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A graph associated to proper non-small ideals of a commutative ring

S. Ebrahimi Atani, S. Dolati Pish Hesari, M. Khoramdel

Abstract. In this paper, a new kind of graph on a commutative ring is introduced and investigated. Small intersection graph of a ring $R$, denoted by $G(R)$, is a graph with all non-small proper ideals of $R$ as vertices and two distinct vertices $I$ and $J$ are adjacent if and only if $I \cap J$ is not small in $R$. In this article, some interrelation between the graph theoretic properties of this graph and some algebraic properties of rings are studied. We investigated the basic properties of the small intersection graph as diameter, girth, clique number, cut vertex, planar property and independence number. Further, it is shown that the independence number of a small graph of a ring $R$ is equal to the number of its maximal ideals and the domination number of small graph is at most 2.

Keywords: small ideal; small intersection graph; clique number; independence number; domination number; planar property

Classification: 05C40, 05C25, 13A15

1. Introduction

In 1988, Beck [3] introduced the concept of the zero-divisor graph. Since then, others have introduced and studied many researches in this area. One of the most important graphs which have been studied is the intersection graph. Bosak [5] in 1964 defined the intersection graph of semigroups. In 1969, Csákány and Pollák [8] studied the graph of subgroups of a finite group. In 2009, the intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [6]. The intersection graph of ideals of rings and submodules of modules has been investigated by several authors ([1], [10], [12], [13] and [14]).

In this paper, we introduce small intersection graph of ideals of a commutative ring, a new kind of intersection graph of rings. If the Jacobson radical of a ring $R$ is zero, then the small intersection graph coincides with the intersection graph which is introduced by Chakrabarty, et al. [6]. The small intersection graph helps us to consider the algebraic properties of rings using graph theoretical tools. In our investigation of $G(R)$, maximal ideals play an important role to find some connections between the graph theoretic properties of this graph and some algebraic properties of rings. For instance, see Theorem 2.6 and 3.6.
In Section 2, we show that the small intersection graph of a ring $R$ is connected if and only if $|\text{max}(R)| \neq 2$. Also if $G(R)$ is a connected graph, then $\text{diam}(G(R)) \leq 2$ and $gr(G(R)) = 3$ provided $G(R)$ contains a cycle. For a ring $R$, it is proved that $G(R)$ cannot be a complete $r$-partite graph and $G(R)$ has no cut vertex. Moreover, if $R$ is a ring with finitely many maximal ideals, then $G(R)$ cannot be a complete graph and we give an example of a ring $R$ with infinite maximal ideals such that its small intersection graph is complete. At the end of this section, it is shown that if $G(R)$ contains an end vertex then $|\text{max}(R)| = 2$.

In Section 3, it is shown that if $\omega(G(R))$ is finite, then the number of maximal ideals of $R$ is finite, $R$ is semiperfect and $R$ has finitely many maximal ideals. This enables us to show that, if the set of proper non-small ideals is nonempty and finite, then the set of ideals of $R$ is finite. Also, it is proved that $G(R)$ is a planar graph if and only if either $|\text{max}(R)| = 2$ and $R = R_1 \times R_2$, where $R_i$ $(i = 1, 2)$ is a local principle ideal ring with maximal ideal $M_i$ such that $M_i^n = 0$, for some $n \leq 4$ or $|\text{max}(R)| = 3$ and $R$ is semisimple. Among other results, it is shown that the independence number of a small graph of a ring $R$ is equal to the number of its maximal ideals and the domination number of small graph is at most 2.

Throughout this paper $R$ is a commutative ring with unity. Jacobson radical of $R$, denoted by $J(R)$, is the intersection of all maximal ideals of $R$ and $\text{max}(R)$ denotes the set of all maximal ideals of $R$. An ideal $I$ of $R$ ($I \leq R$) is small (denoted by $I \ll R$) if $I + K = R$, for some ideal $K$ of $R$, implies $K = R$. A module $M$ is said to be hollow module if every proper submodule of $M$ is a small submodule.

Let $I$ be an ideal of a ring $R$. It is said that idempotents of $R/I$ can be lifted, if for every idempotent $a + I \in R/I$, there exists idempotent $e \in R$ such that $a + I = e + I$. A ring $R$ is called semiperfect in case $R/J(R)$ is semisimple and every idempotent of $R/J(R)$ can be lifted (see [15]).

A graph $G$ is called connected, if for any vertices $x$ and $y$ of $G$ there is a path between $x$ and $y$. Otherwise, $G$ is called disconnected. The distance between two distinct vertices $a$ and $b$, denoted by $d(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$, also $d(a, a) = 0$). The diameter of a graph $\Gamma$, denoted by $\text{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph $\Gamma$, denoted by $\text{gr}(\Gamma)$, is the length of a shortest cycle in $\Gamma$, provided $\Gamma$ contains a cycle; otherwise; $\text{gr}(\Gamma) = \infty$. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph $G$, denoted by $w(G)$, is called the clique number of $G$. For $r$ a nonnegative integer, an $r$-partite graph is one whose vertex set can be partitioned into $r$ subsets so that no edge has both ends in any one subset. A complete $r$-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with part sizes $m$ and $n$ is denoted by $K_{m,n}$. We will sometimes call $K_{1,n}$ a star graph. Note that a graph whose
vertices-set is empty is a null graph and a graph whose edge-set is empty is an empty graph.

