Frank J. Hall; Miroslav Rozložník
G-matrices, J -orthogonal matrices, and their sign patterns

Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 3, 653–670

Persistent URL: <http://dml.cz/dmlcz/145863>

Terms of use:

© Institute of Mathematics AS CR, 2016

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

G-MATRICES, J -ORTHOGONAL MATRICES, AND THEIR SIGN PATTERNS

FRANK J. HALL, Atlanta, MIROSLAV ROZLOŽNÍK, Praha

(Received July 31, 2015)

This paper is dedicated to the memory of Professor Miroslav Fiedler; it was an honor to work with him. He was an exceptionally kind person, a wonderful friend, a tremendous inspiration, and a great mathematician.

Abstract. A real matrix A is a G-matrix if A is nonsingular and there exist nonsingular diagonal matrices D_1 and D_2 such that $A^{-T} = D_1 A D_2$, where A^{-T} denotes the transpose of the inverse of A . Denote by $J = \text{diag}(\pm 1)$ a diagonal (signature) matrix, each of whose diagonal entries is $+1$ or -1 . A nonsingular real matrix Q is called J -orthogonal if $Q^T J Q = J$. Many connections are established between these matrices. In particular, a matrix A is a G-matrix if and only if A is diagonally (with positive diagonals) equivalent to a column permutation of a J -orthogonal matrix. An investigation into the sign patterns of the J -orthogonal matrices is initiated. It is observed that the sign patterns of the G-matrices are exactly the column permutations of the sign patterns of the J -orthogonal matrices. Some interesting constructions of certain J -orthogonal matrices are exhibited. It is shown that every symmetric staircase sign pattern matrix allows a J -orthogonal matrix. Sign potentially J -orthogonal conditions are also considered. Some examples and open questions are provided.

Keywords: G-matrix; J -orthogonal matrix; Cauchy matrix; sign pattern matrix

MSC 2010: 15A80, 15A15, 15A23

1. INTRODUCTION

In [9], a new type of matrix was introduced and studied. A real matrix A is a G-matrix if A is nonsingular and there exist nonsingular diagonal matrices D_1 and D_2

This research is supported by the Grant Agency of the Czech Republic under the project 108/11/0853.

such that

$$(1.1) \quad A^{-T} = D_1 A D_2,$$

where A^{-T} denotes the transpose of the inverse of A . Denote by $J = \text{diag}(\pm 1)$ a diagonal (signature) matrix, each of whose diagonal entries is $+1$ or -1 . As in [12], a nonsingular real matrix Q is called *J-orthogonal* if

$$(1.2) \quad Q^T J Q = J,$$

or equivalently, if

$$(1.3) \quad Q^{-T} = J Q J.$$

A (J_1, J_2) -orthogonal matrix is defined as a nonsingular real matrix Q such that

$$(1.4) \quad Q^T J_1 Q = J_2,$$

where $J_1 = \text{diag}(\pm 1)$ and $J_2 = \text{diag}(\pm 1)$ are signature matrices having the same inertia [12]. *J*-orthogonal matrices were studied for example in the context of the group theory [4] or generalized eigenvalue problems [5]. Numerical properties of several orthogonalization techniques with respect to symmetric indefinite bilinear forms have been analyzed recently in [13]. Although *J*-orthogonality has many numerical connections, this particular paper has more of a combinatorial matrix theory point of view.

In Section 2 we lay the foundation of the paper. We show that a matrix A is a G-matrix if and only if A is diagonally (with positive diagonals) equivalent to a (J_1, J_2) -orthogonal matrix. Hence, as we shall see, a matrix A is a G-matrix if and only if A is diagonally (with positive diagonals) equivalent to a column permutation of a *J*-orthogonal matrix.

In Section 3 we review sign pattern matrices and recall from [9] some results on the sign patterns of the G-matrices and the sign patterns of the nonsingular Cauchy (generalized Cauchy) matrices. Section 4 is concerned with the connection of the sign patterns of the G-matrices and the sign patterns of the *J*-orthogonal matrices. In particular, we observe that the sign patterns of the G-matrices are exactly the column permutations of the sign patterns of the *J*-orthogonal matrices.

In Section 5 we give some interesting constructions of certain *J*-orthogonal matrices. We also show that every symmetric staircase sign pattern matrix allows a *J*-orthogonal matrix and we discuss the situation for nonsymmetric staircase sign patterns. Further considerations are made in Section 6, including sign potentially

J -orthogonal conditions. This paper particularly begins an exploration of the sign patterns of the J -orthogonal matrices.

2. G-MATRICES AND J -ORTHOGONAL MATRICES

It was shown in [9] that G-matrices enjoy interesting properties and that many well known special matrices are G-matrices. Two very basic, but useful, properties are the following:

If A is an $n \times n$ G-matrix and D is an $n \times n$ nonsingular diagonal matrix, then both AD and DA are G-matrices, see [9], Theorem 2.4.

If A is an $n \times n$ G-matrix and P is an $n \times n$ permutation matrix, then both AP and PA are G-matrices, see [9], Theorem 2.5.

Obviously, for any nonsingular diagonal matrix A of order at least 2, the matrices D_1 and D_2 are not unique up to scalar multiples. However, it follows from Sylvester's law of inertia that D_1 and D_2 in (1) always have the same inertia, and thus have the same number of positive entries. We now establish several interesting structural properties of G-matrices and characterize the G-matrices A for which the matrices D_1 and D_2 in (1) are unique up to scalar multiples. For the notion of fully indecomposable matrices, we refer the reader to [2].

Theorem 2.1. *Let A be a nonsingular real matrix in block upper triangular form*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ & \ddots & \vdots \\ 0 & & A_{mm} \end{bmatrix},$$

where all the diagonal blocks are square. Then A is a G-matrix if and only if each A_{ii} , $i = 1, \dots, m$, is a G-matrix and all the strictly upper triangular blocks A_{ij} are equal to 0. Furthermore, if A is a G-matrix that has a row (or a column) with no 0 entry, then A is fully indecomposable.

