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ON STABLE CONES OF POLYNOMIALS VIA REDUCED ROUTH PARAMETERS

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A problem of inner convex approximation of a stability domain for continuous-time linear systems is addressed in the paper. A constructive procedure for generating stable cones in the polynomial coefficient space is explained. The main idea is based on a construction of so-called Routh stable line segments (half-lines) starting from a given stable point. These lines (Routh rays) represent edges of the corresponding Routh subcones that form (possibly after truncation) a polyhedral (truncated) Routh cone. An algorithm for approximating a stability domain by the Routh cone is presented.

Keywords: linear systems, Hurwitz stability, convex approximation

Classification: 93C05, 93D09

1. INTRODUCTION

The stability is one of the most important properties in the field of control systems. It arises in various applications and has to be taken into account while studying a system or designing an appropriate controller. The stability property can be analyzed in several ways. In case of linear systems the most intuitively understandable and inherently simple test is based on the location of roots of a characteristic polynomial. Other alternatives include Hurwitz, Routh, and Hermite–Bieler tests [6, 18] or frequency domain based techniques [23].

However, once a system contains uncertainties, these techniques cannot be directly applied. This resulted in the development of a parametric approach [4], which links the study of relationships between roots of a polynomial and its coefficients. The main problem appearing with the parametric approach is that, in general, the stability domain is nonconvex in the coefficient space. This challenge has led to the development of techniques for convex approximation of the stability domain such as based on ellipsoids [5, 9], polytopes [11, 14], hyperrectangles [8, 12], and convex directions [19]. The type of convex approximation of the stability domain depends on the type of system parameters uncertainty, for example, rectangular approximation is suitable for interval parameters, and polytopic approximation is applicable for polytopic uncertainties. This paper deals with conic approximation which may be useful for systems with one dominant uncertainty or with several conic type uncertainties.

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In this paper, we provide a simple and efficient algorithm for the convex approximation of the stability domain by polyhedral Routh cones. The method is based on a new multilinear stability criterion for Hurwitz polynomials relying on the so-called reduced Routh parameters. For discrete-time systems the multilinear stability condition is introduced via reflection coefficients of polynomials [14] and the idea of random generation of stable line segments for stabilizing robust controller design is efficiently used [15, 20]. Here, we have proved that for continuous-time systems a similar approach can be used via reduced Routh parameters. The results of this paper can be understood as extension of those presented in [3] and [16]. For [3, 16], and this paper the multilinear stability condition is the main conception. In [16] the method for polytopic approximation of the stability domain is addressed. In the conference paper [3] the main idea of conic approximation of the stability domain is considered. However, the majority of facts are used without detailed proofs. In this paper, we provide the theoretical justification by giving complete proofs. Furthermore, the relevant additional material is added emphasizing relations between papers [3] and [16].

The paper is organized as follows. Section 2 recalls necessary definitions related to stability of polynomials in the continuous-time case. The notion of the reduced Routh parameters is introduced. The next section is devoted to the description of stable half-lines (Routh rays) of polynomials. The main results, related to the approximation of stability domain by polyhedral Routh cone, are addressed in Section 4. The presented material is illustrated by several numerical examples. Concluding remarks and possible directions for the future research are drawn in Section 5. Supplementary material is collected in the Appendix.

2. REDUCED ROUTH PARAMETERS OF POLYNOMIALS

A polynomial of degree n

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \tag{1}$$

with real coefficients $a_i \in \mathbb{R}$, for $i = 0, \ldots, n$, is said to be continuous-time stable in the Hurwitz sense, if all its roots λ_i, for $i = 1, \ldots, n$, are in the open left-half plane of $\mathbb{C}$, i.e., $\Re(\lambda_i) < 0$. Since polynomial (1) is uniquely defined by its coefficients, for simplicity, sometimes, we use a to denote both the polynomial $a(s)$ and the vector $a = \begin{bmatrix} a_n & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$ of its coefficients, i.e., $a := a(s) = \begin{bmatrix} a_n & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$. Then, the Hurwitz *region* $\mathcal{H}_n$ is defined as $\mathcal{H}_n = \big\{ a \in \mathbb{R}^{n+1} \mid (1) \text{ is Hurwitz} \big\}$.

A *stability boundary* is either the boundary of the stability domain in the coefficient space or the boundary of the root location domain (imaginary axis). The stability of polynomials $a(s)$ can be tested by Routh table, see [7]. Based on this criterion, a method for constructing Hurwitz polynomials can be derived as follows [20]. Start with arbitrary Hurwitz polynomial of degree 2. Since positivity of the coefficients is equivalent to stability of the second-order polynomials, generate arbitrary positive numbers h_0, h_1, h_2 and compose the polynomial $a(s) = h_2 s^2 + h_1 s + h_0$ or $a = \begin{bmatrix} a_2 & a_1 & a_0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} h_2 & h_1 & h_0 \end{bmatrix}^{\mathrm{T}}$. At the kth step, having a Hurwitz polynomial of degree k, i.e., $a(s) = \begin{bmatrix} a_k & a_{k-1} & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$, consider two polynomials of degree $k+1$

$$p(s) = \begin{bmatrix} 0 & a_k & a_{k-1} & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$$

and

$$q(s) = \begin{bmatrix} a_k & 0 & a_{k-2} & 0 & a_{k-4} & 0 & \cdots \end{bmatrix}^{\mathrm{T}}.$$

Generate a positive random number h_{k+1} and compose

$$a(s) = p(s) + \frac{h_{k+1}}{a_k} q(s), \tag{2}$$

which is Hurwitz polynomial of degree $k+1$, according to the Routh rule. Proceeding in this manner up to $k = n$, we obtain a Hurwitz polynomial of degree n, see [21, 22]. Thus, the coefficients a_k of the nth-order polynomial are obtained from the Routh *parameters* h_k, $k = 0, \dots, n$ recursively by increasing k. Furthermore, all Hurwitz polynomials of degree n can be obtained using this construction [20]. Next, we introduce the reduced Routh parameters that are used later in construction of stable line segments.

