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Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 3, 671–679

Persistent URL: <http://dml.cz/dmlcz/145864>

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$\mathcal{D}_{n,r}$ IS NOT POTENTIALLY NILPOTENT FOR $n \geq 4r - 2$

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(Received August 29, 2015)

Dedicated to the memory of Miroslav Fiedler

Abstract. An $n \times n$ sign pattern $\mathcal{A}$ is said to be potentially nilpotent if there exists a nilpotent real matrix B with the same sign pattern as $\mathcal{A}$. Let $\mathcal{D}_{n,r}$ be an $n \times n$ sign pattern with $2 \leq r \leq n$ such that the superdiagonal and the (n, n) entries are positive, the $(i, 1)$ ($i = 1, \dots, r$) and $(i, i - r + 1)$ ($i = r + 1, \dots, n$) entries are negative, and zeros elsewhere. We prove that for $r \geq 3$ and $n \geq 4r - 2$, the sign pattern $\mathcal{D}_{n,r}$ is not potentially nilpotent, and so not spectrally arbitrary.

Keywords: sign pattern; potentially nilpotent pattern; spectrally arbitrary pattern

MSC 2010: 15A18, 05C50

1. INTRODUCTION

A *sign pattern* $\mathcal{A}$ is a matrix whose entries are from the set $\{+, -, 0\}$. Associated with each sign pattern $\mathcal{A} = (a_{ij})$ is a class of real matrices, called the *qualitative class* of $\mathcal{A}$, defined by

$$Q(\mathcal{A}) = \{B = (b_{ij}) : B \text{ is an } n \times n \text{ real matrix, and } \text{sign } b_{ij} = a_{ij} \forall i, j\}.$$

An $n \times n$ sign pattern $\mathcal{A}$ is a *spectrally arbitrary sign pattern* (SAP) if for any given real monic polynomial $f(x)$ with degree n , there exists a real matrix $B \in Q(\mathcal{A})$ with characteristic polynomial $f(x)$. A sign pattern $\mathcal{A}$ is a *minimal SAP* (MSAP) if $\mathcal{A}$ is a SAP, but is not a SAP if one or more nonzero entries are replaced by zero.

An $n \times n$ sign pattern $\mathcal{A}$ is *potentially nilpotent* if there exists $B \in Q(\mathcal{A})$ such that B is nilpotent, i.e., there exists $B \in Q(\mathcal{A})$ with characteristic polynomial $f(x) = x^n$. In particular, each SAP must necessarily be potentially nilpotent.

The research has been supported by Shanxi Scholarship Council of China (No. 2012-070).

A *permutation sign pattern* is a square sign pattern with entries from the set $\{0, +\}$, where the entry $+$ occurs precisely once in each row and in each column. A *signature sign pattern* is a square diagonal sign pattern all of whose diagonal entries are nonzero. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two square sign patterns of the same order. A sign pattern $\mathcal{A}_1$ is said to be *permutationally similar* to $\mathcal{A}_2$ if there exists a permutation sign pattern $\mathcal{P}$ such that $\mathcal{A}_2 = \mathcal{P}^T \mathcal{A}_1 \mathcal{P}$. A sign pattern $\mathcal{A}_1$ is said to be *signature similar* to $\mathcal{A}_2$ if there exists a signature sign pattern $\mathcal{D}$ such that $\mathcal{A}_2 = \mathcal{D} \mathcal{A}_1 \mathcal{D}$. The properties of being potentially nilpotent and spectrally arbitrary are preserved under negation, transposition, signature similarity and permutation similarity. Two sign patterns are said to be *equivalent* if one can be obtained from the other by any combination of these four operations.

Cavers and Vander Meulen [3] considered an interesting $n \times n$ sign pattern $\mathcal{D}_{n,r}$ with $2n$ nonzero entries defined as follows:

- ▷ $\mathcal{D}_{n,r}$ is an $n \times n$ sign pattern with $2 \leq r \leq n$ such that the superdiagonal and the (n, n) entries are positive, the $(i, 1)$, $i = 1, \dots, r$, and $(i, i - r + 1)$, $i = r + 1, \dots, n$, entries are negative, and zeros elsewhere.

They proved that if $n \leq 2r$, then $\mathcal{D}_{n,r}$ is a MSAP. If $r \geq 3$, then $\mathcal{D}_{2r+1,r}$ is not potentially nilpotent.

