary.

Definition 2.2. A continuous function $f : [0, a) \to [0, \infty)$ is a $\mathcal{K}$-function if it is strictly increasing and $f(0) = 0$.

A continuous function $f : [0, a) \times [0, \infty) \to [0, \infty)$ is a $\mathcal{KL}$-function if $f(x, t)$ is a $\mathcal{K}$-function as function of x for every fixed t and $\lim_{t\to\infty} f(x, t) = 0$ for every fixed x.

Definition 2.3. A system

$$\dot{x} = f(x(t), x(t - h), u)$$

with $h > 0$ and initial condition $x(0) = x_0, x(t) = \varphi(t)$ for $t < 0$ is locally input-to-state stable (ISS) if there a $\mathcal{KL}$ function β, a $\mathcal{K}$ function γ and positive constants k_1, k_2 such that

$$\|x(t)\| \le \beta(\|x_0\| + \|x\|_{L^\infty(-h,0)}, t) + \gamma(\|u\|_{L^\infty(0,T)}) \text{ for all } t \ge 0,\ T \in (0, T)$$

and $\|u\|_\infty \le k_1$, $\|x_0\| + \|x\|_{L^\infty(-h,0)} \le k_2$.

The definition of local ISS can be found in [21] for systems without time delay, in [6, 7] for time-delay systems. As global ISS is too strong for our purpose, it is necessary to deal with local ISS.

The following theorem can be found (in a slightly more general version) in [6] as Proposition 1.

Theorem 2.4. Consider the system $\dot{x} = f(x(t), x(t-h)) + Bu(t)$ with the initial conditions as above. If there exist $a > 0$, $b > 0$ and a Lyapunov–Krasovskii functional V satisfying $\dot{V} + aV < b\|u\|^2$ then

$$V(t) \le e^{-at}V(0) + (1 - e^{-at})\frac{b}{a}|u|^2_{[0,t],\infty} \tag{2}$$

which implies input-to-state stability.

3. PROBLEM SETTING

First, we introduce the following notation: if ξ is a function of time t, then this argument is omitted, the subscript h denotes the time delay: $\xi = \xi(t)$ and $\xi_h = \xi(t-h)$. The argument is specified if it is different from t or $t-h$.

The observer problem for the time-delay system is introduced here. The delay occurring in the system is denoted by h, $h > 0$ is assumed. Moreover, this delay is constant and known. The system is described by equations

$$\dot{x} = f(x, x_h) + Bw \tag{3}$$

$$y = Cx \tag{4}$$

where $x \in R^n$ is the state, $y \in R^p$ is the measurable output and, where applicable, $B \in R^{n\times q}$, $w \in R^q$ is the disturbance. Assume also the origin is the equilibrium of the system (3).

The function $f : R^n \to R^n$ is supposed to be polynomial and $B \in R^{n\times q}$. For the sake of simplicity of further computations, the output is supposed to be linear function of the state. In practice, this is often the case. Extension of the results presented in this paper to a case where output depends nonlinearly on the state requires simple but lengthy computations.

The problem is to design an observer for the system (3). It is the system

$$\dot{\hat{x}} = f(\hat{x}, \hat{x}_h) + l(Ce, Ce_h) \tag{5}$$

where $e = x - \hat{x}$ is the observation error. To be specific, the goal is to find the polynomial $l : R^{2p} \to R^n$ so that

$$\lim_{t\to\infty} \|e(t)\| = 0.$$

Remark 3.1. The nonlinear observer can be designed so that it uses more values of the measurable quantity Ce. For example, all values $Ce(\tau)$ for all $\tau \in [t-h, t]$. However, due to practical problems with storing such amount of data, we restrict our attention to the case when the current value Ce and Ce_h are used.

Assumption 3.2. There exists a constant $M_x > 0$ such that $\|\hat{x}(t)\| < M_x$ for each $t > 0$.

In order to simplify the notation let us define

$$\Phi(\hat{x}, e, h) = f(\hat{x} + e, \hat{x}_h + e_h) - f(\hat{x}, \hat{x}_h).$$

The error dynamics obeys the equation

$$\dot{e} = \Phi(\hat{x}, e, h) - l(Ce, Ce_h) + Bw(t). \tag{6}$$

4. OBSERVER ANALYSIS AND DESIGN

4.1. Preliminaries

The Lyapunov–Krasovskii functional (in the form suitable for problems with known and fixed time delay) is defined in this section. It consists of two terms, $V = V_1 + V_2$. Two cases will be distinguished. First one: no disturbances are present, hence convergence of the observation error to zero can be guaranteed while in the second case, disturbances are present and their influence on the overall behavior of the observer is investigated by means of the ISS.