Proof. Assume that A satisfies $A^{-T} = D_1 A D_2$. Note that $D_1 A D_2$ is block upper triangular and the conformally partitioned A^{-T} is block lower triangular. It follows that the strictly upper triangular blocks of A are equal to 0. The rest is clear. $\square$

We now characterize those G-matrices A for which the matrices D_1 and D_2 in (1) are unique up to scalar multiples.

Theorem 2.2. *Let A be a fully indecomposable G-matrix. Then the diagonal matrices D_1 and D_2 satisfying $A^{-T} = D_1 A D_2$ are unique up to scalar multiples.*

Proof. Replacing A with a matrix permutationally equivalent to A if necessary, without loss of generality, we may assume that all the diagonal entries of A are nonzero. Write $D_1 = \text{diag}(x_1, \dots, x_n)$ and $D_2 = \text{diag}(y_1, \dots, y_n)$. It follows that x_i and y_i determine each other, for each $i = 1, \dots, n$. Since A is fully indecomposable, we know that A is irreducible. Hence, for each $i \neq j$, the presence of a directed path from i to j in the directed graph of A [2] shows that x_i and y_j determine each other. If we assume that the $(1,1)$ -entry of D_1 is 1, then all the entries of D_1 and D_2 are uniquely determined. Therefore, D_1 and D_2 are unique up to scalar multiples. $\square$

The following result is then clear.

Theorem 2.3. *Let A be a G-matrix such that $A = A_1 \oplus \dots \oplus A_m$, where A_i is fully indecomposable and is of order n_i , $i = 1, \dots, m$. Suppose that $\widehat{D}_1$ and $\widehat{D}_2$ are two diagonal matrices satisfying $A^{-T} = \widehat{D}_1 A \widehat{D}_2$. Then all the D_1 and D_2 satisfying $A^{-T} = D_1 A D_2$ are given by $D_1 = (c_1 I_{n_1} \oplus \dots \oplus c_m I_{n_m}) \widehat{D}_1$ and $D_2 = (c_1^{-1} I_{n_1} \oplus \dots \oplus c_m^{-1} I_{n_m}) \widehat{D}_2$, where $c_1, \dots, c_m$ are arbitrary nonzero real numbers.*

As is well known, Cauchy matrices are matrices of the form $C = [c_{ij}]$, where $c_{ij} = 1/(x_i + y_j)$ for some numbers x_i and y_j . We shall restrict ourselves to square, say $n \times n$, Cauchy matrices. Of course, such matrices are defined only if $x_i + y_j \neq 0$ for all pairs of indices i, j , and it is well known that C is nonsingular if and only if all the numbers x_i are mutually distinct and all the numbers y_j are mutually distinct. By Observation 1 in [7], every nonsingular Cauchy matrix is a G-matrix.

For *generalized Cauchy matrices* of order n , additional parameters $u_1, \dots, u_n, v_1, \dots, v_n$ are considered:

$$\widehat{C} = \left(\frac{u_i v_j}{x_i + y_j} \right).$$

Note that then $\widehat{C} = D_1 C D_2$, where $D_1 = \text{diag}(u_i)$, $D_2 = \text{diag}(v_j)$, so that $\widehat{C}$ is a G-matrix.

As mentioned in the introduction, in the recent decades and particularly in numerical mathematics, the class of problems appeared where the scalar products were indefinite, see for example [12], [4], [5] or [13].

Of course, every orthogonal matrix is a J -orthogonal matrix, where J is the identity matrix of the same order as Q . And clearly, from (1.3), every J -orthogonal matrix is a G-matrix. On the other hand, a G-matrix can always be transformed to a J -orthogonal matrix.

Definition 2.4. We say that two real matrices A and B are *positive-diagonally equivalent* if there are diagonal matrices D_1 and D_2 with all diagonal entries positive such that $B = D_1 A D_2$.

Theorem 2.5. *A matrix A is a G-matrix if and only if A is positive-diagonally equivalent to a (J_1, J_2) -orthogonal matrix.*

Proof. Let A be a G-matrix, i.e., $A^{-T} = D_1 A D_2$ for some nonsingular diagonal matrices D_1 and D_2 . Consequently, $A^T D_1 A = D_2^{-1}$. Write $D_1 = |D_1|^{1/2} J_1 |D_1|^{1/2}$ and $D_2^{-1} = |D_2|^{-1/2} J_2 |D_2|^{-1/2}$. Thus,

$$(|D_1|^{1/2} A)^T J_1 (|D_1|^{1/2} A) = |D_2|^{-1/2} J_2 |D_2|^{-1/2},$$

which can be written as

$$(|D_1|^{1/2} A |D_2|^{1/2})^T J_1 (|D_1|^{1/2} A |D_2|^{1/2}) = J_2.$$

For $Q = |D_1|^{1/2} A |D_2|^{1/2}$, this is $Q^T J_1 Q = J_2$, so that Q is (J_1, J_2) -orthogonal. Note that due to $A = |D_1|^{-1/2} Q |D_2|^{-1/2}$, A is positive-diagonally equivalent to a (J_1, J_2) -orthogonal matrix.

Conversely, if Q is (J_1, J_2) -orthogonal, then it is a G-matrix and any positive-diagonally equivalent matrix is a G-matrix as well. $\square$

We now have the following.

Theorem 2.6. *A matrix A is a G-matrix if and only if A is positive-diagonally equivalent to a column permutation of a J -orthogonal matrix.*

Proof. As mentioned in [12], the matrices J_1 and J_2 in (1.4) have the same inertia, so that $J_2 = P J_1 P^T$ for some permutation matrix P , and hence $(QP)^T J_1 (QP) = J_1$. It follows that the (J_1, J_2) -orthogonal matrices are the column permutations of the J_1 -orthogonal matrices. Considering this and Theorem 2.5 we get the statement of our theorem. $\square$

3. SIGN PATTERN MATRICES

In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the sign of entries in the matrix. An $m \times n$ matrix whose entries are from the set $\{+, -, 0\}$ is called a *sign pattern matrix* (or sign pattern). For a real matrix B , $\text{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive, negative, or zero entry of B , respectively, by $+$, $-$, or 0 . For a sign pattern matrix A , the *sign pattern class* of A is defined by

$$Q(A) = \{B : \text{sgn}(B) = A\}.$$

We denote the set of $n \times n$ sign pattern matrices by Q_n .