Let $G = (V, E)$ be a graph. The (open) neighborhood $N(v)$ of a vertex $v$ of $V$ is the set of vertices which are adjacent to $v$. For each $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. A set of vertices $S$ in $G$ is a dominating set, if $N[S] = V$. The domination number, $\gamma(G)$, of $G$ is the minimum cardinality of a dominating set of $G$ ([9]).

In a graph $G = (V, E)$, a set $S \subseteq V$ is an independent set if the subgraph induced by $S$ is totally disconnected. The independence number $\alpha(G)$ is the maximum size of an independent set in $G$.

2. Some basic properties of $G(R)$

We begin this section with the following remark which will be used in the next theorems and lemmas.

**Remark 2.1.** (i) Let $R$ be a ring and $I, J$ be two ideals of $R$. If $M$ is a maximal ideal of $R$, then $I \cap J \subseteq M$ implies $I \subseteq M$ or $J \subseteq M$.

(ii) Let $R$ be a ring with $\max(R) = \{M_i\}_{i \in I}$ and $\nu$ be a proper finite subset of $I$. Then $\bigcap_{\nu} M_i$ is not a small ideal of $R$. Otherwise, if $\bigcap_{\nu} M_i \ll R$, then $\bigcap_{\nu} M_i \subseteq M_j$ for each $j \in I \setminus \nu$. Hence $M_i \subseteq M_j$ for some $i \in \nu$, which is a contradiction.

We begin with the key definition of this paper.

**Definition 2.2.** Let $R$ be a ring. The small intersection graph $G(R)$ is the graph with all non small proper ideals of $R$ as vertices and two distinct vertices $I$ and $J$ are adjacent if and only if $I \cap J \not\ll R$.

**Proposition 2.3.** Let $R$ be a ring. Then $G(R)$ is a null graph if and only if $R$ is a local ring.

**Proof:** The proof is clear. $\square$

Since all definitions of graph theory are for non-null graph, in this paper all graphs are considered non-null ([4]).

**Theorem 2.4.** Let $R$ be a ring. Then $G(R)$ is an empty graph if and only if $\max(R) = \{M_1, M_2\}$, where $M_1$ and $M_2$ ($M_1 \neq M_2$) are finitely generated hollow $R$-modules.

**Proof:** Let $G(R)$ be an empty graph. If $|\max(R)| = 1$, then $G(R)$ is a null graph by Proposition 2.3, a contradiction. Suppose, $|\max(R)| \geq 3$ and $M_1, M_2$ and $M_3 \in \max(R)$. By Remark 2.1, $M_1$ and $M_2$ are adjacent, a contradiction. So $|\max(R)| = 2$. Let $\max(R) = \{M_1, M_2\}$ with $M_1 \neq M_2$. We show that $M_1$ and $M_2$ are hollow $R$-modules. Since $R \frac{M_1 + M_2}{M_2} = \frac{M_1}{M_1 \cap M_2}$, $M_1 \cap M_2$ is a maximal submodule of $M_1$. We show that this is the only maximal submodule of $M_1$. Let $I$ be a maximal submodule of $M_1$. If $I \not\ll R$, then $I \cap M_1 = I$ implies $I$ and $M_1$ are adjacent in $G(R)$, a contradiction. So $I \ll R$. Hence $I \subseteq J(R) = M_1 \cap M_2$,
which implies that $I = M_1 \cap M_2$ by maximality of $I$. So $M_1$ is a local $R$-module with maximal submodule $M_1 \cap M_2$. Now, we show that $M_1$ is a finitely generated $R$-module. Let $x \in M_1 \setminus M_2$, so $Rx \nsubseteq R$ because $Rx \nsubseteq M_1 \cap M_2 = J(R)$. If $Rx \neq M_1$, then $Rx \cap M_1 = Rx$ which implies $Rx$ and $M_1$ are adjacent in $G(R)$, a contradiction. So $Rx = M_1$. Hence $M_1$ is a finitely generated local $R$-module. So $M_1$ is a finitely generated hollow $R$-module by [15]. By the similar way $M_2$ is a finitely generated hollow $R$-module.