More recently, Gao et al. in [4] proved that if $r \geq 3$ and $2r + 2 \leq n \leq 4r - 3$, then $\mathcal{D}_{n,r}$ is not potentially nilpotent and thus not a SAP. Garnett and Shader in [5] proved that $\mathcal{D}_{n,2}$ is a SAP for all $n \geq 2$.

To the best of our knowledge, for $r \geq 3$ and $n \geq 4r - 2$, some questions about $\mathcal{D}_{n,r}$ are still open ([2], page 3084). For instance, “Is $\mathcal{D}_{n,r}$ a SAP?”

Our main result answers the above question:

Theorem 1.1. *If $r \geq 3$ and $n \geq 4r - 2$, then $\mathcal{D}_{n,r}$ is not potentially nilpotent and thus not a SAP.*

2. PRELIMINARY

Let $A = (a_{ij})$ be an $n \times n$ matrix. The *directed graph (digraph)* $D(A)$ of A is the directed graph with vertex set $\{1, 2, \dots, n\}$ such that there is a directed edge in $D(A)$ from i to j , denoted by $i \rightarrow j$, if and only if $a_{ij} \neq 0$.

A *directed path* of length $k - 1$ in $D(A)$ is a sequence of $k - 1$ edges $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{k-1} \rightarrow i_k$ such that the vertices are distinct. A *simple cycle* of length k in $D(A)$ consists of a directed path as above together with the additional directed edge $i_k \rightarrow i_1$. A (*composite*) *k-cycle* is a set of simple cycles whose total length is k , and whose index sets are mutually disjoint.

A nonzero product of the form $\gamma = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1}$ in which the index set $\{i_1, i_2, \dots, i_k\}$ consists of distinct indices is called a *simple cycle of length k* of A . A *composite k -cycle* is a product of simple cycles whose total length is k , and whose index sets are mutually disjoint.

A cycle (simple or composite) of a matrix A just corresponds to $\pm$ a term in the principle minor of A based upon the indices appearing in the cycle.

Note that the cycles of a matrix A correspond exactly to the cycles of the digraph $D(A)$ (see [1]).

Let $D_{n,r}$ be a real matrix in $Q(\mathcal{D}_{n,r})$ with $r \geq 3$ and $n \geq 4r-3$. Up to equivalence, we may assume $D_{n,r}$ has the form

$$(2.1) \quad D_{n,r} = \begin{bmatrix} -a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -a_2 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -a_r & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & -a_{r+1} & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 1 \\ 0 & 0 & 0 & -a_n & \dots & 0 & b \end{bmatrix},$$

where b and all the a_i , $i = 1, 2, \dots, n$, are positive.

The digraph $D(D_{n,r})$ is as follows:

Figure 1. The digraph $D(D_{n,r})$.

3. THE CHARACTERISTIC POLYNOMIAL OF $D_{n,r}$

The characteristic polynomial of a real matrix B of order n is given [6] by

$$p_B(t) = t^n - E_1(B)t^{n-1} + E_2(B)t^2 + \dots + (-1)^n E_n(B),$$

where $E_k(B)$ is the sum of the $k \times k$ principal minors of B , $k = 1, 2, \dots, n$. We use $B\{i_1, i_2, \dots, i_k\}$ to denote the $k \times k$ principal minor of B based on the indices $\{i_1, i_2, \dots, i_k\}$.

For $D_{n,r}$ in the form (2.1) with $r \geq 3$ and $n \geq 4r-3$, we have the following results, which are important in the proof of Theorem 1.1.

Lemma 3.1. $E_1(D_{n,r}) = b - a_1$, $E_2(D_{n,r}) = -ba_1 + a_2$.

P r o o f. It is clear that $E_1(D_{n,r}) = \text{tr}(D_{n,r}) = b - a_1$. The only two 2×2 nonzero principal minors of $D_{n,r}$ are $D_{n,r}\{1, 2\} = a_2$ and $D_{n,r}\{1, n\} = -ba_1$. Thus, $E_2(D_{n,r}) = -ba_1 + a_2$. $\square$

Lemma 3.2. $E_r(D_{n,r}) = (-1)^{r-1}ba_{r-1} + (-1)^r \sum_{i=r}^n a_i$.