A sign pattern matrix P is called a *permutation sign pattern* (*generalized permutation sign pattern*) if exactly one entry in each row and column is equal to $+$ ($+$ or $-$) and all the other entries are 0. A *permutation similarity* of the $n \times n$ sign pattern A has the form P^TAP , where P is an $n \times n$ permutation matrix. A *signature pattern* is a diagonal sign pattern matrix, each of whose diagonal entries is $+$ or $-$. A sign pattern B is *signature equivalent* to the sign pattern A provided $B = S_1AS_2$, where S_1 and S_2 are signature patterns. A *signature similarity* of the $n \times n$ sign pattern A has the form SAS , where S is an $n \times n$ signature pattern.

Suppose P is a property referring to a real matrix. A sign pattern A is said to *require* P if every matrix in $Q(A)$ has property P ; A is said to *allow* P if some real matrix in $Q(A)$ has property P . The reader is referred to [3] or [11] for more information on sign pattern matrices.

As in [9], we let $\mathcal{G}_n$ denote the class of all $n \times n$ sign pattern matrices A that allow a G-matrix, that is, there exists a nonsingular matrix $B \in Q(A)$ such that $B^{-T} = D_1BD_2$ for some nonsingular diagonal matrices D_1 and D_2 . The following assertion is Theorem 3.1 of [9]: The class $\mathcal{G}_n$ is closed under

- (i) multiplication (on either side) by a permutation pattern, and
- (ii) multiplication (on either side) by a signature pattern.

The use of these operations in $\mathcal{G}_n$ then produces “equivalent” sign patterns.

Also as in [9], next let $\mathcal{C}_n$ ($\mathcal{GC}_n$) be the class of all sign patterns of the $n \times n$ nonsingular Cauchy (generalized Cauchy) matrices. It should be clear that $\mathcal{C}_n$ ($\mathcal{GC}_n$) is closed under operation (i) (operations (i) and (ii)) above. The classes $\mathcal{C}_n$ and $\mathcal{GC}_n$ are two particular sub-classes of $\mathcal{G}_n$.

The class $\mathcal{C}_n$ is the same as the class of $n \times n$ sign patterns permutation equivalent to a sign pattern of the form

$$\begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & - \\ + & + & + & + & + & + & + & - \\ + & + & + & + & + & + & + & - \\ + & + & + & + & + & - & - & - \\ + & + & + & - & - & - & - & - \\ + & + & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \end{pmatrix},$$

where the part above (below) the staircase is all $+$ ($-$) [9], Theorem 3.2. In this form, whenever there is a minus, then to the right and below there are also minuses. Note that this form includes the all $+$ and all $-$ patterns.

4. G-MATRIX/ J -ORTHOGONAL MATRIX SIGN PATTERNS

Of course, when $J = I_n$, a J -orthogonal matrix is an orthogonal matrix. An old question raised by M. Fiedler in 1964, [8], is the following: what are the sign patterns which allow an orthogonal matrix? Since that time, much research has been done on these sign patterns, [11]. Letting $\mathcal{PO}_n$ denote the class of $n \times n$ sign patterns that allow an orthogonal matrix, we give the connection with G-matrices.

Proposition 4.1. *An $n \times n$ sign pattern A allows a G-matrix with associated diagonal matrices having positive diagonal entries if and only if $A \in \mathcal{PO}_n$.*

Proof. Let A be an $n \times n$ sign pattern. Suppose there exist a nonsingular matrix $B \in Q(A)$ and nonsingular diagonal matrices D_1, D_2 with + diagonal entries such that $B^{-T} = D_1 B D_2$. Let $E_1 = D_1^{1/2}, E_2 = D_2^{1/2}$. Then

$$(E_1 B E_2)^{-1} = (E_1 B E_2)^T.$$

So, $E_1 B E_2$ is an orthogonal matrix in $Q(A)$. Conversely, if C is an orthogonal matrix in $Q(A)$, then $C^{-T} = C = I_n C I_n$. $\square$

Remark 4.2. In [6] the class $\mathcal{T}_n$ of all $n \times n$ sign patterns A for which there exists a nonsingular matrix $B \in Q(A)$ where $B^{-1} \in Q(A^T)$ was studied. There it was asked if the class $\mathcal{T}_n$ is the same as the subclass $\mathcal{PO}_n$. This question is still unanswered.

A more general question than characterizing $\mathcal{PO}_n$ is the following: what are the sign patterns which allow a J -orthogonal matrix? Specifically, it is of interest to find sign patterns which allow a J -orthogonal matrix, but do not allow an orthogonal matrix. We shall let $\mathcal{J}_n$ denote the class of all sign patterns of the $n \times n$ J -orthogonal matrices, that is, the class of $n \times n$ sign patterns that allow a J -orthogonal matrix.

From Theorem 2.6 we immediately have the following connection with G-matrices.

Theorem 4.3. *The sign patterns of the $n \times n$ G-matrices are exactly the column permutations of the sign patterns in $\mathcal{J}_n$.*

Now, the all + (also, all -) $n \times n$ sign pattern is the sign pattern of a nonsingular Cauchy matrix, which is a G-matrix. Thus:

Theorem 4.4. *The all + (also, all -) $n \times n$ sign pattern allows a J -orthogonal matrix (but of course not an orthogonal matrix, unless $n = 1$).*

Remark 4.5. In general, every sign pattern in $\mathcal{C}_n$ ($\mathcal{GC}_n$) is the sign pattern of a nonsingular Cauchy (generalized Cauchy) matrix, which is a G-matrix. So, every such sign pattern is a column permutation of a sign pattern in $\mathcal{J}_n$. This implicitly provides many sign patterns that allow a J -orthogonal matrix, but not an orthogonal matrix.

Finally in this section, we digress to the (J_1, J_2) -orthogonal matrices and utilize Theorem 2.5.

Theorem 4.6. *The sign patterns of the G-matrices are the same as the sign patterns of the (J_1, J_2) -orthogonal matrices.*

In particular, we have the following.