Moreover, the cases of observer analysis (the observer gain l is given, the task is to verify stability of the observer) and observer design (the observer gain l is to be found so that the observation error converges to zero) are treated separately.

The functional V_2 remains the same for both analysis and design cases while the functional V_1 differs in both cases and is defined in the subsequent subsections.

Let $\nu_p, \nu_q > 0$ be an even integers (in practice, these values are defined by the user). Assume q is a s.o.s. polynomial in variables e, e_h with degree up to ν_q. Then the function V_2 is defined as

$$V_2(e) = \int_{t-h}^{t} q(e(\alpha))\,\mathrm{d}\alpha. \tag{7}$$

The functional (7) is changed as follows if the ISS is required

$$V_2^{'}(e) = \int_{t-h}^{t} e^{a(\alpha-t)} q(e(\alpha))\,\mathrm{d}\alpha \tag{8}$$

with some $a > 0$. The derivatives of the functions V_2 and $V_2^{'}$ satisfy the relations

$$\dot{V}_2 = q(e) - q(e_h) \tag{9}$$

$$\dot{V}_2^{'} = -aV_2^{'} + q(e) - q(e_h)e^{-ah}. \tag{10}$$

As shown in [24] and [36], the Lyapunov–Krasovskii functional in this form is suitable to handle the case when the time delay is known. In the case when capability of dealing with an unknown time delay (only an upper bound on this delay would be known), a more general form is required, see [29].

Finding the above functionals is a key part of the solution of the Observer analysis or Observer design problems. However, the solution on the whole Euclidean space might be infeasible or the result might be too conservative. To restrict the convergence region one can employ the following standard procedure: Let the desired region Ω be defined by the following set of N inequalities

$$\Omega = \{(e, \hat{x}) \mid g_i(e, \hat{x}) < 0,\ i = 1, \ldots, N\} \tag{11}$$

where g_i are suitable polynomials. It is assumed $0 \in \Omega$. The following proposition can be found in [24].

Proposition 4.1. The problem of finding a s.o.s. representation of a polynomial p on the domain Ω is equivalent to finding a s.o.s. representation of the polynomial

$$p + s_1 g_1 + \cdots + s_N g_N$$

on the whole Euclidean space while s_i are s.o.s. polynomials.

The following lemma is useful for estimating the terms containing the disturbance w.

Lemma 4.2. Let $\pi : R^{2n} \to R^m$ be a (row) vector function with polynomial elements. Denote the vector containing all monomials that occur in $\pi(e, \hat{x})$ by $\eta(e, \hat{x})$. Let r denote the length of the vector $\eta(e, \hat{x})$. Finally define the matrix $\Pi \in R^{r\times m}$ so that

$$\pi(e, \hat{x}) = \eta(e, \hat{x})^T \Pi.$$

Assume also there are symmetric positive definite matrices $R \in R^{r\times r}$, $S \in R^{n\times n}$ such that

$$\begin{pmatrix} R & \Pi B \\ B^T\Pi^T & S \end{pmatrix} > 0.$$

Then

$$|\pi(e, \hat{x})Bw| \leq \eta^T(e, \hat{x})R\eta(e, \hat{x}) + w^T Sw.$$

4.2. Observer analysis

Let p be a s.o.s. polynomial up to a degree ν_p in variables $e, \hat{x}$ such that $p(\xi, \zeta) = 0$ ($\xi, \zeta \in R^n$) implies $\xi = 0$ for all ζ and

$$V_{a,1} = p(e, \hat{x}). \tag{12}$$

The Lyapunov–Krasovskii functional is then

$$V = V_{a,1} + V_2 \tag{13}$$

without presence of disturbances. In the opposite case, this functional is changed into

$$V = V_{a,1} + V_2^{'}. \tag{14}$$

Let the function $\mathcal{P}$ be defined as

$$\mathcal{P}(e, \hat{x}) = (\nabla_e p(e, \hat{x}), \nabla_x p(e, \hat{x}))$$

(the symbols ∇_e and ∇_x mean the gradients with respect to the first and last n variables, respectively) and $V_1 = V_{a,1}$. Then, the derivative of the Lyapunov–Krasovskii functional satisfies the following relations:

1. Without disturbances,

$$\dot{V}_{a,1} = \mathcal{P}(e, \hat{x}) \begin{pmatrix} \Phi(\hat{x}, e, h) - l(Ce, Ce_h) \\ f(\hat{x}, \hat{x}_h) + l(Ce, Ce_h) \end{pmatrix}. \tag{15}$$