Definition 2.1. The *reduced* Routh parameters w_j for normed polynomials $a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + 1$ are defined as follows

$$\begin{aligned} w_0 &= h_0 = 1, \\ w_1 &= h_1, \\ w_2 &= h_2, \\ w_j &= \frac{h_j}{h_{j-1}}, \quad j = 3, \dots, n. \end{aligned} \tag{3}$$

From (2) and (3) relations for recursive generation of normed Hurwitz polynomials of order $k+1$, for $k > 2$, can be obtained as

$$a(s) = p(s) + w_{k+1} q(s).$$

Denote the degree of a polynomial by superscript to get

$$a^{k+1} = \begin{bmatrix} w_k a_k^k & a_k^k & a_{k-1}^k + w_k a_{k-2}^k & a_{k-2}^k a_{k-3}^k + w_k a_{k-4}^k & \cdots & 1 \end{bmatrix}^{\mathrm{T}}, \tag{4}$$

where $a^k = \begin{bmatrix} a_k^k & a_{k-1}^k & \cdots & 1 \end{bmatrix}^{\mathrm{T}}$. Using matrix notation, equation (4) becomes

$$a^{k+1} = W_k a^k, \tag{5}$$

where W_k is a $(k+1) \times k$ matrix of the form

$$W_k = w_k \begin{bmatrix} J_k \\ \vdots \\ 0^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0^{\mathrm{T}} \\ \vdots \\ I_k \end{bmatrix}$$

with I_k being the $k \times k$ unit matrix and J_k being the $k \times k$ diagonal matrix $J_k = \mathrm{diag}\{1, 0, 1, 0, \ldots\}$, i. e.,

$$W_k = \begin{bmatrix} w_k & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & w_k & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

From recursive relation (5) it follows that $a^n = W_k^n a^k$, where $W_k^n = W_n W_{n-1} \cdots W_k$, $k = n, \ldots, 3$ or

$$a^n = W_3^n a^2 = W_n W_{n-1} \cdots W_3 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix}. \tag{6}$$

Lemma 2.2. The elements in (6), can be calculated using the direct formula

$$a_l^n = \sum_{i_0=1}^{n} \sum_{i_1=1}^{i_0} \cdots \sum_{i_{n-l}=1}^{i_{n-l-1}} \prod_{j=0}^{n-l} \overline{w}_{i_j} \ \mathrm{mod}\ (i_j + n - l - j, 2), \tag{7}$$

where $l = 1, \ldots, n$ is the index number of the corresponding row in (6), $n > 2$, and mod $(\alpha, 2)$ is the usual modulus operation that returns either 1 or 0 depending on whether the number α is odd or even, respectively. Elements $\overline{w}_{i_j}$ in (7) correspond to entries of the matrix W_k as

$$\overline{w}_{i_j} := \begin{cases} w_2/\overline{w}_1 & \text{for } i_j = 2, \\ w_{i_j} & \text{otherwise.} \end{cases} \tag{8}$$

P r o o f. See the detailed explanation in [3] for the proof. □

The inverse mapping from polynomial coefficients a_k to the reduced Routh parameters w_k, $k = n, \ldots, 1$ can be recursively found starting from w_n via (5) as

$$\begin{aligned} w_j &= \frac{a_j^j}{a_{j-1}^j}, \quad j = n, \ldots, 3, \\ w_2 &= a_2^2, \\ w_1 &= a_1^2. \end{aligned} \tag{9}$$

Note that in (9) parameters a_j^j and a_{j-1}^j can be found explicitly as

$$\begin{aligned} a_{k-i-1}^{k-1} &= a_{k-i-1}^k, & i &= 0, \ldots, 2\lfloor (k-2)/2 \rfloor, \\ a_{k-i-2}^{k-1} &= a_{k-i-2}^k - w_k a_{k-i-3}^k, & i &= 0, \ldots, 2\lfloor (k-3)/2 \rfloor \end{aligned}$$

with $a_0 = 1$ or in the matrix form as $a^{k-1} = \overline{W}_k a^k$, where $\overline{W}_k$ is a $k \times k$ matrix

$$\overline{W}_k = I_k - w_k \begin{bmatrix} 0 & \overline{J}_{k-1} \\ \vdots & \vdots \\ 0 & 0^{\mathrm{T}} \end{bmatrix}$$

and $\overline{J}_k$ is a $k \times k$ diagonal matrix, i.e., $\overline{J}_k = \mathrm{diag}\{0, 1, 0, 1, \ldots\}$.

Proposition 2.3. A normed polynomial $a(s)$ with $a_0 = 1$ is Hurwitz stable if and only if $w_k > 0$, $k = 1, \ldots, n$.