P r o o f. All r -cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$i \rightarrow i+1 \rightarrow \dots \rightarrow i+r-1 \rightarrow i, \quad i = 1, 2, \dots, n-r+1, \\ (1 \rightarrow 2 \rightarrow \dots \rightarrow r-1 \rightarrow 1) \cup (n \rightarrow n).$$

The corresponding nonzero $r \times r$ principal minors of $D_{n,r}$ are

$$D_{n,r}\{i, i+1, \dots, i+r-1\} = (-1)^r a_{i+r-1}, \quad i = 1, 2, \dots, n-r+1, \\ D_{n,r}\{1, 2, \dots, r-1, n\} = (-1)^{r-1}ba_{r-1}.$$

So, $E_r(D_{n,r}) = (-1)^{r-1}ba_{r-1} + (-1)^r \sum_{i=r}^n a_i$. $\square$

Lemma 3.3. $E_{r+1}(D_{n,r}) = (-1)^r b \sum_{i=r}^{n-1} a_i - (-1)^r a_1 \sum_{i=r+1}^n a_i$.

P r o o f. All $(r+1)$ -cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$(i \rightarrow i+1 \rightarrow \dots \rightarrow i+r-1 \rightarrow i) \cup (n \rightarrow n), \quad i = 1, 2, \dots, n-r, \\ (1 \rightarrow 1) \cup (i \rightarrow i+1 \rightarrow \dots \rightarrow i+r-1 \rightarrow i), \quad i = 2, 3, \dots, n-r+1.$$

The corresponding nonzero $(r+1) \times (r+1)$ principal minors of $D_{n,r}$ are

$$D_{n,r}\{i, i+1, \dots, i+r-1, n\} = (-1)^r a_{i+r-1}b, \quad i = 1, 2, \dots, n-r, \\ D_{n,r}\{1, i, i+1, \dots, i+r-1\} = -a_1(-1)^r a_{i+r-1}, \quad i = 2, 3, \dots, n-r+1.$$

So, $E_{r+1}(D_{n,r}) = (-1)^r b \sum_{i=r}^{n-1} a_i - (-1)^r a_1 \sum_{i=r+1}^n a_i$. $\square$

Lemma 3.4. $E_{r+2}(D_{n,r}) = (-1)^{r+1}a_1b \sum_{i=r+1}^{n-1} a_i + (-1)^r a_2 \sum_{i=r+2}^n a_i$.

P r o o f. All $(r + 2)$ -cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$(1 \rightarrow 1) \cup (i \rightarrow i + 1 \rightarrow \dots \rightarrow i + r - 1 \rightarrow i) \cup (n \rightarrow n), \quad i = 2, 3, \dots, n - r,$$

$$(1 \rightarrow 2 \rightarrow 1) \cup (i \rightarrow i + 1 \rightarrow \dots \rightarrow i + r - 1 \rightarrow i), \quad i = 3, 4, \dots, n - r + 1.$$

The corresponding nonzero $(r + 2) \times (r + 2)$ principal minors of $D_{n,r}$ are

$$D_{n,r}\{1, i, i + 1, \dots, i + r - 1, n\} = (-1)^{r+1} a_1 b a_{i+r-1}, \quad i = 2, 3, \dots, n - r,$$

$$D_{n,r}\{1, 2, i, i + 1, \dots, i + r - 1\} = (-1)^r a_2 a_{i+r-1}, \quad i = 3, 4, \dots, n - r + 1.$$

$$\text{So, } E_{r+2}(D_{n,r}) = (-1)^{r+1} a_1 b \sum_{i=r+1}^{n-1} a_i + (-1)^r a_2 \sum_{i=r+2}^n a_i. \quad \square$$

$$\mathbf{Lemma\ 3.5.} \quad E_{2r}(D_{n,r}) = -a_{r-1} b \sum_{i=2r-1}^{n-1} a_i + \sum_{i=r}^{n-r} \sum_{j=i+r}^n a_i a_j.$$

P r o o f. All $2r$ -cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$(1 \rightarrow 2 \rightarrow \dots \rightarrow r - 1 \rightarrow 1) \cup (i \rightarrow i + 1 \rightarrow \dots \rightarrow i + r - 1 \rightarrow i) \cup (n \rightarrow n),$$

$$i = r, r + 1, \dots, n - r,$$

$$(i \rightarrow i + 1 \rightarrow \dots \rightarrow i + r - 1 \rightarrow i) \cup (j \rightarrow j + 1 \rightarrow \dots \rightarrow j + r - 1 \rightarrow j),$$