2. Presence of the disturbance w changes the latter result into:

$$\dot{V}_{a,1}^{'} = \mathcal{P}(e, \hat{x}) \begin{pmatrix} \Phi(\hat{x}, e, h) - l(Ce, Ce_h) + Bw \\ f(\hat{x}, \hat{x}_h) + l(Ce, Ce_h) \end{pmatrix}. \tag{16}$$

In the latter case, denote the vector of all monomials occurring in $\nabla_e p(e,\hat{x})$ by $\eta_a(e,\hat{x})$. Also denote by Π_a the matrix satisfying $\nabla_e p(e,\hat{x}) = \eta_a^T(e,\hat{x})\Pi_a$. Lemma 4.1. guarantees existence of a matrices X, Z such that

$$\begin{aligned} |\nabla_e p(e,\hat{x})Bw| &\leq \eta_a^T(e,\hat{x})X\eta_a(e,\hat{x}) + w^T Z w, \\ 0 &< \begin{pmatrix} X & \Pi_a B \\ B^T\Pi_a^T & Z \end{pmatrix}. \end{aligned}$$

To sum up, the algorithm for verifying stability of a polynomial observer l can be written as follows:

1. If no ISS is to be guaranteed:

Algorithm 4.3. Find

- a s.o.s. polynomial p in the variables $e, \hat{x}$, a s.o.s. polynomial q in variable e,
- s.o.s. polynomials $s_1, \ldots, s_N$ in the variables $e, e_h, \hat{x}, \hat{x}_h$

such that

$$\begin{aligned} &-\mathcal{P}(e,\hat{x})\begin{pmatrix} \Phi(\hat{x},e,h) - l(Ce,Ce_h) \\ f(\hat{x},\hat{x}_h) + l(Ce,Ce_h) \end{pmatrix} \\ &-q(e) + q(e_h) + s_1 g_1 + \cdots + s_N g_N \end{aligned} \tag{17}$$

is a s.o.s. polynomial.

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2. If ISS is required:

Algorithm 4.4. Find

- a s.o.s. polynomial p in the variables $e, \hat{x}$, a s.o.s. polynomial q in variable e,
- s.o.s. polynomials $s_1, \ldots, s_N$ in the variables $e, e_h, \hat{x}, \hat{x}_h$,
- positive definite matrices X, Z

such that

$$\begin{aligned} &-\mathcal{P}(e,\hat{x})\begin{pmatrix} \Phi(\hat{x},e,h) - l(Ce,Ce_h) \\ f(\hat{x},\hat{x}_h) + l(Ce,Ce_h) \end{pmatrix} - \eta_a^T(e,\hat{x})X\eta_a(e,\hat{x}) \\ &-q(e) + e^{-ah}q(e_h) + aV_{a,1} + s_1 g_1 + \cdots + s_N g_N \end{aligned} \tag{18}$$

is a s.o.s. polynomial,

$$\begin{pmatrix} X & \Pi_a B \\ B^T\Pi_a^T & Z \end{pmatrix} > 0.$$

$\|Z\|$ is minimized and $\nabla_e p(e,\hat{x}) = \eta_a^T(e,\hat{x})\Pi_a$.

4.3. Observer design

The observer to be designed cannot depend on unmeasurable quantities. As the function V_1 also influences the observer gain it is necessary to specify its form to satisfy this requirement. First, assume that

$$C = (I_{p\times p}, 0_{p\times(n-p)}). \tag{19}$$

Let $\bar{P} \in R^{n\times n}$ be a symmetric positive matrix and $S : R^p \times R^n \to R^{p\times p}$ be a matrix function whose elements are polynomials up to a degree ν_p vanishing at the origin together with their derivatives, $S(\xi,\zeta)$ is symmetric positive semidefinite for all (ξ,ζ). Define the matrix function $P : R^p \times R^n \to R^{n\times n}$ as

$$P(\xi,\zeta) = \bar{P} + \begin{pmatrix} S(\xi,\zeta) & 0_{p\times(n-p)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times(n-p)} \end{pmatrix}. \tag{20}$$

Remark 4.5. If the matrix C has not the form assumed above but has full rank, then one can find a matrix $\bar{C} \in R^{(n-p)\times n}$ such that $\operatorname{rank}(C^T, \bar{C}^T) = n$ and transform the states using $\bar{x} = (C^T, \bar{C}^T)^T x$. Hence, without loss of generality, the matrix C can be assumed to attain the form (19).