Proof. *Necessity*: Assume that a normed polynomial $a(s)$ of order n is stable in the Hurwitz sense. Then, according to Routh stability criterion, all the Routh parameters h_k of stable polynomial $a(s)$ must be positive real numbers $h_k > 0$, $k = 1, \ldots, n$. Thus, (3) yields $w_k > 0$, $k = 1, \ldots, n$.

Sufficiency: From (3) it follows

$$\begin{aligned} h_0 &= 1, \\ h_1 &= w_1, \\ h_2 &= w_2, \\ h_j &= w_j h_{j-1}, \quad j = 3, \ldots, n. \end{aligned}$$

Observe that, if $w_k > 0$, for $k = 1, \ldots, n$, then all Routh parameters of the polynomial $a(s)$ are positive $h_k > 0$, $k = 0, \ldots, n$. Hence, the polynomial $a(s)$ is Hurwitz stable.

□

Proposition 2.4. The mapping (5) from the reduced Routh parameters w_k, for $k = 1, \ldots, n$ to the normed polynomial coefficients a_k^n, $k = 1, \ldots, n$ with $a_0 = 1$ is a one-to-one mapping if $w_k > 0$, $k = 1, \ldots, n$.

Proof. According to the construction procedure, defined by (2), the mapping between the Routh parameters h_k, $k = 1, \ldots, n$ and the polynomial coefficients a_k^n, $k = 1, \ldots, n$ for $a_0 = 1$ is one-to-one, see [20]. Observe that mapping (3) between the reduced Routh parameters w_k, $k = 1, \ldots, n$ and the Routh parameters h_k, $k = 1, \ldots, n$ is one-to-one by $h_0 = 1$ as well. Hence, it remains to note that the composition of two injective functions is injective, and conclusion follows. □

3. STABLE ROUTH RAYS OF POLYNOMIALS

In this section we introduce the stable line segments (half-lines) of polynomials that can be obtained starting from the reduced Routh parameters w_k, $k = 1, \ldots, n$ of a Hurwitz polynomial $a \in \mathcal{H}_n \subset \mathbb{R}^{n+1}$.

Theorem 3.1. Through an arbitrary Hurwitz stable point

$$a = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}^{\mathrm{T}}$$

with reduced Routh parameters $w_k > 0$, $k = 1, \ldots, n$ one can draw n stable half-lines $\mathcal{R}_k(a) \subset \mathcal{H}_n$ such that

$$\mathcal{R}_k(a) = \{a \mid w_k \in (0, \infty), w_j = \text{const},\ j \neq k;\ k, j \in \{1, \ldots, n\}\}. \tag{10}$$

Proof. Observe that all points of the line $\mathcal{R}_k(a)$ are Hurwitz stable, since

1. $n-1$ reduced Routh parameters w_j, $j \in \{1, \ldots, n\}$, $j \neq k$ are assumed to be fixed and positive $w_j > 0$;
2. the kth reduced Routh parameters $w_k > 0$, according to assumption $w_k \in (0, \infty)$.

Next, we have to prove that $\mathcal{R}_k(a)$ is a line segment (half-line). It is easy to see that mapping (5) is multilinear. If $n-1$ reduced Routh parameters w_j, $j \in \{1, \ldots, n\}$, $j \neq k$ are fixed, then mapping (5) turns out to be linear with respect to the kth reduced Routh parameter w_k. The latter means that for each $k = 1, \ldots, n$ there is a half-line $\mathcal{R}_k(a)$, and altogether n half-lines $\mathcal{R}_k(a) \subset \mathcal{H}_n$. □

Definition 3.2. The half-lines $\mathcal{R}_k(a)$, $k = 1, \ldots, n$ defined by (10) are called *Routh rays* of the polynomial $a(s)$. Moreover, their endpoints $v_k(a)$ such as

$$v_k(a) = a(w_k = 0)$$

are supposed to be the *Routh sources* of the polynomial $a(s)$.

Proposition 3.3. (Multilinear stability criterion) If a is a Hurwitz stable polynomial with reduced Routh parameters $w_k(a)$, $k = 1, \ldots, n$, then all the Routh rays $\mathcal{R}_k(a)$ are Hurwitz stable.

Proof. The proof follows directly from Theorem 3.1. □

According to Proposition 2.3, all Routh sources $v_k(a)$ of Hurwitz (stable) polynomials $a(s)$ are placed on the stability boundary. This means that some of the roots $\lambda_j(v_k)$, $j = 1, \ldots, n$, $k = 1, \ldots, n$ are placed on the imaginary axis. Using mapping (6) the following theorem can be formulated, regarding roots of Routh sources.

Theorem 3.4. All the Routh sources $v_j(a)$, $j = 2, \ldots, n-1$ of a Hurwitz polynomial $a(s)$ of the order n have at least two roots at the origin

$$\lambda_1(v_j) = \lambda_2(v_j) = 0, \quad j = 2, \ldots, n-1$$

and the last Routh source $v_n(a)$ has at least one root at the origin

$$\lambda_1(v_n) = 0.$$

P r o o f . To prove the theorem, the direct formula (7) from Lemma 2.2 is used. Indeed, take in (7) for $l = 1$ and $l = 2$ indices as $i_0 = n, i_1 = n-1, \dots, i_{n-2} = 2, i_{n-1} = 1$ and $i_0 = n-1, i_1 = n-2, \dots, i_{n-3} = 2, i_{n-2} = 1$, respectively. This yields the first two elements a_1^n, a_2^n of (6) given as

$$\begin{aligned} a_1^n &= \overline{w}_n \overline{w}_{n-1} \cdots \overline{w}_2 \overline{w}_1, \\ a_2^n &= \overline{w}_{n-2} \overline{w}_{n-3} \cdots \overline{w}_2 \overline{w}_1 \end{aligned}$$

or, using (8), in the simplified form as

$$\begin{aligned} a_1^n &= w_n w_{n-1} \cdots w_2, \\ a_2^n &= w_{n-2} w_{n-3} \cdots w_2. \end{aligned}$$