$$i = 1, 2, \dots, n - 2r + 1, \quad j = i + r, i + r + 1, \dots, n - r + 1.$$

The corresponding nonzero $2r \times 2r$ principal minors of $D_{n,r}$ are

$$D_{n,r}\{1, 2, \dots, r - 1, i, i + 1, \dots, i + r - 1, n\} = -a_{r-1} b a_{i+r-1},$$

$$i = r, r + 1, \dots, n - r,$$

$$D_{n,r}\{i, i + 1, \dots, i + r - 1, j, j + 1, \dots, j + r - 1\} = a_{i+r-1} a_{j+r-1},$$

$$i = 1, 2, \dots, n - 2r + 1, \quad j = i + r, i + r + 1, \dots, n - r + 1.$$

$$\text{So, } E_{2r}(D_{n,r}) = -a_{r-1} b \sum_{i=2r-1}^{n-1} a_i + \sum_{i=r}^{n-r} \sum_{j=i+r}^n a_i a_j. \quad \square$$

$$\mathbf{Lemma\ 3.6.} \quad E_{2r+1}(D_{n,r}) = b \sum_{i=r}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j - a_1 \sum_{i=r+1}^{n-r} \sum_{j=i+r}^n a_i a_j.$$

P r o o f. All $(2r + 1)$ -cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$(i \rightarrow i + 1 \rightarrow \dots \rightarrow i + r - 1 \rightarrow i) \cup (j \rightarrow j + 1 \rightarrow \dots \rightarrow j + r - 1 \rightarrow j) \cup (n \rightarrow n),$$

$$i = 1, 2, \dots, n - 2r, \quad j = i + r, i + r + 1, \dots, n - r,$$

$$(1 \rightarrow 1) \cup (i \rightarrow i + 1 \rightarrow \dots \rightarrow i + r - 1 \rightarrow i) \cup (j \rightarrow j + 1 \rightarrow \dots \rightarrow j + r - 1 \rightarrow j),$$

$$i = 2, 3, \dots, n - 2r + 1, \quad j = i + r, i + r + 1, \dots, n - r + 1.$$

The corresponding nonzero $(2r+1) \times (2r+1)$ principal minors of $D_{n,r}$ are

$$\begin{aligned} D_{n,r}\{i, i+1, \dots, i+r-1, j, j+1, \dots, j+r-1, n\} &= ba_{i+r-1}a_{j+r-1}, \\ i &= 1, 2, \dots, n-2r, \quad j = i+r, i+r+1, \dots, n-r, \\ D_{n,r}\{1, i, i+1, \dots, i+r-1, j, j+1, \dots, j+r-1\} &= -a_1a_{i+r-1}a_{j+r-1}, \\ i &= 2, 3, \dots, n-2r+1, \quad j = i+r, i+r+1, \dots, n-r+1. \end{aligned}$$

$$\text{So, } E_{2r+1}(D_{n,r}) = b \sum_{i=r}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j - a_1 \sum_{i=r+1}^{n-r} \sum_{j=i+r}^n a_i a_j. \quad \square$$

$$\textbf{Lemma 3.7.} \quad E_{2r+2}(D_{n,r}) = -a_1 b \sum_{i=r+1}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j + a_2 \sum_{i=r+2}^{n-r} \sum_{j=i+r}^n a_i a_j.$$

Proof. All $(2r+2)$ -cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$\begin{aligned} &(1 \rightarrow 1) \cup (i \rightarrow i+1 \rightarrow \dots \rightarrow i+r-1 \rightarrow i) \cup (j \rightarrow j+1 \rightarrow \dots \rightarrow j+r-1 \rightarrow j) \\ &\quad \cup (n \rightarrow n), \quad i = 2, 3, \dots, n-2r, \quad j = i+r, i+r+1, \dots, n-r, \\ &(1 \rightarrow 2 \rightarrow 1) \cup (i \rightarrow i+1 \rightarrow \dots \rightarrow i+r-1 \rightarrow i) \cup (j \rightarrow j+1 \rightarrow \dots \rightarrow j+r-1 \rightarrow j), \\ &\quad i = 3, 4, \dots, n-2r+1, \quad j = i+r, i+r+1, \dots, n-r+1. \end{aligned}$$