Corollary 4.7. *If $A \in \mathcal{GC}_n$ (in particular, if $A \in \mathcal{C}_n$), then A allows a (J_1, J_2) -orthogonal matrix.*

From [9] we know that every 2×2 $(+, -)$ sign pattern is a matrix in $\mathcal{GC}_2$ and that every 3×3 $(+, -)$ sign pattern is a matrix in $\mathcal{GC}_3$. Hence:

Corollary 4.8. *For $n \leq 3$, every $n \times n$ $(+, -)$ sign pattern allows a (J_1, J_2) -orthogonal matrix.*

5. CONSTRUCTION OF CERTAIN J -ORTHOGONAL MATRICES

It follows from Theorem 4.4 that there exists a 2×2 matrix with all $+$ sign pattern that is a J -orthogonal matrix. It is also clear that the all $+$ sign pattern does not allow an orthogonal matrix with respect to the standard inner product, where $J = I_2$. For example, the symmetric matrix $Q_2 = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$ is J -orthogonal with respect to the matrix $J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ due to

$$Q_2^T J_2 Q_2 = Q_2 J_2 Q_2 = J_2,$$

but there is no 2×2 matrix Q_2 with all positive entries that satisfies $Q_2^T Q_2 = I_2$. We arrive at the following result.

Theorem 5.1. *If we take the 2×2 sign pattern matrix $A_2 = \begin{pmatrix} + & + \\ + & + \end{pmatrix}$ and for each $n = 1, 2, \dots$ define recursively the $2^{n+1} \times 2^{n+1}$ sign pattern matrix*

$$(5.1) \quad A_{2^{n+1}} = \begin{pmatrix} A_{2^n} & -A_{2^n} \\ -A_{2^n} & A_{2^n} \end{pmatrix}$$

then each sign pattern matrix A_{2^n} allows a J -orthogonal matrix and does not allow an orthogonal matrix.

P r o o f. As was already pointed out the statement is true for $n = 1$. Inductively, if there exists a $2^n \times 2^n$ matrix Q_{2^n} such that $Q_{2^n}^T J_{2^n} Q_{2^n} = J_{2^n}$ and we define

$$Q_{2^{n+1}} = \begin{pmatrix} \sqrt{2}Q_{2^n} & -Q_{2^n} \\ -Q_{2^n} & \sqrt{2}Q_{2^n} \end{pmatrix}$$

then $Q_{2^{n+1}}^T J_{2^{n+1}} Q_{2^{n+1}} = J_{2^{n+1}}$, i.e. the matrix $Q_{2^{n+1}}$ is J -orthogonal with respect to the matrix $J_{2^{n+1}}$ given as

$$J_{2^{n+1}} = \begin{pmatrix} J_{2^n} & 0 \\ 0 & -J_{2^n} \end{pmatrix}.$$

It is also clear from the definition that $\text{sgn}(Q_{2^{n+1}}) = A_{2^{n+1}}$. Moreover, the sign pattern $A_{2^{n+1}}$ does not allow orthogonality since its first two rows or columns are equal. $\square$

Remark 5.2. Note that $Q_2 = \begin{pmatrix} \sqrt{2} & -1 \\ -1 & \sqrt{2} \end{pmatrix}$ is also J -orthogonal with respect to the matrix $J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We could alternatively take the matrix A_2 as $A_2 = \begin{pmatrix} + & - \\ - & + \end{pmatrix}$ as the starting point in Theorem 5.1. Then, the sign pattern A_2 also allows a J -orthogonal matrix, but does not allow an orthogonal matrix. The sign pattern matrices $A_{2^{n+1}}$ can still be defined as in (5.1). The proof of Theorem 5.1 works in the same way and we generate a different sequence of sign patterns that allow J -orthogonality but not orthogonality.

We now return to the staircase patterns.

Theorem 5.3. *Each symmetric staircase sign pattern matrix allows a J -orthogonal matrix.*

P r o o f. Let us recall that each symmetric staircase sign pattern matrix A corresponds to the symmetric Cauchy matrix $C = [c_{ij}]$ with $c_{ij} = 1/(x_i + x_j)$, where the numbers x_i are ordered so that $x_1 > x_2 > \dots > x_n$. It follows then from [10] that if we define the diagonal matrix D as $D = \text{diag}(d_i)$ with

$$(5.2) \quad d_i = 2x_i \prod_{k \neq i} \frac{x_i + x_k}{x_i - x_k},$$

then indeed $C^{-T} = DCD$. If we write $D = |D| \text{diag}(\text{sign}(d_i))$, then the matrix $Q = |D|^{1/2} C |D|^{1/2}$ is J -orthogonal with respect to the matrix $J = \text{diag}(\text{sign}(d_i))$ satisfying $Q^T J Q = Q J Q = J$. It is clear from the construction that the sign pattern of Q coincides with the sign pattern A as $\text{sgn}(Q) = \text{sgn}(C) = A$. $\square$

Remark 5.4. The i -th diagonal entry of J defined in Theorem 5.3 is actually equal to the sign of d_i from (5.2). If we denote by m_i the number of negative signs in the i -th row of the staircase sign pattern matrix corresponding to the numbers $x_1 > x_2 > \dots > x_n$, then $m_n \geq \dots \geq m_2 \geq m_1 \geq 0$. It is clear that $x_i + x_k < 0$ for $k = n - m_i + 1, \dots, n$ and $x_i - x_k < 0$ for $k = 1, \dots, i - 1$. Taking into account all negative terms in (5.2) we get that the sign of d_i is equal to $(-1)^{m_i+i-1}$ for $i = 1, \dots, n$.

Remark 5.5. The all $+$ $n \times n$ sign pattern corresponds to the situation with $m_i = 0$ for $i = 1, \dots, n$. Consequently, the signs in J alternate according to $(-1)^{i-1}$ for $i = 1, \dots, n$. The all $-$ $n \times n$ sign pattern corresponds to the situation with $m_i = n$ for $i = 1, \dots, n$. Consequently, the signs in J alternate according to $(-1)^{n+i-1}$ for $i = 1, \dots, n$.