Hence, according to Definition 3.2, from the previous equations it follows $\lambda_1(v_j) = \lambda_2(v_j) = 0$, for $j = 2, \dots, n-1$, and $\lambda_1(v_n) = 0$. □

4. STABLE ROUTH CONES OF POLYNOMIALS

Next, we study the stability of polynomials with conic uncertainty [10] by means of Routh rays. We define so-called Routh cones[1] in the polynomial coefficient space $a \in \mathbb{R}^n$ starting from the reduced Routh parameter space $w \in \mathbb{R}^n$. Let $a^* \in \mathcal{H}_n$ be arbitrary stable polynomial of the order n and w^* its reduced Routh parameters.

Definition 4.1.

1. A subset $\mathcal{K}_i(a^*)$ of normed polynomials $a(s)$ of the degree n with coefficients $a \in \mathbb{R}^n$ is said to be a Routh cone of a polynomial $a^*(s)$ if it is closed under positive scalar multiplication of one of its reduced Routh parameters w_i^*, $i \in \{1, \dots, n\}$, i.e., $a(w_i = \alpha w_i^*) \in \mathcal{K}_i$ when $a \in \mathcal{K}_i$ and $\alpha > 0$, where all the other reduced Routh parameters w_j, $j \neq i$, $j \in \{1, \dots, n\}$ are fixed $w_j = w_j^*$.

2. If P is a subset of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$, then
$$\mathcal{K}_i(P) = \{a(w_i = \alpha w_i);\ a \in P, \alpha > 0, i \in \{1, \dots, n\}\}$$
is called the Routh cone generated by P.

3. A convex cone $\mathcal{K}(a^*)$ of normed polynomials $a(s)$ of the degree n with coefficients $a \in \mathbb{R}^n$ is said to be a polyhedral Routh cone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that
$$\mathcal{K}(a^*) = \left\{ \sum_{i=1}^n \beta_i a(\alpha_i w_i^*);\ \alpha_i > 1, 0 < \beta_i < 1, \sum_{i=1}^n \beta_i = 1, w_j = w_j^* = \text{const}, j \neq i, i = 1, \dots, n \right\}.$$

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[1]Note that the notion cone is used in consonance with results in [10]. In our paper definition of the Routh cone coincides with that of the Routh ray.

4. A convex cone $\mathcal{K}_{i,j}(a^*)$ of normed polynomial $a(s)$ of the degree n with coefficients $a \in \mathbb{R}^n$ is said to be a polyhedral Routh i,j-subcone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that

$$\mathcal{K}_{i,j}(a^*) = \big\{\beta_i a(w_i = \alpha_i w_i^*, w_j = w_j^*) + \beta_j a(w_j = \alpha_j w_j^*, w_i = w_i^*); \\ \alpha_i, \alpha_j > 1, 0 < \beta_i, \beta_j < 1, \beta_i + \beta_j = 1, \\ w_k = w_k^* = \text{const}, k \neq i, j;\ i, j, k \in \{1, \dots, n\}\big\}.$$

5. A convex set $\overline{\mathcal{K}}^n_{j,k}(a^*)$ of normed polynomials $a(s)$ of the degree n with coefficients $a \in \mathbb{R}^n$ is said to be a truncated polyhedral Routh cone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that

$$\overline{\mathcal{K}}^n_{j,k}(a^*) = \left\{\sum_{i=1}^n \beta_i a(\alpha_i w_i^*);\ \alpha_i > 1, i \neq j, k; \right. \\ 1 < \alpha_j < \overline{\alpha_j}, 1 < \alpha_k < \overline{\alpha_k};\ 0 < \beta_i < 1, \sum_{i=1}^n \beta_i = 1, \\ \left. w_h = w_h^* = \text{const}, h \neq i, i = 1, \dots, n \right\}.$$

Remark 4.2. According to Theorem 3.1, it is possible to draw n stable Routh rays $\mathcal{R}_i(a^*)$ through an arbitrary stable point a^*. In [16] it was shown that if the point is not placed on the boundary of stability domain, then there are *positive* and *negative* directions with respect to a^*. The positive part of a Routh ray corresponds to $\alpha_i \in (1, \infty)$ while the negative to $\alpha_i \in (0, 1)$, and for $\alpha_i = 1$ rays intersect at the point a^*. In this paper notions of Routh rays and Routh cones $\mathcal{K}_i(a^*)$ coincide for positive direction. Therefore, the point a^* should be understood as a vertex of the polyhedral Routh cone.

Proposition 4.3. An arbitrary subset P of normed polynomials $a(s)$ of the degree n, $a(s) \in \mathbb{R}^n$ has n Routh cones $\mathcal{K}_i(P)$, $i = 1, \dots, n$ generated by P. If the subset P is stable, then all Routh cones $\mathcal{K}_i(P)$ generated by P are stable.