Define the functions $V_{d,1}$ and s_{ij}, $i,j = 1,\dots,p$ as

$$\begin{aligned} V_{d,1}(e) &= \tfrac{1}{2} e^T P(Ce,\hat{x}) e, \\ S(Ce,\hat{x}) &= \begin{pmatrix} s_{11}(Ce,\hat{x}) & \dots & s_{1p}(Ce,\hat{x}) \\ \vdots & & \vdots \\ s_{p1}(Ce,\hat{x}) & \dots & s_{pp}(Ce,\hat{x}) \end{pmatrix} \end{aligned} \tag{21}$$

and functions $\bar{\rho}, \bar{\sigma} : R^{p\times n} \to R^{n\times n}$ by

$$\begin{aligned} \rho(Ce,\hat{x}) &= \tfrac{1}{2} \begin{pmatrix} \nabla_{Ce} s_{11}(Ce,\hat{x}) & \dots & \nabla_{Ce} s_{1p}(Ce,\hat{x}) \\ \vdots & & \vdots \\ \nabla_{Ce} s_{p1}(Ce,\hat{x}) & \dots & \nabla_{Ce} s_{pp}(Ce,\hat{x}) \end{pmatrix} \left(I_{p\times p} \otimes Ce\right) \\ \sigma(Ce,\hat{x}) &= \tfrac{1}{2} \begin{pmatrix} \nabla_{x} s_{11}(Ce,\hat{x}) & \dots & \nabla_{x} s_{1p}(Ce,\hat{x}) \\ \vdots & & \vdots \\ \nabla_{x} s_{p1}(Ce,\hat{x}) & \dots & \nabla_{x} s_{pp}(Ce,\hat{x}) \end{pmatrix} \left(I_{n\times n} \otimes Ce\right) \\ \bar{\rho}(Ce,\hat{x}) &= \begin{pmatrix} \rho(Ce,\hat{x}) & 0_{p\times(n-p)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times(n-p)} \end{pmatrix} \\ \bar{\sigma}(Ce,\hat{x}) &= \begin{pmatrix} \sigma(Ce,\hat{x}) & 0_{p\times(n-p)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times(n-p)} \end{pmatrix} \end{aligned}$$

where the symbol $I_{m\times m}$ stands for the identity matrix of dimension m, $0_{l\times m}$ denotes the zero matrix of dimension $l \times m$, ∇_{Ce} means the gradient with respect to first p variables while ∇_x denotes the gradient with respect to the last n variables. Both are supposed to be row vectors. In this case, define

$$\mathcal{P}(e,\hat{x}) = \Big(e^T P(Ce,\hat{x}) + e^T \bar{\rho}(Ce,\hat{x}), e^T \bar{\sigma}(Ce,\hat{x})\Big).$$

First, let us note that the observer design problem is not linear in the optimization parameters – the elements of P and l. In order to establish convexity of the problem let us define the function Λ by

$$\Lambda(Ce, Ce_h) = \Big(-P(Ce,\hat{x}) - \bar{\rho}(Ce,\hat{x}) + \bar{\sigma}(Ce,\hat{x})\Big) l(Ce, Ce_h). \tag{22}$$

As in the observer analysis case, the terms describing the disturbance are estimated using the Lemma 4.1. Denote the vector containing all monomials in $e^T(P(Ce,\hat{x}) + \bar{\rho}(Ce,\hat{x}))$ by $\eta_d^T(Ce,\hat{x})$ and let $e^T(P(Ce,\hat{x}) + \bar{\rho}(Ce,\hat{x})) = \eta_d^T(Ce,\hat{x})\Pi_d$. If there exist symmetric positive definite matrices X, Z such that

$$\begin{pmatrix} X & \Pi_d B \\ B^T \Pi_d^T & Z \end{pmatrix} > 0$$

then $e^T(P(Ce,\hat{x}) + \rho(Ce,\hat{x}))Bw \le \eta_d^T(Ce,\hat{x}) X \eta_d(Ce,\hat{x}) + w^T Z w$.

1. If no ISS is to be guaranteed, the following s.o.s. problem reads:

Algorithm 4.6. Find

- a s.o.s. polynomial q in the variable e,
- s.o.s. polynomials $s_1, \dots, s_N$ in the variables $e, e_h, \hat{x}, \hat{x}_h$,
- the matrix $P(\xi,\zeta) \in R^{p\times n}$ such that $P(\xi,\zeta)$ is positive definite on the set Ω defined above.

such that

$$-\mathcal{P}(e,\hat{x}) \begin{pmatrix} \Phi(\hat{x}, e, h) \\ f(\hat{x}, \hat{x}_h) \end{pmatrix} - \Lambda(Ce, Ce_h) - q(e) + q(e_h) + s_1 g_1 + \cdots + s_N g_N \tag{23}$$

is a s.o.s. polynomial.