Example 5.6. The symmetric sign patterns

$$\begin{pmatrix} + & + & + & + \\ + & - & - & - \\ + & - & - & - \\ + & - & - & - \end{pmatrix}, \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & - \\ + & + & - & - \end{pmatrix}, \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & - \end{pmatrix},$$

$$\begin{pmatrix} + & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}, \begin{pmatrix} + & + & - & - \\ + & + & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}, \begin{pmatrix} + & + & + & - \\ + & + & + & - \\ + & + & + & - \\ - & - & - & - \end{pmatrix}$$

allow J -orthogonal matrices but do not allow orthogonal matrices.

Remark 5.7. One can consider also nonsymmetric staircase sign patterns. Let us recall that each nonsymmetric staircase sign pattern matrix A corresponds to the (nonsymmetric) Cauchy matrix $C = [c_{ij}]$ with $c_{ij} = 1/(x_i + y_j)$, where the numbers x_i and y_j are ordered so that $x_1 > x_2 > \dots > x_n > 0$ and $y_1 > y_2 > \dots > y_n$. It follows then from [10] that if we define the diagonal matrices D_1 and D_2 as $D_1 = \text{diag}(u_i)$ and $D_2 = \text{diag}(v_j)$ with

$$(5.3) \quad u_i = (x_i + y_i) \prod_{k \neq i} \frac{x_i + y_k}{x_i - x_k}, \quad v_j = (x_j + y_j) \prod_{k \neq j} \frac{y_j + x_k}{y_j - y_k},$$

then indeed $C^{-T} = D_1 C D_2$. If we write $D_1 = |D_1| J_1$ and $D_2 = |D_2| J_2$ where $J_1 = \text{diag}(\text{sign}(u_i))$ and $J_2 = \text{diag}(\text{sign}(v_j))$ there exists a permutation matrix P such that it provides the transformation $J_2 = P J_1 P^T$. Then the matrix $Q = |D_1|^{1/2} C |D_2|^{1/2} P$ is J -orthogonal with respect to J_1 satisfying $Q^T J_1 Q = J_1$. The sign pattern of Q is equal to a column permutation of the sign pattern A as $\text{sgn}(Q) = \text{sgn}(CP) = AP$.

It is easy to see from the construction that the i -th diagonal entry of J_1 is actually equal to the sign of u_i from (5.3). If we denote by m_i the number of negative signs in the i -th row of the staircase sign pattern matrix corresponding to the numbers $x_1 > x_2 > \dots > x_n > 0$ and $y_1 > y_2 > \dots > y_n$, then $m_n \geq \dots \geq m_2 \geq m_1 \geq 0$. It is clear that $x_i + y_k < 0$ for $k = n - m_i + 1, \dots, n$ and $x_i - x_k < 0$ for $k = 1, \dots, i - 1$. Taking into account all negative terms in (5.3) we get that the sign of u_i is equal to $(-1)^{m_i+i-1}$ for $i = 1, \dots, n$. Similarly the j -th diagonal entry of J_2 is equal to the sign of v_j from (5.3). If we denote by n_j the number of negative signs in the j -th columns of the staircase sign pattern matrix corresponding to the numbers $x_1 > x_2 > \dots > x_n > 0$ and $y_1 > y_2 > \dots > y_n$, then $n_n \geq \dots \geq n_2 \geq n_1 \geq 0$. It is clear that $y_j + x_k < 0$ for $k = n - n_j + 1, \dots, n$ and $y_j - y_k < 0$ for $k = 1, \dots, j - 1$. Taking into account all negative terms in (5.3) we get that the sign of u_j is equal to $(-1)^{n_j+j-1}$ for $j = 1, \dots, n$.

Example 5.8. Note that there exist also nonsymmetric staircase sign patterns such as

$$\begin{pmatrix} + & + & - & - \\ + & + & - & - \\ + & + & - & - \\ + & + & - & - \end{pmatrix}, \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$$

that allow J -orthogonal matrices but do not allow orthogonal matrices. In such cases we have $(m_i - n_i) \bmod 2 = 0$ and this leads to $J_2 = J_1$ in Remark 5.7 with the permutation matrix P equal to the identity $P = I_4$.

Example 5.9. Note that the nonsymmetric staircase sign patterns

$$\begin{pmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & + & - \end{pmatrix}, \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & - & - & - \end{pmatrix}, \\ \begin{pmatrix} + & - & - & - \\ + & - & - & - \\ + & - & - & - \\ - & - & - & - \end{pmatrix}, \begin{pmatrix} + & + & + & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$$

also allow J -orthogonal matrices but do not allow orthogonal matrices. The situation is more complicated for these four sign patterns as $P = [e_1, e_3, e_2, e_4] \neq I_4$ but we still have $\text{sgn}(Q) = \text{sgn}(CP) = AP = A$.

6. SIGN POTENTIALLY J -ORTHOGONAL CONDITIONS

First, we develop some conditions for J -orthogonal matrices which extend the sign potentially orthogonal (SPO) conditions. As in [6], we use the symbol $\#$ to denote an *ambiguous* quantity, namely, $\# = (+) + (-)$. We define a *generalized sign pattern matrix* $A = (a_{ij})$ as a $(+, -, 0, \#)$ matrix, and the sign pattern class of such an $n \times n$ matrix is given by

$$Q(A) = \{B = (b_{ij}) \in M_n(\mathbb{R}) : a_{ij} = \# \text{ or } a_{ij} = \text{sgn}(b_{ij})\}.$$

Note that every sign pattern matrix is also a generalized sign pattern matrix. We denote the set of $n \times n$ generalized sign pattern matrices by $\overline{Q}_n$. We say two patterns $A, A' \in \overline{Q}_n$ are *compatible* if, for all $i, j \in \{1, 2, \dots, n\}$, either $a_{ij} = a'_{ij}$, or one of a_{ij} and a'_{ij} is $\#$. Equivalently, A and A' are compatible if and only if $Q(A) \cap Q(A') \neq \emptyset$. We write $A \overset{c}{\leftrightarrow} A'$ when A and A' are compatible. For example,

$$\begin{pmatrix} \# & 0 \\ + & - \end{pmatrix} \overset{c}{\leftrightarrow} \begin{pmatrix} - & \# \\ + & \# \end{pmatrix}.$$

Let A be an $n \times n$ sign pattern matrix. If $A \in \mathcal{J}_n$, then there exists $B \in Q(A)$ such that

$$\begin{aligned} B^T J B &= J, \\ (B^T J)(B J) &= I, \\ (B J)(B^T J) &= I, \\ B J B^T &= J. \end{aligned}$$

With a slight abuse of notation, we will identify J with $\text{sgn}(J)$. Thus the *sign potentially J -orthogonal* (SPJO) conditions are that

$$A^T J A \overset{c}{\leftrightarrow} J$$

and

$$A J A^T \overset{c}{\leftrightarrow} J$$

for some $(+, -)$ signature pattern J .