Conversely, let $\max(R) = \{M_1, M_2\}$, where $M_1, M_2$ are finitely generated hollow $R$-modules. By a similar argument as above, $M_1 \cap M_2$ is a maximal submodule of $M_1$ and $M_2$. Since $M_1$ and $M_2$ are local, $M_1 \cap M_2$ is the only maximal submodule of $M_1$ and $M_2$. Let $I \neq M_1, M_2$ be a non-small ideal of $R$. Then $I \subseteq M_1$ or $I \subseteq M_2$. Suppose, without loss of generality, $I \subseteq M_1$. Since $M_1$ is a finitely generated local $R$-module, $I \subseteq M_1 \cap M_2 = J(R)$. So $I \ll R$, a contradiction. So the only non-small ideals of $R$ are $M_1$ and $M_2$ which are not adjacent. So $G(R)$ is an empty graph.

In the following we give an example of a ring $R$ with empty $G(R)$.

**Example 2.5.** Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. It is clear that $\max(R) = \{0 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus 0\}$ and $J(R) = 0$. By drawing the $G(R)$, we see that $G(R)$ is an empty graph with two vertices and $M_1, M_2$ are hollow.

We are now in a position to show a finer relationship between the number of maximal ideals of $R$ and the connectivity of $G(R)$.

**Theorem 2.6.** Let $R$ be a ring. The following statements are equivalent:

(i) $G(R)$ is not connected;

(ii) $|\max(R)| = 2$;

(iii) $G(R) = G_1 \cup G_2$, where $G_1, G_2$ are two disjoint complete subgraphs of $G(R)$.

**Proof:** (i) $\Rightarrow$ (ii) Assume that $G(R)$ is not connected. Let $G_1$ and $G_2$ be two components of $G(R)$ and $I$, $J$ be two ideals of $R$ such that $I \subseteq G_1$ and $J \subseteq G_2$. Let $M_1$, $M_2$ be maximal ideals of $R$ such that $I \subseteq M_1$ and $J \subseteq M_2$. If $M_1 = M_2$, then $I - M_1 - J$ is a path in $G(R)$ which is a contradiction. So $M_1 \neq M_2$. If $M_1 \cap M_2 \ll R$, then $I - M_1 - M_2 - J$ is a path between $G_1$ and $G_2$, which is a contradiction. Hence $M_1 \cap M_2 \ll R$, which gives $|\max(R)| = 2$.

(ii) $\Rightarrow$ (iii) Let $|\max(R)| = 2$ and $J(R) = M_1 \cap M_2$, where $M_1, M_2$ are two maximal ideals of $R$. Let $G_i = \{I_i \subseteq R : I_i \subseteq M_i \text{ and } I_i \ll R\}$ for $i = 1, 2$. Let $I, J$ be elements of $G_1$. If $I$ and $J$ are not adjacent then $I \cap J \ll R$, which implies $I \cap J \subseteq M_1 \cap M_2$. Hence $I \cap J \subseteq M_2$, which gives $I \subseteq M_2$ or $J \subseteq M_2$ by Remark 2.1. So $I \ll R$ or $J \ll R$, a contradiction. So $G_1$ is a complete subgraph of $G(R)$. By the similar way $G_2$ is a complete subgraph of $G(R)$. Now, we show that there is no path between $G_1$ and $G_2$. Suppose, on the contrary, $I$ and $J$ are adjacent for some ideals $I \in G_1$ and $J \in G_2$ (note that each vertex in $G(R)$ is contained in $G_1$ or $G_2$). Since $I \cap J \subseteq M_1 \cap M_2 = J(R)$, so $I \cap J \ll R$, a contradiction with adjacency of $I$ and $J$. So none of elements of $G_1$ and $G_2$
are adjacent. Hence $G(R) = G_1 \cup G_2$, where $G_i$’s are disjoint complete subgraph of $G(R)$.

(iii) $\Rightarrow$ (i) It is clear.

In the following we provide an example of a ring $R$ with two maximal ideals such that $G(R)$ is not connected.

**Example 2.7.** Let $R = \mathbb{Z}_4 \oplus \mathbb{Z}_4$. It is clear that $\max(R) = \{2\mathbb{Z}_4 \oplus \mathbb{Z}_4, \mathbb{Z}_4 \oplus 2\mathbb{Z}_4\}$ and $V(G(R)) = \{2\mathbb{Z}_4 \oplus \mathbb{Z}_4, \mathbb{Z}_4 \oplus 2\mathbb{Z}_4, 0 \oplus \mathbb{Z}_4, \mathbb{Z}_4 \oplus 0\}$. An inspection shows that $G(R)$ is not connected and $G(R) = G_1 \cup G_2$, where $G_1 = \{2\mathbb{Z}_4 \oplus \mathbb{Z}_4, 0 \oplus \mathbb{Z}_4\}$ and $G_2 = \{\mathbb{Z}_4 \oplus 2\mathbb{Z}_4, \mathbb{Z}_4 \oplus 0\}$.