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2. If ISS is required:

Algorithm 4.7. For a fixed $a > 0$ find

- s.o.s. polynomial q in the variables e,
- s.o.s. polynomials $s_1, \dots, s_N$ in the variables $e, e_h, \hat{x}, \hat{x}_h$,

- symmetric positive definite matrices X, Z,
- the matrix $P(\xi,\zeta) \in R^{n\times n}$ such that $P(\xi,\zeta)$ is positive definite on the set Ω defined above

such that

$$-\mathcal{P}(Ce,\hat{x})\begin{pmatrix}\Phi(\hat{x},e,h)\\ f(\hat{x},\hat{x}_h)\end{pmatrix} - \Lambda(Ce,Ce_h) - \eta_d^T(Ce,\hat{x})X\eta_d(Ce,\hat{x}) \tag{24}$$
$$-q(e)+q(e_h)e^{-ah}+aV_{d,1}+s_1g_1+\cdots+s_Ng_N$$

is a s.o.s. polynomial,

$$\begin{pmatrix} X & \Pi_d B\\ B^T\Pi_d^T & Z\end{pmatrix} > 0$$

and $\|Z\|$ is minimized.

The observer l is recovered by

$$l(Ce,Ce_h) = \Big(-P(Ce,\hat{x}) - \bar{\rho}(Ce,\hat{x}+\bar{\sigma}(Ce,\hat{x})\Big)^{-1}\Lambda(Ce,Ce_h).$$

4.4. Discussion

Lemma 4.8. 1. If $w=0$, there exists a neighborhood U of the origin such that the observer satisfying (17) or (23) guarantees $\lim_{t\to\infty}|e(t)|=0$ if initial conditions of the system and the observer are in U .

2. In presence of disturbances, there exist constants $C,c>0$ (C being dependent on initial conditions) and an open set $0\in\bar{\Omega}\subset\Omega$ such that if $(e(t),\hat{x}(t))\in\bar{\Omega}$ then

$$|e|_{[0,t],\infty} \le c|w|_{[0,t],\infty} + Ce^{-at}.$$

P r o o f. Denote $\mathcal{P}=\mathcal{P}(e,\hat{x})$, $\eta(e,\hat{x})=\eta_a(e,\hat{x})$, $V_1=V_{a,1}$ for the observer analysis case and $\mathcal{P}=\mathcal{P}(Ce,\hat{x})$, $\eta(e,\hat{x})=\eta_d(Ce,\hat{x})$, $V_1=V_{d,1}$ for the observer design case.

ad 1. The relations (17) and (23) imply $\dot{V}_1+\dot{V}_2<0$ for $e\neq 0$ (with $V_1=V_{a,1}$ for the observer analysis case and $V_1=V_{d,1}$ for the observer design case). Hence $V_1\to 0$ which, due to assumptions on the polynomial p or the function P, implies $e(t)\to 0$.

ad 2. Let

$$W=\mathcal{P}\begin{pmatrix}\Phi(\hat{x},e,h)-l(Ce,Ce_h))\\ f(\hat{x},\hat{x}_h)+l(Ce,Ce_h)\end{pmatrix} + \eta^T(e,\hat{x})X\eta(e,\hat{x}) + aV_1.$$

Then $\dot{V}_1 \le W + w^TZw$. On the set Ω, this together with (10) implies

$$\dot{V}_1+\dot{V}_2' + a(V_1+V_2') \le W + q(e) - q(e_h)e^{-ah} + w^TZw.$$

Taking (18) or (24) into account one arrives at $\dot{V}_1+\dot{V}_2'+a(V_1+V_2')\le w^TZw$. This, positive definiteness of $P(Ce,\hat{x})$ on a neighborhood of $e=0$ and (2) yields the result.

□

Remark 4.9. The set Ω defined in (11) is merely the set where nonpositivity of the derivative of the Lyapunov–Krasovskii functional is guaranteed. It is by no means the domain of attraction - it is a subset of the set Ω. Hence convergence is guaranteed on a subset U of this set.

Remark 4.10. In practice, maximal degrees of all polynomials are chosen by the user a-priori. Often, predefining a certain structure of the polynomials (such as lack of certain monomials) can be helpful to achieve better computational efficiency.