The corresponding nonzero $(2r+2) \times (2r+2)$ principal minors of $D_{n,r}$ are

$$\begin{aligned} D_{n,r}\{1, i, i+1, \dots, i+r-1, j, j+1, \dots, j+r-1, n\} &= -a_1 ba_{i+r-1}a_{j+r-1} \\ i &= 2, 3, \dots, n-2r, \quad j = i+r, i+r+1, \dots, n-r, \\ D_{n,r}\{1, 2, i, i+1, \dots, i+r-1, j, j+1, \dots, j+r-1\} &= a_2 a_{i+r-1} a_{j+r-1} \\ i &= 3, 4, \dots, n-2r+1, \quad j = i+r, i+r+1, \dots, n-r+1. \end{aligned}$$

$$\text{So, } E_{2r+2}(D_{n,r}) = -a_1 b \sum_{i=r+1}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j + a_2 \sum_{i=r+2}^{n-r} \sum_{j=i+r}^n a_i a_j. \quad \square$$

4. PROOF OF THEOREM 1.1

We will prove the theorem by contradiction.

Let $r \geq 3$ and $n \geq 4r-2$. Suppose $\mathcal{D}_{n,r}$ is potentially nilpotent. Then there exists $D_{n,r} \in Q(\mathcal{D}_{n,r})$ in the form (2.1) such that $E_k(D_{n,r}) = 0$ for $k = 1, 2, \dots, n$.

By Lemma 3.1, we have $E_1(D_{n,r}) = b - a_1 = 0$ and $E_2(D_{n,r}) = -ba_1 + a_2 = 0$. Thus, $b = a_1$ and $a_2 = ba_1$.

By Lemma 3.2, we have $E_r(D_{n,r}) = (-1)^{r-1}ba_{r-1} + (-1)^r \sum_{i=r}^n a_i = 0$. Thus,

$$(4.1) \quad ba_{r-1} = \sum_{i=r}^n a_i.$$

By Lemma 3.3, we have $E_{r+1}(D_{n,r}) = (-1)^r b \sum_{i=r}^{n-1} a_i - (-1)^r a_1 \sum_{i=r+1}^n a_i = 0$. Thus, $b \sum_{i=r}^{n-1} a_i = a_1 \sum_{i=r+1}^n a_i$. Since $b = a_1$, we get $a_r = a_n$.

By Lemma 3.4, we have $E_{r+2}(D_{n,r}) = (-1)^{r+1}a_1b \sum_{i=r+1}^{n-1} a_i + (-1)^r a_2 \sum_{i=r+2}^n a_i = 0$. Thus, $a_1b \sum_{i=r+1}^{n-1} a_i = a_2 \sum_{i=r+2}^n a_i$. Since $ba_1 = a_2$, we get $a_{r+1} = a_n$.

By Lemma 3.6, we have $E_{2r+1}(D_{n,r}) = b \sum_{i=r}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j - a_1 \sum_{i=r+1}^{n-r} \sum_{j=i+r}^n a_i a_j = 0$. Thus, $b \sum_{i=r}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j = a_1 \sum_{i=r+1}^{n-r} \sum_{j=i+r}^n a_i a_j$. Since $b = a_1$, we get

$$\sum_{i=r}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j = \sum_{i=r+1}^{n-r} \sum_{j=i+r}^n a_i a_j,$$

and so,

$$a_r(a_{2r} + a_{2r+1} + \dots + a_{n-1}) = a_n(a_{r+1} + a_{r+2} + \dots + a_{n-r}).$$

Noting that $a_r = a_n$, we have

$$(4.2) \quad a_{2r} + a_{2r+1} + \dots + a_{n-1} = a_{r+1} + a_{r+2} + \dots + a_{n-r}.$$

By Lemma 3.7, we have

$$E_{2r+2}(D_{n,r}) = -a_1b \sum_{i=r+1}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j + a_2 \sum_{i=r+2}^{n-r} \sum_{j=i+r}^n a_i a_j = 0.$$

Thus $a_1b \sum_{i=r+1}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j = a_2 \sum_{i=r+2}^{n-r} \sum_{j=i+r}^n a_i a_j$. Since $ba_1 = a_2$, we get

$$\sum_{i=r+1}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j = \sum_{i=r+2}^{n-r} \sum_{j=i+r}^n a_i a_j,$$

and so,

$$a_{r+1}(a_{2r} + a_{2r+1} + \dots + a_{n-1}) = a_n(a_{r+2} + a_{r+3} + \dots + a_{n-r}).$$

Noting that $a_{r+1} = a_n$, we have

$$(4.3) \quad a_{2r+1} + a_{2r+2} + \dots + a_{n-1} = a_{r+2} + a_{r+3} + \dots + a_{n-r}.$$

Combining (4.2) and (4.3), we obtain $a_{r+1} = a_{2r}$.