Proof. According to Theorem 3.1, through an arbitrary point $a \in P \subset \mathbb{R}^n$ it is possible to draw half-lines $\mathcal{R}_i(a)$ such that $w_i \in (0, \infty)$, $i = 1, \dots, n$. If polynomials $a \in P$ are stable, then all half-lines $\mathcal{R}_i(a)$ are stable, i. e., Routh cone $\mathcal{K}_i(P)$ is stable. □

Proposition 4.4. The n-times Routh cone of the polynomial $a(s) = 1$, i. e., $a = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} \in \mathcal{R}^n$, generates the whole stability domain $\mathcal{A}$ in polynomial coefficient space, $\mathcal{A} \subset \mathbb{R}^n$.

Proof. Starting from the origin $a = 0$ it is possible to find the Routh ray $\mathcal{R}_1(0)$ which is placed on the stability boundary, since all the points $a \in \mathcal{R}_1(0)$ have $w_j = 0$,

$j = 2, \ldots, n$. The Routh cone $\mathcal{K}_{1,2}(0) = \mathcal{K}_2(\mathcal{R}_1(0))$ is also placed on the stability boundary, since all the points $a \in \mathcal{K}_{1,2}(0)$ have $w_j = 0$, $j = 3, \ldots, n$ and $w_i \in (0, \infty)$, $i = 1, 2$. Similarly, for all the points $a \in \mathcal{K}_{1,\ldots,n-1}(0)$ it follows that $w_j = 0$, $j = n$ and $w_i \in (0, \infty)$, $i = 1, \ldots, n-1$. Finally, the Routh cone $K_{1,\ldots,n}(0)$ contains points a with $w_i \in (0, \infty)$, $i = 1, \ldots, n$, i. e., $\mathcal{K}_{1,\ldots,n}(0) = \mathcal{A}$. □

Theorem 4.5. (Artemchuk et al. [3]) If all the polyhedral Routh subcones $\mathcal{K}_{i,j}(a^*)$, $i, j \in \{1, \ldots, n\}$ of a stable polynomial $a^*(s)$ are stable, then the polyhedral Routh cone $\mathcal{K}(a^*)$ is stable.

Let $\Gamma = \{1, \ldots, n\}$ be a set of integers. Rewrite it as $\Gamma = \gamma_1 \cup \gamma_2$, where γ_1 and γ_2 are sets that contain indices corresponding to ordinary and truncated Routh subcones, respectively, with $\dim \gamma_1 = m_1$ and $\dim \gamma_2 = m_2$ such that $m_1 + m_2 = n$.

Theorem 4.6. (Artemchuk et al. [3]) A truncated polyhedral Routh cone $\overline{\mathcal{K}}_{i_j}^{n}(a^*)$ such that $i_j \in \gamma_2$ and $j = 1, \ldots, m_2$ of a stable polynomial $a^*(s)$ is stable if the following conditions hold:

1. the polyhedral Routh subcones $\mathcal{K}_{r,s}(a^*)$, $r, s \in \gamma_1$ are stable;
2. the line segments $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$, $u, v \in \gamma_2$ are stable, where

$$S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v) = \operatorname{conv}\{a(w_u = \overline{\alpha}_{u,\min} w_u^*), a(w_v = \overline{\alpha}_{v,\min} w_v^*), w_i = w_i^*, i \neq u, v\}$$

   and $\overline{\alpha}_{u,\min} = \min_u \overline{\alpha}_u$.

Proposition 4.7. (Artemchuk et al. [3]) For $n = 3$ the polyhedral Routh cone $\mathcal{K}(a^*)$ of an arbitrary stable polynomial $a^*(s)$ is stable.

Example 4.8. Consider an Unmanned Free-Swimming Submersible vehicle [13] for which the relation of pitch angle to elevator surface angle can be represented by the transfer function

$$H(s) = \frac{-0.125(s + 0.435)}{(s + 1.23)(s^2 + 0.226s + 0.0169)}.$$

Since the poles $\lambda_1 = -1.23$, $\lambda_{2,3} = -0.113 \pm 0.0643i$ have negative real parts, it immediately follows that the nominal system $H(s)$ is stable. The goal is to construct the stable polyhedral cone in the coefficient space starting from the nominal characteristic polynomial (denominator of $H(s)$)

$$a^*(s) = s^3 + 1.456s^2 + 0.2949s + 0.028.$$

Normalize the polynomial $a^*(s)$ dividing it by free term 0.028 to get

$$a^*(s) = 35.7143s^3 + 52s^2 + 10.5321s + 1$$

or

$$a^3 = \begin{bmatrix} a_3^3 & a_2^3 & a_1^3 & 1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 35.7143 & \underbrace{52 \quad 10.5321 \quad 1}_{\overline{a}^3} \end{bmatrix}^{\mathrm{T}}.$$

The reduced Routh parameters can be found using recursive relation (9) as follows. Start from

$$w_3^* = \frac{a_3^3}{a_2^3} = \frac{35.7143}{52} = 0.6868.$$

Next, find the second-order polynomial

$$a^2 = \begin{bmatrix} a_2^2 \\ a_1^2 \\ 1 \end{bmatrix} = \overline{W}_3 \overline{a}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -0.6868 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 52 \\ 10.5321 \\ 1 \end{bmatrix} = \begin{bmatrix} 52 \\ 9.8453 \\ 1 \end{bmatrix}$$

and, therefore,

$$w^* = \begin{bmatrix} w_3^* & w_2^* & w_1^* & w_0^* \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0.6868 & 52 & 9.8453 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Then, according to Definition 4.1, Routh cones can be calculated as

$$\mathcal{K}_i = \underbrace{\begin{bmatrix} w_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{bmatrix}}_{W_3} \cdot \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix}.$$