Remark 4.11. Thanks to the special structure of the function P defined by (20), the observer gain l depends only on the observable quantity Ce, Ce_h and the estimate $\hat{x}$. The invertibility of the matrix $-P(Ce,\hat{x}) - \bar{\rho}(Ce,\hat{x}) + \bar{\sigma}(Ce,\hat{x})$ is not guaranteed. This issue requires some further analysis. However, if the function V_1 is quadratic (which implies $P(Ce,\hat{x}) = \bar{P}$), invertibility is guaranteed as the matrix $\bar{P}$ is regular by assumption. In this case, the function $V_{a,1}$ and $V_{d,1}$ change into

$$V_{a,1} = \bar{P}(\Phi(\hat{x},e,h) - l(Ce,Ce_h),\ V_{d,1} = \bar{P}\Phi(\hat{x},e,h) - \Lambda,$$

with $\Lambda = \bar{P}l(Ce,Ce_h)$. For majority of practical applications, a quadratic function P is sufficient.

5. EXAMPLES

5.1. Example

As an example the system

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1(t) \\ \dot{x}_2 &= -x_1 - x_{h,1} + 0.25x_1^3 - 0.5x_{h,1}^3 + w_2(t) \end{aligned}$$

is used. The delay $h = 0.5$. The value of the constant a was set as $a = 1$, the measurable output is x_1.

The observer is in the form

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + l_1(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= -\hat{x}_1 - \hat{x}_{h,1} + 0.25\hat{x}_1^3 - 0.5\hat{x}_{h,1}^3 + l_2(x_1 - \hat{x}_1). \end{aligned}$$

The polynomial Λ is sought in the form

$$\Lambda(\xi) = \begin{pmatrix} l_{11}\xi + l_{13}\xi^3 \\ l_{21}\xi + l_{23}\xi^3 \end{pmatrix}$$

while

$$q(\xi_1,\xi_2) = q_1\xi_1^2 + q_{12}\xi_1\xi_2 + q_2\xi_2^2 + q_3\xi_1^4 + q_4\xi_1^2\xi_2^2 + q_5\xi_2^4.$$

Numerical simulations with other forms of the above mentioned polynomials show that the only terms indicated above are significant. Values of other terms were close to zero.

It was assumed that $x(t)$ does not exceed 6 at any time:

$$\begin{array}{lcl}(\hat{x}_1+e_1)^2+(\hat{x}_2+e_2)^2 & \leq & 36\\ (\hat{x_{h1}}+e_{h1})^2+(\hat{x_{h2}}+e_{h2})^2 & \leq & 36\end{array}$$

hence, the behavior of the observer outside of this bound is not guaranteed - see the Remark 4.9.

Initial values of the observed system were chosen as $x(t)=(\frac{4}{3},2)^T$ for $t\leq 0$. The initial values of the observer were all equal to zero. The norm of disturbances was bounded by 5. Maximal eigenvalue of the matrix Z was 27.8.

The resulting observer is described by

$$l(e_1)=\begin{pmatrix}-3.73e_1-17.41e_1^3\\ -11.42e_1-23.18e_1^3\end{pmatrix}.$$

The following figures illustrate the results. Figure 1 shows the ability of the observer to reconstruct the state x_2. In this case, no disturbance was added. Figure 2 illustrates the effect of disturbances added to the state of the observed system. Figures 3, 4 illustrate the transition phase in more details and also provide a comparison with a linear observer. This observer was constructed by taking the linear terms $(l_{11},l_{21})^T(x_1-\hat{x}_1)$ into account. One can see that the observation error is significantly larger for this linear observer. Finally, comparison of these observation errors is depicted in Figure 5.

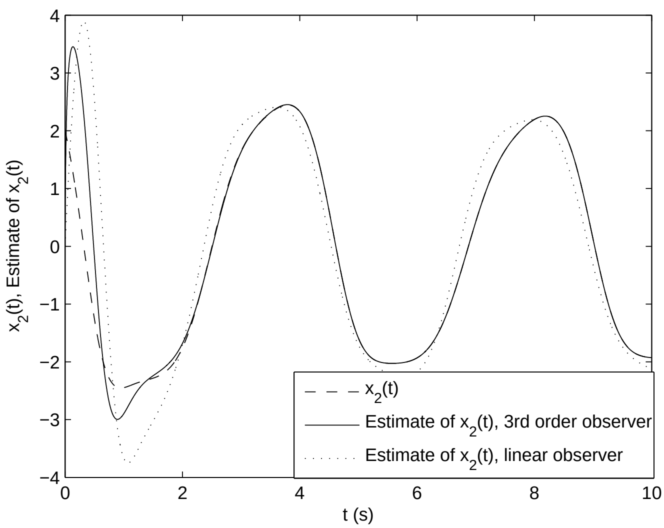

Fig. 1. State x_2.