By Lemma 3.5 and (4.1), that is, $ba_{r-1} = \sum_{i=r}^n a_i$, we have

$$E_{2r}(D_{n,r}) = -a_{r-1}b \sum_{i=2r-1}^{n-1} a_i + \sum_{i=r}^{n-r} \sum_{j=i+r}^n a_i a_j = -\sum_{i=r}^n a_i \sum_{j=2r-1}^{n-1} a_j + \sum_{i=r}^{n-r} a_i \sum_{j=i+r}^n a_j.$$

By the assumption $n \geq 4r - 2$, we have $n - r - 1 \geq 3r - 3 > r$. Thus the above expression can be written as

$$\begin{aligned} E_{2r}(D_{n,r}) &= \sum_{i=r}^{n-r} a_i \sum_{j=i+r}^n a_j - \sum_{i=r}^n a_i \sum_{j=2r-1}^{n-1} a_j \\ &= \sum_{i=r}^{n-r-1} a_i \left(\sum_{j=i+r}^n a_j - \sum_{j=2r-1}^{n-1} a_j \right) + a_{n-r} a_n \\ &\quad - \sum_{i=n-r}^{n-1} a_i \sum_{j=2r-1}^{n-1} a_j - a_n \sum_{j=2r-1}^{n-1} a_j \\ &= \sum_{i=r}^{n-r-1} a_i \left(a_n - \sum_{j=2r-1}^{i+r-1} a_j \right) + a_{n-r} a_n - \sum_{i=n-r}^{n-1} a_i \sum_{j=2r-1}^{n-1} a_j - a_n \sum_{j=2r-1}^{n-1} a_j \\ &= a_n \sum_{i=r}^{n-r} a_i - \sum_{i=r}^{n-r-1} a_i \sum_{j=2r-1}^{i+r-1} a_j - \sum_{i=n-r}^{n-1} a_i \sum_{j=2r-1}^{n-1} a_j - a_n \sum_{j=2r-1}^{n-1} a_j. \end{aligned}$$

By (4.2),

$$a_n \sum_{i=r}^{n-r} a_i - a_n \sum_{j=2r-1}^{n-1} a_j = a_n(a_r - a_{2r-1}).$$

Then according to the assumption that $a_i > 0$ for $i = 1, 2, \dots, n$ and the known result $a_r = a_{r+1} = a_{2r} = a_n$, we have

$$\begin{aligned} E_{2r}(D_{n,r}) &= a_n(a_r - a_{2r-1}) - \sum_{i=r}^{n-r-1} a_i \sum_{j=2r-1}^{i+r-1} a_j - \sum_{i=n-r}^{n-1} a_i \sum_{j=2r-1}^{n-1} a_j \\ &< a_n(a_r - a_{2r-1}) - a_{r+1}(a_{2r-1} + a_{2r}) \\ &= a_n(a_n - a_{2r-1}) - a_n(a_{2r-1} + a_n) = -2a_n a_{2r-1} < 0, \end{aligned}$$

contradicting the identity $E_{2r}(D_{n,r}) = 0$. So $\mathcal{D}_{n,r}$ is not potentially nilpotent. $\square$

Acknowledgment. The authors are very grateful to the referee for many valuable suggestions.

References

- [1] *R. A. Brualdi, H. J. Ryser*: Combinatorial Matrix Theory. Encyclopedia of Mathematics and Its Applications 39, Cambridge University Press, Cambridge, 1991.
- [2] *M. Catral, D. D. Olesky, P. van den Driessche*: Allow problems concerning spectral properties of sign pattern matrices: a survey. *Linear Algebra Appl.* 430 (2009), 3080–3094.
- [3] *M. S. Cavers, K. N. Vander Meulen*: Spectrally and inertially arbitrary sign patterns. *Linear Algebra Appl.* 394 (2005), 53–72.
- [4] *Y. Gao, Z. Li, Y. Shao*: A note on spectrally arbitrary sign patterns. *JP J. Algebra Number Theory Appl.* 11 (2008), 15–35.
- [5] *C. Garnett, B. L. Shader*: A proof of the T_n conjecture: Centralizers, Jacobians and spectrally arbitrary sign patterns. *Linear Algebra Appl.* 436 (2012), 4451–4458.
- [6] *R. A. Horn, C. R. Johnson*: Matrix Analysis. Cambridge University Press, Cambridge, 1985.

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