Cone $\mathcal{K}_1$: Take $w_1 = \alpha_1 w_1^*, w_2 = w_2^*, w_3 = w_3^*, 1 < \alpha_1 < \infty$, and

$$a^2 = \begin{bmatrix} 52 \\ 9.8453\alpha_1 \\ 1 \end{bmatrix}.$$

Then,

$$\mathcal{K}_1 = \begin{bmatrix} 0.6868 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0.6868 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 52 \\ 9.8453\alpha_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 35.7136 \\ 52 \\ 9.8453\alpha_1 + 0.6868 \\ 1 \end{bmatrix}.$$

Cone $\mathcal{K}_2$: Take $w_1 = w_1^*, w_2 = \alpha_2 w_2^*, w_3 = w_3^*, 1 < \alpha_2 < \infty$, and

$$a^2 = \begin{bmatrix} 52\alpha_2 \\ 9.8453 \\ 1 \end{bmatrix}.$$

Then,

$$\mathcal{K}_2 = \begin{bmatrix} 0.6868 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0.6868 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 52\alpha_2 \\ 9.8453 \\ 1 \end{bmatrix} = \begin{bmatrix} 35.7136\alpha_2 \\ 52\alpha_2 \\ 10.5321 \\ 1 \end{bmatrix}.$$

Cone $\mathcal{K}_3$: Take $w_1 = w_1^*, w_2 = w_2^*, w_3 = \alpha_3 w_3^*,\ 1 < \alpha_3 < \infty$, and

$$a^2 = \begin{bmatrix} 52 \\ 9.8453 \\ 1 \end{bmatrix}.$$

Then,

$$\mathcal{K}_3 = \begin{bmatrix} 0.68682\alpha_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0.68682\alpha_3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 52 \\ 9.8453 \\ 1 \end{bmatrix} = \begin{bmatrix} 35.7136\alpha_3 \\ 52 \\ 0.6868\alpha_3 + 9.8453 \\ 1 \end{bmatrix}.$$

Let $a \in \mathcal{K}(a^*)$ be an inner point of the polyhedral Routh cone $\mathcal{K}(a^*)$. Then, the convex combination can be expressed as

$$a = \beta_1 \mathcal{K}_1(a^*) + \beta_2 \mathcal{K}_2(a^*) + \beta_3 \mathcal{K}_3(a^*),$$

where $0 < \beta_i < 1, \sum_{i=1}^{3} \beta_i = 1$ or in the explicit form as

$$a = \begin{bmatrix} 35.7136(\beta_1 + \beta_2\alpha + \beta_3\alpha) \\ 52(\beta_1 + \beta_2\alpha + \beta_3) \\ 9.8453(\beta_1\alpha + \beta_2 + \beta_3) + 0.6868(\beta_1 + \beta_2 + \beta_3\alpha) \\ 1 \end{bmatrix}.$$

From (9) it follows

$$\begin{aligned} w_3 &= \frac{0.6868(\beta_1 + \beta_2\alpha + \beta_3\alpha)}{\beta_1 + \beta_2\alpha + \beta_3}, \\ w_2 &= 52(\beta_1 + \beta_2\alpha + \beta_3), \\ w_1 &= \frac{511.956(\beta_1\alpha + \beta_2 + \beta_3)(\beta_1 + \beta_2\alpha + \beta_3) + 35.7136(1-\alpha)^2\beta_2\beta_3}{52(\beta_1 + \beta_2\alpha + \beta_3)}. \end{aligned}$$

Observe that $a^*(s)$ is stable. Then, it follows from Proposition 2.3 that $w_i^* > 0$, $i = 1, 2, 3$. It remains to show that the reduced Routh parameters w_i, $i = 1, 2, 3$ are also positive. This trivially follows from the fact that $\alpha_i > 1$ and $0 < \beta_i < 1$ with $\sum_{i=1}^{3} \beta_i = 1$. Therefore, the constructed polyhedral Routh cone

$$\mathcal{K}(a^*) = \Big\{ \beta_1 \mathcal{K}_1(a^*) + \beta_2 \mathcal{K}_2(a^*) + \beta_3 \mathcal{K}_3(a^*) \mid \alpha_i > 1, 0 < \beta_i < 1, \sum_{i=1}^{3} \beta_i = 1, i = 1, 2, 3 \Big\}$$

is stable.

Proposition 4.9. The polyhedral subcones $K_{i,j}(a^*)$, $i, j \in \{1, 2, 3\}$ of an arbitrary stable polynomial $a^*(s)$ of order n are stable.

Proof. See Appendix. □

The following algorithm allows to generate stable truncated polyhedral Routh cones for a given initial polynomial.

Algorithm:

Step 1. Start from a given n degree stable polynomial $a(s)$, or

$$a_n = \begin{bmatrix} a_n^n & a_{n-1}^n & \cdots & a_1^n & 1 \end{bmatrix}.$$

Step 2. Find the reduced Routh parameters w_k, $k = n, \dots, 1$ of the polynomial $a(s)$ by solving (9).

Step 3. Find by (10) the Routh rays $\mathcal{R}_k(a)$, $k = 1, \dots, n$ of the polynomial $a(s)$.