These are necessary conditions for $A \in \mathcal{J}_n$. If these conditions do not hold, then $A \notin \mathcal{J}_n$. When $J = I$, we get the normal SPO conditions for orthogonal matrices, see for example [6]. The SPJO conditions are not sufficient for an $n \times n$ sign pattern matrix to allow J -orthogonality.

Example 6.1. Let

$$A = \begin{pmatrix} + & + & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & + \\ + & + & + & + \end{pmatrix}.$$

It is easily checked that A satisfies the SPJO conditions with $J = \text{diag}(+, -, +, -)$. Other signature patterns also work, such as $J = \text{diag}(+, +, +, -)$. However, suppose that A allows a J -orthogonal matrix. Since every J -orthogonal matrix is a G-matrix, A then allows a G-matrix. But then by Theorem 2.1, A would have to be block-diagonal, which is a contradiction. Thus, $A \notin \mathcal{J}_4$.

For sign vectors $c, x \in \{+, -, 0\}^n$, we have that $c^T x \xleftrightarrow{c} 0$ if at least one of the following holds:

- (1) for each i , we have $c_i = 0$ or $x_i = 0$, or
- (2) there are indices i, j with $c_i = x_i \neq 0$ and $c_j = -x_j \neq 0$.

For a set of sign vectors $S \subseteq \{+, -, 0\}^n$, the *orthogonal complement* of S is

$$S^\perp = \{c \in \{+, -, 0\}^n : c^T x \xleftrightarrow{c} 0 \text{ for all } x \in S\}.$$

Specifically, if $c, x \in \{+, -\}^n$, we have only the second condition.

Theorem 6.2. *If A is an $n \times n$ $(+, -)$ sign pattern matrix and $n \geq 6$, then A satisfies the SPJO conditions.*

Proof. Let $A = (a_{ij})$ be an $n \times n$ $(+, -)$ sign pattern matrix. We need to show that there exists a $(+, -)$ signature pattern J such that

$$(6.1) \quad A^T J A \xleftrightarrow{c} J$$

and

$$(6.2) \quad A J A^T \xleftrightarrow{c} J.$$

Observe that $A^T J A$ and $A J A^T$ are symmetric generalized sign pattern matrices. So, we need only to find a J which fulfils the upper-triangular part of the compatible conditions.

Let $J = \text{diag}(\omega_1, \dots, \omega_n)$. Note that (6.1) and (6.2) may be restated as

$$(6.3) \quad \sum_{k=1}^n \omega_k a_{ki} a_{kj} \xleftrightarrow{c} \delta_{ij} \omega_j \quad \forall i, j$$

and

$$(6.4) \quad \sum_{k=1}^n \omega_k a_{ik} a_{jk} \stackrel{c}{\leftrightarrow} \delta_{ij} \omega_j \quad \forall i, j.$$

Then, for $i = j$, (6.3) and (6.4) automatically hold for any J . For the $i < j$ positions, (6.3) and (6.4) each yield $n(n-1)/2$ linear expressions in J . Letting $v = (\omega_1, \dots, \omega_n)^T$, we have

$$C_1 v \stackrel{c}{\leftrightarrow} 0$$

and

$$C_2 v \stackrel{c}{\leftrightarrow} 0$$

to solve simultaneously, where C_1 and C_2 are $n(n-1)/2 \times n$ $(+, -)$ sign patterns. Let S be the set of rows of C_1 together with the set of rows of C_2 . Let S' be $S \cup (-S)$. To find a possible J , we choose a $(+, -)$ n -vector v such that $v \notin S'$. For $n \geq 6$, $2n(n-1) < 2^n$, so that such a choice of v is always possible. Then for any $c \in S'$, v will be different from c in at least one component and different from $-c$ in at least one component. Hence, $c^T v \stackrel{c}{\leftrightarrow} 0$, i.e., $v \in (S')^\perp$. Letting $J = \text{diag}(v)$, we have a signature pattern that fulfils (6.1) and (6.2). $\square$

If we allow zero entries, then Theorem 6.2 may fail. For example, an $n \times n$ sign pattern A with a zero column does not satisfy $A^T J A \stackrel{c}{\leftrightarrow} J$ and an $n \times n$ sign pattern A with a zero row does not satisfy $A J A^T \stackrel{c}{\leftrightarrow} J$, for any signature pattern J .

The following is straightforward.

Lemma 6.3. *The class $\mathcal{J}_n$ is closed under the following operations:*

- i) *negation;*
- ii) *transposition;*
- iii) *permutation similarity;*
- iv) *multiplication (on either side) by a signature pattern;*
- v) *signature equivalence.*

The use of these operations yields “equivalent” sign patterns. We now investigate the question of whether the $(+, -)$ $n \times n$ sign patterns always allow a J -orthogonal matrix.

Remark 6.4. It was observed in [6] that for $n \leq 4$, the SPO patterns are the same as the sign patterns in $\mathcal{PO}_n$, and that this is also the case for $(+, -)$ sign patterns of order 5, see [1] and [14]. So, regarding the above question with $n \leq 5$, we need only to consider non-SPO patterns.