**Theorem 2.8.** Let $R$ be a ring and $G(R)$ be a connected graph, then $\text{diam}(G(R)) \leq 2$.

**Proof:** Let $I$ and $J$ be two non-adjacent vertices of $G(R)$. So $I \cap J \ll R$. Let $I \subseteq M_1$ and $J \subseteq M_2$ for some maximal ideals $M_1, M_2$ of $R$. If $I \cap M_2 \ll R$, then $I - M_2 - J$ is a path in $G(R)$, hence $d(I, J) = 2$. By the similar way if $J \cap M_1 \ll R$, then $d(I, J) = 2$. Suppose $I \cap M_2 \ll R$ and $J \cap M_1 \ll R$. Since $G(R)$ is connected, $|\max(R)| \geq 3$ by Theorem 2.6. Let $M_3 \in \max(R)$. Since $I \cap J \ll R$, so $I \cap J \subseteq J(R) \subseteq M_3$ which implies $I \subseteq M_3$ or $J \subseteq M_3$.

Suppose, without loss of generality, $I \subseteq M_3$. Now, we show that $J \cap M_3 \ll R$. If $J \cap M_3 \ll R$, then $J \cap M_3 \subseteq J(R) \subseteq M_1$, which implies $J \subseteq M_1$. Hence $J = J \cap M_1 \ll R$, a contradiction. So $J \cap M_3 \ll R$. Thus $I - M_3 - J$ is a path in $G(R)$, so $d(I, J) = 2$.

**Theorem 2.9.** Let $R$ be a ring. If $G(R)$ contains a cycle, then $\text{gr}(G(R)) = 3$.

**Proof:** If $|\max(R)| = 2$, then $G(R)$ is a union of two disjoint complete subgraph by Theorem 2.6. Thus if $G(R)$ contains a cycle, then $\text{gr}(G(R)) = 3$. If $|\max(R)| \geq 3$, then by Remark 2.1, $M_1 - M_2 - M_3 - M_1$ is a cycle in $G(R)$, where $M_i \in \max(R)$. So $\text{gr}(G(R)) = 3$.

A vertex $x$ of a connected graph $G$ is a cut vertex of $G$ if there are vertices $y$ and $z$ of $G$ such that $x$ is in every path from $y$ to $z$ (and $x \neq y, x \neq z$). Equivalently, for a connected graph $G$, $x$ is a cut vertex of $G$ if $G - \{x\}$ is not connected.

**Theorem 2.10.** Let $R$ be a ring with $G(R)$ connected. Then $G(R)$ has no cut vertex.

**Proof:** Let $I$ be a cut vertex of $G(R)$, so $G(R) \setminus \{I\}$ is not connected. Thus there exist vertices $J, K$ such that $I$ lies on every path from $K$ to $J$. By Theorem 2.8, the shortest path from $I$ to $J$ is of length 2. So $J - I - K$ is a path between $J, K$. Hence $J \cap K \ll R, J \cap I \ll R$ and $K \cap I \ll R$. At first we show that $I$ is a maximal ideal of $R$. If not, there exists an ideal $L$ of $R$ such that $I \subset L$ (as $I$ is non-small ideal, $L$ is non-small). Since $J \cap I \subseteq J \cap L$ and $J \cap I \ll R, J \cap L \ll R$. By a similar way $K \cap L \ll R$. So $J - L - K$ is a path in $G(R) \setminus \{I\}$, a contradiction. So $I$ is a maximal ideal of $R$. We claim that there exists a maximal ideal $M_i \neq I$ of $R$ such that $J \subseteq M_i$. Otherwise, if $J \subseteq M_i$ for each
Example 2.13. Let $G$ be a graph such that $K \not\subseteq M_j$. Now, we show that for each $M_t \in \max(R)$, $K \subseteq M_t$ or $J \subseteq M_t$. Because $J \cap K \ll R$, so $J \cap K \subseteq J(R) \subseteq M_t$ for each $M_t \in \max(R)$. So $J \subseteq M_t$ or $K \subseteq M_t$ for each $M_t \in \max(R)$. Since $G(R)$ is connected, $|\max(R)| \geq 3$ by Theorem 2.6. Now, let $I \neq M_i$, $M_j \in \max(R)$ such that $K \not\subseteq M_j$ and $J \not\subseteq M_j$. So $K \subseteq M_j$ and $J \subseteq M_i$. Hence $J - M_i - M_j - K$ is a path in $G(R) \setminus \{I\}$, a contradiction. So $G(R)$ has no cut vertex. □

Theorem 2.11. Let $R$ be a ring. Then $G(R)$ cannot be a complete $r$-partite graph ($r \in \mathbb{N}$).