Step 4. Check the stability of all the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$ with $i, j \in \{4, \dots, n\}$ of the polynomial $a(s)$ by Hurwitz Segment Lemma [1, p.81]. By Proposition 4.9 the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{1, 2, 3\}$ are stable. If all the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{4, \dots, n\}$ are stable, then by Theorem 4.5 the polyhedral Routh cone $\mathcal{K}(a)$ is stable.

Step 5. If some of the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{4, \dots, n\}$ are not stable, then find the stable line segments $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$ using Theorem 4.6 with appropriate values of $\overline{\alpha}_{u,\min} = \min_u \overline{\alpha}_u$ and $\overline{\alpha}_{v,\min} = \min_v \overline{\alpha}_v$.

Step 6. According to Theorem 4.6 the stable truncated polyhedral Routh cone $\overline{\mathcal{K}}^n(a)$ of the polynomial $a(s)$ is determined by the stable polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{1, \dots, n\}$ and the stable line segments $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$.

Example 4.10. Consider the fourth-order system [17]

$$H(s) = \frac{s^3 + 7s^2 + 24s + 24}{s^4 + 10s^3 + 35s^2 + 50s + 24}.$$

The nominal system $H(s)$ is stable, since the poles are $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$, $\lambda_4 = -4$. Our goal is to construct the stable polyhedral Routh cone around the nominal characteristic polynomial

$$a^*(s) = s^4 + 10s^3 + 35s^2 + 50s + 24.$$

Proceed in the same manner as in Example 4.8. Thus, first normalize the polynomial $a^*(s)$ dividing it by the free term 24 and then calculate the reduced Routh parameters as

$$w^* = \begin{bmatrix} w_4^* & w_3^* & w_2^* & w_1^* & w_0^* \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0.1 & 0.33 & 1.25 & 1.75 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Then, according to Definition 4.1, Routh cones can be calculated as

$$\mathcal{K}_i = \underbrace{\begin{bmatrix} w_4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & w_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{W_4} \cdot \underbrace{\begin{bmatrix} w_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{bmatrix}}_{W_3} \cdot \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix},$$

yielding

$$\begin{aligned}
\mathcal{K}_1 &= \begin{bmatrix}0.0417 & 0.4167 & 1.283 + 0.175\alpha_1 & 0.333 + 1.75\alpha_1 & 1\end{bmatrix}^{\mathrm{T}}, \\
\mathcal{K}_2 &= \begin{bmatrix}0.0417\alpha_2 & 0.4167\alpha_2 & 0.2083 + 1.25\alpha_2 & 2.0833 & 1\end{bmatrix}^{\mathrm{T}}, \\
\mathcal{K}_3 &= \begin{bmatrix}0.0417\alpha_3 & 0.4167\alpha_3 & 1.425 + 0.033\alpha_3 & 1.75 + 0.333\alpha_3 & 1\end{bmatrix}^{\mathrm{T}}, \\
\mathcal{K}_4 &= \begin{bmatrix}0.0417\alpha_4 & 0.4167 & 1.25 + 0.2083\alpha_4 & 2.0833 & 1\end{bmatrix}^{\mathrm{T}}
\end{aligned}$$

with $1 < \alpha_i < \infty$, $i = 1, \dots, 4$.

Next, between obtained Routh cones it is possible to draw six polyhedral Routh subcones. According to Proposition 4.9, the polyhedral Routh subcones $\mathcal{K}_{1,2}(a^*)$, $\mathcal{K}_{1,3}(a^*)$, and $\mathcal{K}_{2,3}(a^*)$ are stable. In addition, according to the Edge Theorem, $\mathcal{K}_{2,4}(a^*)$ and $\mathcal{K}_{3,4}(a^*)$ are stable as well. The remaining subcone $\mathcal{K}_{1,4}(a^*)$ is not stable, whereas the truncated polyhedral Routh subcone $\overline{\mathcal{K}}_{1,4}(a^*)$ is stable, for example, for $\overline{\alpha}_1 = \overline{\alpha}_4 = 6.2$.

5. DISCUSSION

This paper proposes the method for convex approximation of stability domain by the polyhedral Routh cone $\mathcal{K}(a^*)$. The main idea is based on the new multilinear stability criterion for Hurwitz polynomials relying on the reduced Routh parameters. The results presented in the paper extend those from [3] by giving rigorous mathematical proofs and providing additional theoretical material. Furthermore, Section 3 and Remark 4.2, in particular, explain how the results from [3] and [16] are related via Routh rays.

It was shown in Proposition 4.7 that for the particular case of the third-order system, the Routh cone of an arbitrary polynomial a^* is always stable. However, for higher order systems is remains an open challenge. Therefore, we state the following hypotheses that require theoretical proofs.

Conjecture 5.1. For $n = 4$ the polyhedral Routh cone $\mathcal{K}(a^*)$ of a stable polynomial $a^*(s)$ is stable if the polyhedral Routh subcone $\mathcal{K}_{1,4}(a^*)$ is stable.

Conjecture 5.2. The polyhedral Routh cone $\mathcal{K}(a^*)$ of a stable polynomial $a^*(s)$ of order n is stable if the polyhedral Routh subcones $K_{1,j}(a^*)$, $j = 4, ..., n$ are stable.

The convex inner approximation of the stability region and the multilinear stability conditions can be used, for example, to design an output controller of a fixed-order via quadratic programming approach so that the closed-loop poles are robustly assigned in the approximated region [2, 15]. This will make another direction for the future research.