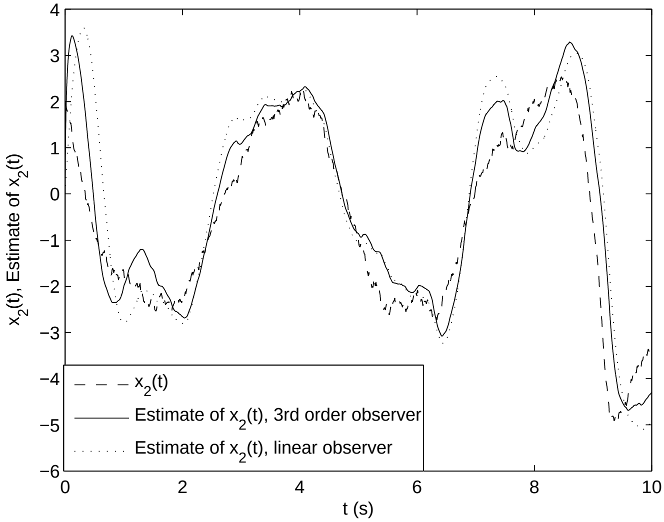

Fig. 2. State x_2 with disturbance.

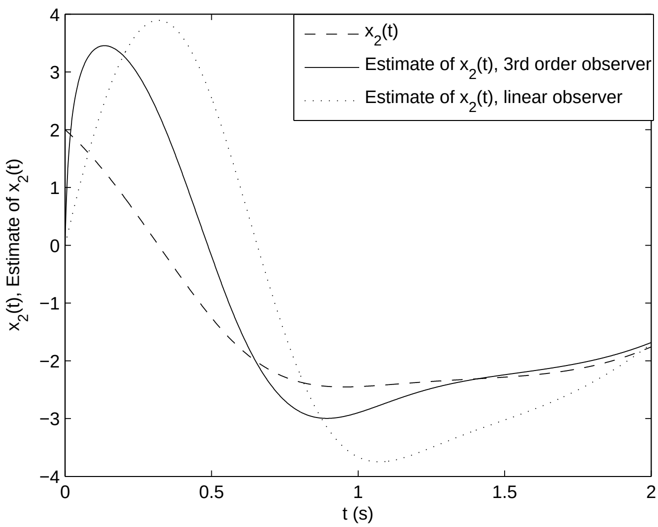

Fig. 3. State x_2, detail and comparison with the linear observer.

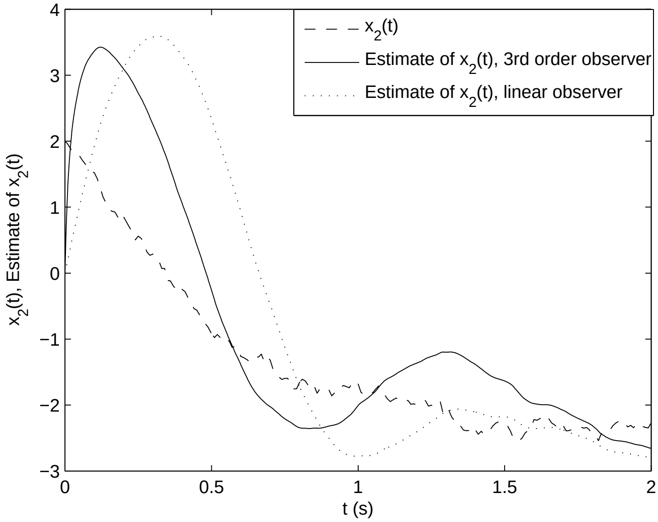

Fig. 4. State x_2, detail and comparison with the linear observer, disturbance added.