Proof: Let $G(R)$ be a complete $r$-partite graph with $r$ parts $V_1, V_2, \ldots, V_r$. By Remark 2.1, $M_i$ and $M_j$ are adjacent, for each $M_i, M_j \in \max(R)$. Hence each $V_i$ contains at most one maximal ideal of $R$. So by Pigeon hole principle $|\max(R)| \leq r$. Now, we show that $|\max(R)| = r$. Suppose, on the contrary, $\max(R) = \{M_1, M_2, \ldots, M_t\}$, where $t < r$. Let $M_i \in V_i$ for $1 \leq i \leq t$. So $V_{t+1}$ contains no maximal ideal. Since $|\max(R)|$ is finite, $\bigcap_{j \neq i} M_j \not\ll R$, by Remark 2.1. Since $\bigcap_{j \neq i} M_j \cap M_i = J(R) \ll R$, so $\bigcap_{j \neq i} M_j$ and $M_i$ are not adjacent. Hence $\bigcap_{j \neq i} M_j \in V_i$, because $M_i \in V_i$. Let $I$ be a vertex in $V_{t+1}$ and $I \subseteq M_k$ for some $M_k \in \max(R)$. So $I$ is adjacent to $M_k$. Since $G(R)$ is a complete $r$-partite graph and $M_k \in V_k$, so $I$ is adjacent to all elements of $V_k$. Thus $I$ is adjacent to $\bigcap_{j \neq k} M_j$, which is a contradiction, because $I \cap (\bigcap_{j \neq k} M_j) \subseteq M_k \cap (\bigcap_{j \neq k} M_j) = J(R) \ll R$. Hence $|\max(R)| = r$. Now, consider the ideal $J = \bigcap_{i=3}^{r} M_i$. By Remark 2.1, $J \not\ll R$. Since $J \cap M_1 = \bigcap_{i \neq 2} M_i \not\ll R$, $J$ is adjacent to $M_1$. By the similar way $J$ is adjacent to $M_2$. So $J \not\subseteq V_1, V_2$. Moreover, $J \cap M_i = J \not\ll R$, for each $3 \leq i \leq r$. So $J$ is adjacent to all maximal ideals $M_i$ of $R$. So $J \not\subseteq V_i$ for each $1 \leq i \leq r$, which is a contradiction. □

Theorem 2.12. Let $R$ be a ring with finitely many maximal ideals. Then

(i) there is no vertex in $G(R)$ which is adjacent to every other vertex;

(ii) $G(R)$ cannot be a complete graph.

Proof: (i) Let $\max(R) = \{M_1, M_2, \ldots, M_t\}$. Suppose, on the contrary, there exists a vertex $I$ in $G(R)$ such that $I$ is adjacent to every other vertex. Let $I \subseteq M_i$. By Remark 2.1, $K = \bigcap_{j \neq i} M_j$ is not a small ideal of $R$. Since $I$ is adjacent to every vertex, $I$ and $K$ are adjacent. Thus $I \cap K \not\ll R$. But $I \cap K \subseteq M_i \cap (\bigcap_{j \neq i} M_j) = J(R)$. So $I \cap K \ll R$, a contradiction. Thus there is no vertex in $G(R)$ which is adjacent to every other vertex.

(ii) By the similar argument as in (i), $G(R)$ cannot be a complete graph. □

The following example shows that the condition “$\max(R)$ is finite” in Theorem 2.12 is not superficial.

Example 2.13. Let $R = \mathbb{Z}$. It is clear that $\max(R)$ is infinite and the only small ideal of $R$ is $\{0\}$. Since for every non-zero ideals $I$ and $J$ of $R$, $I \cap J \neq \{0\}$, thus
I and J are adjacent in G(R). So G(R) is a complete graph and each vertex is adjacent to every other vertex.

**Theorem 2.14.** Let R be a ring. Then the following statements hold:

(i) G(R) contains an end vertex if and only if |max(R)| = 2 and G(R) = G₁ ∪ G₂, where G₁, G₂ are two disjoint complete subgraph of G(R) and |V(Gᵢ)| = 2 for some i = 1, 2;

(ii) G(R) cannot be a star graph.

**Proof:** (i) Let I be an end vertex of G(R). Suppose, |max(R)| ≥ 3. By Remark 2.1, for each Mᵢ ∈ max(R), Mᵢ is adjacent to every other maximal ideals of R, so deg(Mᵢ) ≥ 2. Hence I is not a maximal ideal of R. Without loss of generality, suppose I ⊆ M₁, hence I and M₁ are adjacent. Since deg(I) = 1, so the only vertex of G(R) which is adjacent to I is M₁ and there is no maximal ideal Mᵢ ≠ M₁ of R such that I ⊆ Mᵢ. Thus I ∩ M₂ ⊆ R. So I ∩ M₂ ⊆ Mⱼ for each Mⱼ ≠ M₁, M₂. Thus I ⊆ Mⱼ, which is a contradiction. So |max(R)| = 2. By Theorem 2.6, G(R) = G₁ ∪ G₂, where G₁, G₂ are complete subgraph of G(R). Let I ∈ Gᵢ. Since Gᵢ is a complete subgraph of G(R) and deg(I) = 1, |V(Gᵢ)| = 2. The converse is clear.