APPENDIX

Proof of Proposition 4.9

P r o o f. By (5) we obtain the following Routh cones $\mathcal{K}_i(a^*)$, $i = 1, 2, 3$ for the polynomial $a^*(s)$, $a \in \mathbb{R}^n$

$$\mathcal{K}_1(a^*) = W_4^n(a^*) \begin{bmatrix} w_2^* w_3^* \\ \alpha w_1^* + w_3^* \\ 1 \end{bmatrix}, \quad \mathcal{K}_2(a^*) = W_4^n(a^*) \begin{bmatrix} \alpha w_2^* w_3^* \\ \alpha w_2^* \\ w_1^* + w_3^* \\ 1 \end{bmatrix},$$

$$\mathcal{K}_3(a^*) = W_4^n(a^*) \begin{bmatrix} \alpha w_2^* w_3^* \\ w_2^* \\ w_1^* + \alpha w_3^* \\ 1 \end{bmatrix},$$

where $W_4^n(a^*) := W_n(a^*) \cdots W_4(a^*)$ and $\alpha > 1$.

For $a \in \mathcal{K}_{1,2}(a^*)$ there exist constants $\alpha > 1$ and $0 < \beta < 1$ such that for an arbitrary $a \in \mathcal{K}_{1,2}(a^*)$

$$a = \beta a(w_1 = \alpha w_1^*) + (1 - \beta) a(w_2 = \alpha w_2^*),$$

where $a(w_1 = \alpha w_1^*) \in \mathcal{K}_1$ and $a(w_2 = \alpha w_2^*) \in \mathcal{K}_2$. The above relation can be rewritten in the explicit way as

$$a = W_n(a^*) \cdots W_4(a^*) \begin{bmatrix} (\beta + (1-\beta)\alpha) w_2^* w_3^* \\ (\beta + (1-\beta)\alpha) w_2^* \\ (\beta\alpha + 1 - \beta) w_1^* + w_3^* \\ 1 \end{bmatrix}.$$

Observe that the reduced Routh parameters $w_n, \ldots, w_4$ of a polynomial $a(s)$ are determined by the product of matrix multiplication $W_n(a^*) \cdots W_4(a^*)$, i. e., $w_i = w_i^*$, $i = 4, \ldots, n$. For the reduced Routh parameters w_i, $i = 1, \ldots, 3$ of the polynomial $a \in \mathcal{K}_{1,2}(a^*)$, using (9), it follows

$$\begin{aligned} w_2 w_3 &= (\beta + (1-\beta)\alpha) w_2^* w_3^*, \\ w_2 &= (\beta + (1-\beta)\alpha) w_2^*, \\ w_1 + w_3 &= (\beta\alpha + 1 - \beta) w_1^* + w_3^* \end{aligned}$$

or

$$\begin{aligned} w_1 &= (\beta\alpha + 1 - \beta) w_1^*, \\ w_2 &= (\beta + (1-\beta)\alpha) w_2^*, \\ w_3 &= w_3^*. \end{aligned}$$

Note that $\alpha > 1$, $0 < \beta < 1$, and $w_i^* > 0$, $i = 1, \ldots, n$. Then, $w_i > 0$, $i = 1, \ldots, n$, i. e., $a \in \mathcal{K}_{1,2}(a^*)$ is stable.

In the similar manner we obtain for $a \in \mathcal{K}_{1,3}(a^*)$ the reduced Routh parameters $w_n, \ldots, w_4$, $w_i = w_i^*$, $i = 4, \ldots, n$. For w_i, $i = 1, \ldots, 3$ of the polynomial $a \in \mathcal{K}_{1,3}(a^*)$ we obtain by (9) the following relations

$$\begin{aligned} w_2 w_3 &= (\beta + (1-\beta)\alpha) w_2^* w_3^*, \\ w_2 &= w_2^*, \\ w_1 + w_3 &= (\beta\alpha + 1 - \beta) w_1^* + (\beta + (1-\beta)\alpha) w_3^* \end{aligned}$$

or

$$\begin{aligned} w_1 &= (\beta\alpha + 1 - \beta) w_1^* > 0, \\ w_2 &= w_2^* > 0, \\ w_3 &= (\beta + (1-\beta)\alpha) w_3^* > 0. \end{aligned}$$

Finally, for $a \in \mathcal{K}_{2,3}(a^*)$ we obtain the reduced Routh parameters $w_i = w_i^*$, $i = 4, \ldots, n$ and for w_i, $i = 1, \ldots, 3$

$$\begin{aligned} w_2 w_3 &= (\beta\alpha + (1-\beta)\alpha) w_2^* w_3^*, \\ w_2 &= (\beta\alpha + (1-\beta)) w_2^*, \\ w_1 + w_3 &= w_1^* + (\beta + (1-\beta)\alpha) w_3^* \end{aligned}$$

that yield

$$\begin{aligned} w_1 &= w_1^* + \frac{(\beta(1-\beta)(1-\alpha)^2) w_3^*}{\beta\alpha + (1-\beta)} > 0, \\ w_2 &= (\beta\alpha + 1 - \beta) w_2^* > 0, \\ w_3 &= \frac{\alpha w_3^*}{\beta\alpha + 1 - \beta} > 0. \end{aligned}$$

Hence, all polyhedral subcones $\mathcal{K}_{i,j}(a^*)$, $i, j \in \{1, 2, 3\}$ of an arbitrary stable polynomial $a^*(s)$ of order n are stable. □

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