By what we have previously done, all the $(+, -)$ sign patterns of orders 1 or 2 allow a J -orthogonal matrix. By Theorem 5.3, every symmetric staircase pattern allows a J -orthogonal matrix. By Remark 6.4, for $n \leq 5$, every $n \times n$ $(+, -)$ SPO sign pattern allows orthogonality. If A is a 3×3 $(+, -)$ sign pattern, by signature multiplications, A is equivalent to a sign pattern of the form

$$\begin{pmatrix} + & + & + \\ + & & \\ + & A_1 & \end{pmatrix}$$

where A_1 is a 2×2 $(+, -)$ sign pattern. By analyzing the 16 choices for A_1 , it can be seen that A is equivalent to at least one of the following: a symmetric staircase pattern; a SPO pattern; the pattern

$$\hat{A} = \begin{pmatrix} + & + & + \\ + & + & + \\ + & - & - \end{pmatrix}.$$

By Remark 5.7 it follows that this nonsymmetric staircase pattern allows a J -orthogonal matrix since the counts $m_1 = m_2 = 0$, $m_3 = 2$ and $n_1 = 0$, $n_2 = n_3 = 1$ lead to $P = [e_1, e_3, e_2]$ that satisfies $\hat{A} = \hat{A}P$, similarly to Example 5.9. So, $A \in \mathcal{J}_3$. We arrive at the following result.

Proposition 6.5. *If A is a 3×3 $(+, -)$ sign pattern, then $A \in \mathcal{J}_3$.*

This result improves Corollary 4.8. In fact, given a 3×3 $(+, -)$ sign pattern A , we can easily enough construct $B \in Q(A)$ that is J -orthogonal.

Example 6.6. If

$$A = \begin{pmatrix} + & - & - \\ + & + & - \\ + & - & - \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & -1 \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}, \quad J = \text{diag}(1, 1, -1),$$

then $B \in Q(A)$ and $B^T J B = J$.

Given a 4×4 $(+, -)$ sign pattern we can proceed similarly.

Example 6.7. If

$$A = \begin{pmatrix} + & - & + & + \\ - & + & + & - \\ + & - & - & + \\ - & - & + & - \end{pmatrix}, \quad B = \begin{pmatrix} \frac{14}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & \frac{1}{2\sqrt{6}} & \frac{17}{2\sqrt{6}} \\ -\frac{11}{\sqrt{15}} & \frac{4}{\sqrt{15}} & \frac{1}{\sqrt{6}} & -\frac{7}{\sqrt{6}} \\ \frac{14}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & -\frac{3}{\sqrt{6}} & \frac{9}{\sqrt{6}} \\ -\frac{16}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & \frac{3}{2\sqrt{6}} & -\frac{21}{2\sqrt{6}} \end{pmatrix},$$

$J = \text{diag}(1, -1, 1, -1)$, then $B \in Q(A)$ and $B^T J B = J$.

The same can be done for 5×5 $(+, -)$ sign patterns.

Open Question 6.8. Is every $n \times n$ $(+, -)$ sign pattern in $\mathcal{J}_n$?

Now we return to the SPJO conditions. If A is a 4×4 $(+, -)$ sign pattern, by signature multiplications, A is equivalent to a sign pattern of the form

$$\begin{pmatrix} + & + & + & + \\ + & & & \\ + & & A_1 & \\ + & & & \end{pmatrix}$$

where A_1 is a 3×3 $(+, -)$ sign pattern. We denote the columns and rows of A_1 as follows:

$$A_1 = (c_1 \quad c_2 \quad c_3) = \begin{pmatrix} r_1^T \\ r_2^T \\ r_3^T \end{pmatrix}.$$

The SPJO conditions (6.1) and (6.2) for the matrix A have then the form of linear expressions in diagonal elements of J so that $Cv \stackrel{c}{\leftrightarrow} 0$ and $J = \text{diag}(v)$, where

$$(6.5) \quad C = \begin{pmatrix} + & c_1^T \\ + & c_2^T \\ + & c_3^T \\ + & c_1^T \circ c_2^T \\ + & c_1^T \circ c_3^T \\ + & c_2^T \circ c_3^T \\ + & r_1^T \\ + & r_2^T \\ + & r_3^T \\ + & r_1^T \circ r_2^T \\ + & r_1^T \circ r_3^T \\ + & r_2^T \circ r_3^T \end{pmatrix}.$$

By observing (6.5), it can be seen that any permutation of the rows or columns of A_1 leads to the same SPJO conditions for A .

Assume that columns c_1, c_2, c_3 are mutually different and assume the same for the vectors r_1, r_2, r_3 . Then it is clear that none of the vectors $c_1^T \circ c_2^T, c_1^T \circ c_3^T, c_2^T \circ c_3^T, r_1^T \circ r_2^T, r_1^T \circ r_3^T$ and $r_2^T \circ r_3^T$ is equal to $(+ \ + \ +)^T$. Assuming that none of $c_1, c_2, c_3, r_1, r_2, r_3$ is equal to $(+ \ + \ +)^T$, we have $Cv \stackrel{c}{\leftrightarrow} 0$ with $J = \text{diag}(v)$ and $v = (+ \ + \ + \ +)^T$, so that A satisfies the SPO conditions. If at least one of vectors $c_1, c_2, c_3, r_1, r_2, r_3$ is equal to $(+ \ + \ +)^T$, the matrix A has two identical columns $(+ \ + \ + \ +)^T$ or rows $(+ \ + \ + \ +)$ (and thus it does not satisfy the

SPO conditions). Assume without loss of generality that $c_1 = (+ \ + \ +)^T$. Then $c_1 \circ c_2 = c_2$ and $c_1 \circ c_3 = c_3$. In addition, either $r_1 = c_1$ and thus also $r_1 \circ r_2 = r_2$ and $r_1 \circ r_3 = r_3$, or none of r_1, r_2, r_3 is equal to c_1 , but then either $r_1 = r_2$ or $r_1 \circ r_2 = r_3$. All these cases lead to at most 7 different conditions in (6.5) so that there exists a vector v satisfying $Cv \stackrel{c}{\leftrightarrow} 0$.