(ii) Let G(R) be a star graph. So G(R) contains an end vertex. So |max(R)| = 2 by (i). By Theorem 2.6, G(R) is not connected, which is a contradiction. So G(R) cannot be a star graph.

As we see in Example 2.7, G(R) contains an end vertex.

For every nonnegative integer r, the graph G is called r-regular if the degree of each vertex of G is equal to r.

**Theorem 2.15.** Let R be a ring. Then the following holds:

(i) if I and J are two vertices of G(R) such that I ⊆ J, then deg(I) ≤ deg(J);

(ii) if G(R) is an r-regular graph, then |max(R)| = 2 and |V(G(R))| = 2r + 2.

**Proof:** (i) Let I and J be two vertices of G(R) such that I ⊆ J. Let K be a vertex adjacent to I. So I ∩ K ≪ R, which implies J ∩ K ≪ R. Thus K is adjacent to J. Hence deg(I) ≤ deg(J).

(ii) Let G(R) be an r-regular graph. So for each Mᵢ ∈ max(R), deg(Mᵢ) = r. By Remark 2.1, Mᵢ is adjacent to all maximal ideals of R, hence max(R) is finite. Suppose |max(R)| ≥ 3. Then deg(M₁ ∩ M₂) ≤ deg(M₁) by (i) and deg(M₁ ∩ M₂) ≠ deg(M₁), because I = ∩ⱼ≠₂ Mⱼ is adjacent to M₁ but I is not adjacent to M₁ ∩ M₂ (note that I ≪ R by Remark 2.1). Thus deg(M₁ ∩ M₂) < r, a contradiction. So |max(R)| ≤ 2. If |max(R)| = 1, then R is a local ring hence G(R) is a null graph, which is a contradiction. So |max(R)| = 2 and G(R) is a union of two disjoint complete subgraph G₁, G₂ by Theorem 2.6. Let max(R) = {M₁, M₂} and Mᵢ ∈ Gᵢ. Since deg(Mᵢ) = r, so |Gᵢ| = r + 1. By the similar way |G₂| = r + 1. Hence |V(G(R))| = 2r + 2. □
3. Clique number, independence number, domination number and planar property

In this section, we will investigate clique number, independence number, domination number and planar property of the small graph. Now we start with the following proposition.

Proposition 3.1. Let \( R \) be a ring. The following statements hold.

(i) \( \omega(G(R)) \geq |\max(R)| \).

(ii) If \( \omega(G(R)) < \infty \), then the number of maximal ideals of \( R \) is finite.

(iii) \( \omega(G(R)) = 1 \) if and only if \( \max(R) = \{M_1, M_2\} \), where \( M_1 \) and \( M_2 \) are finitely generated hollow \( R \)-modules.

(iv) If the number of maximal ideals of \( R \) is finite, then \( \omega(G(R)) \geq 2^{\max(R) - 1} - 1 \).

Proof: (i) By Remark 2.1, the subgraph of \( G(R) \) with vertex set \( \{M_i\}_{M_i \in \max(R)} \) is a complete subgraph of \( G(R) \). Hence \( \omega(G(R)) \geq |\max(R)| \).

(ii) It is clear by (i).

(iii) It is clear by Theorem 2.4.

(iv) Let \( \max(R) = \{M_1, M_2, \ldots, M_t\} \) and for each \( 1 \leq i \leq t \), set \( A_i = \{M_1, M_2, \ldots, M_{i-1}, M_{i+1}, \ldots, M_t\} \). Let \( P(A_i) \) be the power set of \( A_i \). For each \( X \in P(A_i) \), set \( T_X = \bigcap_{T \in X} T \). Then by Remark 2.1, the subgraph of \( G(R) \) with vertex set \( \{T_X \}_{X \in P(A_i)} \) is a complete subgraph of \( G(R) \). Since \( |P(A_i) \setminus \{\} = 2^{\max(R) - 1} - 1 \), so \( |\{T_X \}_{X \in P(A_i)}| = 2^{\max(R) - 1} - 1 \). Hence \( \omega(G(R)) \geq 2^{\max(R) - 1} - 1 \).

For any ring \( R \), we use \( \mathbb{I}(R) \) and \( \mathbb{NSI}(R) \) to denote the set of ideals of \( R \) and the set of proper non-small ideals of \( R \), respectively.

We now state our next theorem, which gives us some information on the structure of the rings for which their small intersection graphs have finite clique number.

Theorem 3.2. Let \( R \) be a ring. If \( \omega(G(R)) < \infty \), then the following holds.

(i) \( R \) is semiperfect.

(ii) \( R = R_1 \times R_2 \times \cdots \times R_t \) where \( t \geq 2 \), \( (R_i, M_i) \) is a local ring and \( G(R) \) is finite.

(iii) \( \omega(G(R)) \geq \max\{1, j \neq i |(\prod_{j=1}^t |\mathbb{I}(R_i)|) - 1 : 1 \leq i \leq t\} \).