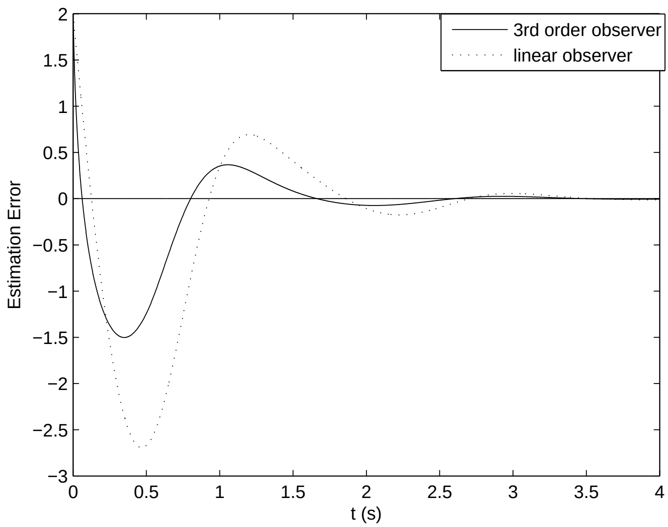

Fig. 5. Observation error for the polynomial and linear observers.

5.2. Example

This example compares the observer proposed in this paper with the observer for non-linear time delay systems proposed in [10]. The observed system is the same as in the cited paper and has the form

$$\begin{aligned} \dot{x}_1 &= -3x_2 + 0.5x_{h,1}x_{2h} \\ \dot{x}_2 &= -x_{h,1}^2 x_{h,2} + u \\ y &= x_1 \end{aligned}$$

where $u(t) = \sin 2t$. The time delay was chosen $h = 0.2$ and initial conditions are $x(t) = (1, -1.5)^T$ for $t \in [-h, 0]$. The observer is described by the equations

$$\begin{aligned} \dot{\hat{x}}_1 &= -3\hat{x}_2 + 0.5\hat{x}_{h,1}\hat{x}_{h,2} + v_1 \\ \dot{\hat{x}}_2 &= -\hat{x}_{h,1}^2 \hat{x}_{h,2} + u + v_2 \\ v_1 &= k_1(x_1 - \hat{x}_1) \\ v_2 &= k_2(x_1 - \hat{x}_1) - \frac{1}{6}(\hat{x}_{h,2}v_{h,1} + \hat{x}_{h,1}v_{h,2}) \end{aligned}$$

with constants k_1, k_2 chosen as $k_1 = -3$, $k_2 = \frac{2}{3}$ which are the same values as in [10]. This example serves as a reference case.

For the polynomial s.o.s.-based observer, third order polynomials were used:

$$l(e_1) = \begin{pmatrix} l_{11}e_1 + l_{12}e_1^2 + l_{13}e_1^3 \\ l_{21}e_1 + l_{22}e_1^2 + l_{23}e_1^3 \end{pmatrix}.$$

In order to achieve a fair comparison, the linear terms are equal to linear terms in the reference case. Hence, the variables l_{11}, l_{21} were not subject of the s.o.s. computation, the differences in the behavior of both observers is solely due to nonlinear terms. The polynomials g_i were chosen so that $\dot{V} < 0$ on the set $\|x_h\|^2 + \|e_h\|^2 \le 9$. The results of s.o.s. computation are

$$l(e_1) = \begin{pmatrix} -3e_1 + 0.0417e_1^2 - 0.5051e_1^3 \\ \frac{2}{3}e_1 + 0.0439e_1^2 - 0.7934e_1^3 \end{pmatrix}.$$

The error estimates are compared in Figure 6.

6. CONCLUSIONS

A sum-of-squares based method for observer design was presented. It is suitable for polynomial systems with time delays. It extends the known results for linear time-delay systems as it uses similar techniques like the Lyapunov–Krasovskii functionals. Viability of the proposed method for solving practical problems was demonstrated by examples.

Observers for systems with a variable time delay or multiple time delays will be treated in future. This case will require to use more general Lyapunov–Krasovskii functionals. Also, influences of uncertainties in the system description will be investigated by means of input-to-state stability.

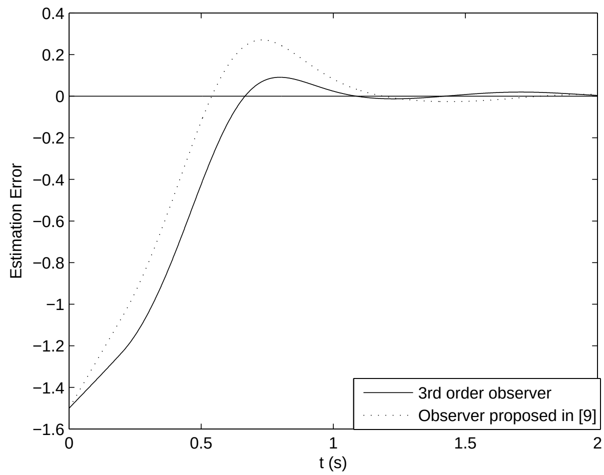

Fig. 6. Observation error for the proposed polynomial observer and the reference observer.

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