It remains to treat the cases of at least two identical columns or rows in the submatrix A_1 . The case $c_1 = c_2 = c_3$ leads to three vectors r_1, r_2, r_3 that are equal to $(+ \ + \ +)^T$ or to $(- \ - \ -)^T$. Here, $c_1 \circ c_2 = c_1 \circ c_3 = c_2 \circ c_3 = (+ \ + \ +)^T$. Therefore, it is easy to find v such that $(+ \ + \ + \ +)v \stackrel{c}{\leftrightarrow} 0$, $(+ \ - \ - \ -)v \stackrel{c}{\leftrightarrow} 0$ and $(+ \ c_1^T)v \stackrel{c}{\leftrightarrow} 0$. For the next case, assume without loss of generality that $c_1 = c_2 \neq c_3$, so that $c_1 \circ c_2 = (+ \ + \ +)^T$ and $c_2 \circ c_3 = c_1 \circ c_3$. Then it is not difficult to show that at least one of the vectors r_1, r_2, r_3 must be equal to $(+ \ + \ +)^T$ or $(- \ - \ -)^T$, or, all the three vectors r_1, r_2, r_3 are the same, or two are the same and the third is negative of those two (in which cases our desired result easily holds). Then, without loss of generality, $r_1 = (+ \ + \ +)^T$ so that $r_1 \circ r_2 = r_2$ and $r_1 \circ r_3 = r_3$, or $r_1 = (- \ - \ -)^T$ so that $r_1 \circ r_2 = -r_2$ and $r_1 \circ r_3 = -r_3$. All these cases also lead to at most 7 different conditions in (6.5) so that there exists a vector v satisfying $Cv \stackrel{c}{\leftrightarrow} 0$.

We have proved the following.

Proposition 6.9. *If A is a 4×4 $(+, -)$ sign pattern, then A satisfies the SPJO conditions.*

The case for $n = 5$ can be handled in a generally similar way. We omit the proof.

Proposition 6.10. *If A is a 5×5 $(+, -)$ sign pattern, then A satisfies the SPJO conditions.*

In view of Proposition 6.9, Proposition 6.10, and Theorem 6.2, we have all the cases covered (the cases $n = 1$ and $n = 2$ are trivial).

Theorem 6.11. *For all $n \geq 1$, each $n \times n$ $(+, -)$ sign pattern A satisfies the SPJO conditions.*

We finish with some more open questions.

Open Question 6.12. Let A be an $n \times n$ $(+, -)$ sign pattern and A_1 a principal submatrix of A . Are there relations between signature patterns J satisfying the SPJO conditions for A and the signature patterns J_1 satisfying the SPJO conditions for A_1 ?

Open Question 6.13. Let A be an $n \times n$ $(+, -)$ sign pattern that satisfies the SPJO conditions. Are there some sufficient conditions on submatrices of A to ensure that $A \in \mathcal{J}_n$?

7. CONCLUDING REMARKS

In this paper we have established connections between G -matrices and J -orthogonal matrices, and we have begun an exploration of the sign patterns of the J -orthogonal matrices. This opens an interesting new topic for further research and there are many questions still to be resolved. We will continue this investigation in a follow-up paper.

Acknowledgment. The authors wish to express their sincere appreciation to Professor Miroslav Fiedler for his consultation. The authors greatly appreciate the valuable suggestions of the referee.

References

- [1] *L. B. Beasley, D. J. Scully*: Linear operators which preserve combinatorial orthogonality. *Linear Algebra Appl.* *201* (1994), 171–180.
- [2] *R. A. Brualdi, H. J. Ryser*: Combinatorial Matrix Theory. Encyclopedia of Mathematics and Its Applications 39, Cambridge University Press, Cambridge, 1991.
- [3] *R. A. Brualdi, B. L. Shader*: Matrices of Sign-Solvable Linear Systems. Cambridge Tracts in Mathematics 116, Cambridge University Press, Cambridge, 1995.
- [4] *J. Della-Dora*: Numerical linear algorithms and group theory. *Linear Algebra Appl.* *10* (1975), 267–283.
- [5] *L. Elsner*: On some algebraic problems in connection with general eigenvalue algorithms. *Linear Algebra Appl.* *26* (1979), 123–138.
- [6] *C. A. Eschenbach, F. J. Hall, D. L. Harrell, Z. Li*: When does the inverse have the same sign pattern as the transpose? *Czech. Math. J.* *49* (1999), 255–275.
- [7] *M. Fiedler*: Notes on Hilbert and Cauchy matrices. *Linear Algebra Appl.* *432* (2010), 351–356.
- [8] *M. Fiedler*, ed.: Theory of Graphs and Its Applications. Proc. Symp., Smolenice, 1963. Publishing House of the Czechoslovak Academy of Sciences, Praha, 1964.
- [9] *M. Fiedler, F. J. Hall*: G -matrices. *Linear Algebra Appl.* *436* (2012), 731–741.
- [10] *M. Fiedler, T. L. Markham*: More on G -matrices. *Linear Algebra Appl.* *438* (2013), 231–241.
- [11] *F. J. Hall, Z. Li*: Sign pattern matrices. Handbook of Linear Algebra. Chapman and Hall/CRC Press, Boca Raton, 2013.
- [12] *N. J. Higham*: J -orthogonal matrices: properties and generation. *SIAM Rev.* *45* (2003), 504–519.
- [13] *M. Rozložník, F. Okulicka-Dtůžewská, A. Smoktunowicz*: Cholesky-like factorization of symmetric indefinite matrices and orthogonalization with respect to bilinear forms. *SIAM J. Matrix Anal. Appl.* *36* (2015), 727–751.
- [14] *C. Waters*: Sign patterns that allow orthogonality. *Linear Algebra Appl.* *235* (1996), 1–13.

Authors' addresses: Frank J. Hall, Department of Mathematics and Statistics, Georgia State University, 30 Pryor St, Atlanta, Georgia 30303, USA, e-mail: fhall@gsu.edu; Miroslav Rozložník, Institute of Computer Science, Czech Academy of Sciences, Pod Vodárenskou věží 2, 182 07 Praha 8, Czech Republic, e-mail: miro@cs.cas.cz.