(iv) \( R \) is artinian.

Proof: (i) Let \( R \) be a ring such that \( \omega(G(R)) < \infty \). Then by Proposition 3.1, \( \max(R) \) is finite. Hence \( R/J(R) \) is semisimple. Now, we show that idempotent of \( R/J(R) \) can be lifted. Let \( a + J(R) \) be a nonzero idempotent of \( R/J(R) \). As \( a \notin J(R) \), \( a^n \notin J(R) \) for each \( n \in \mathbb{N} \). Hence \( Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \cdots \) is a descending chain of non-small proper ideals of \( R \) (if \( Ra^n = R \), then \( a + J(R) = 1 + J(R) \)). Since \( \omega(G(R)) < \infty \), there exists \( n \in \mathbb{N} \) such that \( Ra^n = Ra^{n+1} \). Hence \( a^n = a^{n+1}r \) for some \( r \in R \). Let \( e = a^r r^n \). Then \( e = (a^{n+1}r)r^n = a^{n+1}r^{n+1} \). This implies that \( e = e^2 \) and \( a + J(R) = a^n + J(R) = a^{n+1}r + J(R) = (a^{n+1} + J(R))(r + J(R)) = \).
Let \( R \) be a ring such that \( \text{NSI}(R) \neq \emptyset \). Then \( \text{NSI}(R) \) is finite if and only if \( \mathcal{I}(R) \) is finite.

**Proof:** Let \( \text{NSI}(R) \neq \emptyset \). Then \( G(R) \) is non-null. If \( \text{NSI}(R) \) is finite, then \( \omega(G(R)) \) is finite and so \( G(R) \) is finite, by Theorem 3.2. Hence \( \mathcal{I}(R) \) is finite. The converse is clear.

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930, that says that a graph is planar if and only if it contains no subdivision of \( K_5 \) or \( K_{3,3} \) [4]. In the following theorem, rings for which their small intersection graph is planar are characterized.

**Theorem 3.4.** Let \( R \) be a ring. Then \( G(R) \) is a planar graph if and only if one of the following cases occurs.

(i) \(|\max(R)| = 2 \) and \( R = R_1 \times R_2 \), where \( R_i \ (i = 1, 2) \) is a local principle ideal ring with maximal ideal \( M_i \) such that \( M_i^n = 0 \) for some \( n \leq 4 \).

(ii) \(|\max(R)| = 3 \) and \( R \) is semisimple.
Proof: Let \( G(R) \) be a planar graph. Then \( G(R) \) contains no \( K_5 \) as subgraph and so \( \omega(G(R)) \leq 4 \). By Remark 2.1, \( |\text{max}(R)| \leq 3 \). Since \( G(R) \) is not a null graph, \( |\text{max}(R)| \neq 1 \). So \( |\text{max}(R)| = 2 \) or 3.

By Theorem 3.2, \( R \) is a direct product of local ring. If \( |\text{max}(R)| = 2 \), then \( R = R_1 \times R_2 \) where \( R_i \) is a local ring with maximal ideal \( M_i \) (\( i = 1, 2 \)). Let \( \{x_1, x_2, \ldots, x_n\} \) be a minimal generating set for \( M_1 \). If \( n \geq 3 \), then

\[
\{0 \times R_2, x_1 R_1 \times R_2, x_2 R_1 \times R_2, x_3 R_1 \times R_2, M_1 \times R_2\}
\]

is a clique with five elements in \( G(R) \), a contradiction. Hence \( n \leq 2 \). Let \( M_1 = xR_1 + yR_1 \), where \( \{x, y\} \) is a minimal generating set for \( M_1 \). Then \( xR_1, yR_1 \) and \( x + y \) are distinct ideals of \( R_1 \) and

\[
\{0 \times R_2, xR_1 \times R_2, yR_1 \times R_2, (x + y)R_1 \times R_2, M_1 \times R_2\}
\]

is a clique with five elements in \( G(R) \), a contradiction. Hence \( M_1 \) is principle and so \( R_1 \) is a principle ideal ring. This implies that

\[
\mathbb{I}(R_1) = \{M_1^i : 1 \leq i \leq n\},
\]

where \( n \) is the smallest number such that \( M_1^n = 0 \) and \( n \leq 4 \).

Similarly, \( R_2 \) is a principle ideal ring and

\[
\mathbb{I}(R_2) = \{M_2^i : 1 \leq i \leq n\},
\]

where \( n \) is the smallest number such that \( M_2^n = 0 \) and \( n \leq 4 \). Hence (i) holds.

If \( |\text{max}(R)| = 3 \), then \( R = R_1 \times R_2 \times R_3 \) where \( R_i \) is a local ring with maximal ideal \( M_i \), for each \( 1 \leq i \leq 3 \). If \( R_1 \) is not a field, then \( M_1 \neq 0 \) and

\[
\{0 \times 0 \times R_3, 0 \times R_2 \times R_3, M_1 \times 0 \times R_3, M_1 \times R_2 \times R_3, R_1 \times 0 \times R_3\}
\]

is a clique with five elements in \( G(R) \), a contradiction. Hence \( R_1 \) is a field. Similarly, \( R_2 \) and \( R_3 \) are fields. Hence \( R \) is semisimple and (ii) holds.

Conversely, assume that (i) holds. Then \( G(R) = G_1 \cup G_2 \), where \( G_1, G_2 \) are two disjoint complete subgraphs of \( G(R) \) by Theorem 2.6. By (i), \( G_i \cong K_n \) (a complete graph with \( n \) vertices) where \( n \leq 4 \) for each \( i = 1, 2 \). Hence \( G(R) \) is planar. If (ii) holds, then by drawing \( G(R) \), it is clear that \( G(R) \) is planar. \( \square \)

In the following theorem, for a ring \( R \), the domination number of \( G(R) \) is determined.

**Theorem 3.5.** Let \( R \) be a ring. Then the following hold:

(i) \( \gamma(G(R)) \leq 2 \);

(ii) \( \max(R) \) is infinite if and only if \( \gamma(G(R)) = 1 \);

(iii) \( \max(R) \) is finite if and only if \( \gamma(G(R)) = 2 \).

**Proof:** (i) As \( G(R) \) is non-null, \( |\text{max}(R)| \geq 2 \). Set \( S = \{M_1, M_2\} \) where \( M_1, M_2 \in \text{max}(R) \). Let \( I \) be a vertex of \( G(R) \). If \( I \subseteq M_1 \) or \( I \subseteq M_2 \), then \( I \cap M_1 \not\subseteq R \) or \( I \cap M_2 \not\subseteq R \). Hence \( I \) is adjacent to \( M_1 \) or \( M_2 \). Assume
that $I \not\subseteq M_1$ and $I \not\subseteq M_2$. If $I$ is not adjacent to $M_1$, then $I \cap M_1 \ll R$. So $I \cap M_1 \preceq M_2$. This gives $I \subseteq M_2$, a contradiction. Similarly, $I$ is adjacent to $M_2$. Hence $\gamma(G(R)) \leq 2$.

(ii) If max($R$) is infinite, then $R/J(R)$ is not semisimple. Hence $R/J(R)$ has an essential ideal $I/J(R)$, where $I$ is an ideal of $R$. So $I$ is not small and for each ideal $K$ of $R$ with $J(R) \subset K$ we have $K \cap I \not\ll R$. Let $P$ be a proper non-small ideal of $R$. As $I \cap (P + J(R)) = J(R) + I \cap P \not\ll R$, $I \cap P \ll R$. Hence $I$ is adjacent to every other vertex of $G(R)$, and so $\gamma(G(R)) = 1$.

Conversely, assume that $\gamma(G(R)) = 1$. Hence there is an ideal which is adjacent to every other vertex of $G(R)$. So max($R$) is infinite by Theorem 2.12.

(iii) It is clear from Theorem 2.12 and (ii). □

In the following theorem, it is shown that the independence number of $G(R)$ is equal to $|\max(R)|$, for a ring $R$ with a finite number of maximal ideals.

**Theorem 3.6.** Let $R$ be a ring with a finite number of maximal ideals. Then $\alpha(G(R)) = |\max(R)|$.

**Proof:** Suppose that max($R$) is finite and max($R$) = \{M_1, M_2, \ldots, M_n\}. As \(\bigcap_{j=1, i \neq j}^{n} M_j\) is an independent set in $G(R)$, $n \leq \alpha(G(R))$. Let $\alpha(G(R)) = m$ and $S = \{I_1, I_2, \ldots, I_m\}$ be a maximal independent set in $G(R)$. For each $I \in S$, $I \not\ll R$. Hence $I \not\subseteq M$ for some $M \in \max(R)$. If $m > n$, then by Pigeon hole principle, there exist $1 \leq i, j \leq n$ and $M \in \max(R)$ such that $I_i \not\subseteq M$ and $I_j \not\subseteq M$. Hence $I_i \cap I_j \not\subseteq M$. As $S$ is an independent set in $G(R)$, $I_i$ and $I_j$ are not adjacent and $I_i \cap I_j \ll R$. Hence $I_i \cap I_j \not\subseteq M$, a contradiction. This proves that $\alpha(G(R)) = |\max(R)|$. If $\alpha(G(R)) = \infty$, then by a similar argument as above (Pigeon hole principle), we have a contradiction. Hence $\alpha(G(R)) = |\max(R)|$. □

In the ring $R = \mathbb{Z}$, it can be easily seen that $|\max(R)| = \infty$ and $\alpha(G(R)) = 0$. So the condition “$\max(R)$ is finite” is not superficial in Theorem 3.6.

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