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APPROXIMATION PROPERTIES FOR MODIFIED (p,q)-BERNSTEIN-DURRMEYER OPERATORS

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Abstract. We introduce modified (p,q)-Bernstein-Durrmeyer operators. We discuss approximation properties for these operators based on Korovkin type approximation theorem and compute the order of convergence using usual modulus of continuity. We also study the local approximation property of the sequence of positive linear operators $D^*_{n,p,q}$ and compute the rate of convergence for the function f belonging to the class $\mathrm{Lip}_M(\gamma)$.

Keywords: (p,q)-integer; (p,q)-Bernstein-Durrmeyer operator; q-Bernstein-Durrmeyer operator; modulus of continuity; positive linear operator; Korovkin type approximation theorem

MSC 2010: 41A10, 41A25, 41A36

1. Introduction and preliminaries

During the last two decades, the applications of q-calculus emerged as a new area in the field of approximation theory. The rapid development of q-calculus has led to the discovery of various generalizations of several polynomials involving q-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations. Using q-integers, Lupaş [13] introduced the first q-analogue of the classical Bernstein operators and investigated its approximating and shape preserving properties. Another q-generalization of the classical Bernstein polynomial is due to Phillips (see [21]). Several generalization of well known positive linear operators based on q-integers were introduced and their approximation properties have been studied by several researchers. Recently, Mursaleen et al. introduced (p,q)-calculus in approximation theory and constructed the (p,q)-analogue of Bernstein operators (see [17]) and (p,q)-analogue of Bernstein-Stancu operators

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(see [16]). Most recently, the (p,q)-analogue of some more operators has been studied in [4], [2], [3], [7], [14], [15], [18], [19], [20] and [24]. The (p,q)-integer was introduced in order to generalize or unify several forms of q-oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras (see [8]). Let us recall certain notations and definitions of (p,q)-calculus. Let $0 < q < p \leqslant 1$. For each nonnegative integer k, n, $n \geqslant k \geqslant 0$, the (p,q)-integer $[k]_{p,q}$, (p,q)-factorial $[k]_{p,q}!$ and (p,q)-binomial are defined by

$$[k]_{p,q} := \frac{p^k - q^k}{p-q},$$
$$[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q}\dots 1, & k \geqslant 1, \\ 1, & k = 0 \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}!\,[n-k]_{p,q}!}.$$

In case of $p = 1$, the above notations reduce to q-analogues and one can easily see that $[n]_{p,q} = p^{n-1}[n]_{q/p}$. Further, the (p,q)-power basis is defined as

$$(x \oplus a)^n_{p,q} := (x+a)(px+qa)(p^2x+q^2a)\dots(p^{n-1}x+q^{n-1}a)$$

and

$$(x \ominus a)^n_{p,q} := (x-a)(px-qa)(p^2x-q^2a)\dots(p^{n-1}x-q^{n-1}a).$$

Also the (p,q)-derivative of a function f, denoted by $D_{p,q}f$, is defined by

$$(D_{p,q}f)(x) := \frac{f(px)-f(qx)}{(p-q)x},\ x \neq 0, \quad (D_{p,q}f)(0) := f'(0),$$

provided that f is differentiable at 0. The formula for the (p,q)-derivative of a product is

$$D_{p,q}(u(x)v(x)) := D_{p,q}(u(x))v(qx) + D_{p,q}(v(x))u(qx).$$

For more details on (p,q)-calculus, we refer the readers to [11] and the references therein. Let f be an arbitrary function and $a \in \mathbb{R}$. The (p,q)-integral of f on $[0,a]$ is defined as:

$$\int_0^a f(t)\,\mathrm{d}_{p,q}t = (q-p)a\sum_{k=0}^{\infty} f\Big(\frac{p^k}{q^{k+1}}a\Big)\frac{p^k}{q^{k+1}} \quad \text{if } \Big|\frac{p}{q}\Big| < 1,$$
$$\int_0^a f(t)\,\mathrm{d}_{p,q}t = (p-q)a\sum_{k=0}^{\infty} f\Big(\frac{q^k}{p^{k+1}}a\Big)\frac{q^k}{p^{k+1}} \quad \text{if } \Big|\frac{q}{p}\Big| < 1.$$

For $0 < q < p \leqslant 1$ and $s, t \in \mathbb{R}^+$, (p,q)-beta integral is defined as:

$$\beta_{p,q}(t,s) = \int_0^1 x^{t-1}(1-qt)_{p,q}^{s-1}\,\mathrm{d}_{p,q}x. \tag{1.1}$$

In 2008, Gupta [10] introduced q-analogue of Durrmeyer operators as

$$D_{n,q}(f;x) = [n+1]_q \sum_{k=0}^{n} q^{-k} b_{n,k}^q(x) \int_0^1 b_{n,k}^q(q;qt) f(t)\,\mathrm{d}_q t, \tag{1.2}$$

where

$$b_{n,k}^q(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k}.$$

In 2015, Mursaleen et al. [17] introduced (p,q)-Bernstein operators and theirs variant. For $0 < q < p \leqslant 1$, $n \in \mathbb{N}$ and $f \in C[0,1]$, (p,q)-Bernstein operators are defined as

$$B_{n,p,q}(f;x) = p^{-n(n-1)/2} \sum_{k=0}^{n} b_{n,k}^{(p,q)}(x) f\Big(\frac{p^{-k}[k]_{p,q}}{[n]_{p,q}}\Big), \tag{1.3}$$

where $b_{n,k}^{(p,q)}(x)$ is a basis of (p,q)-Bernstein given as:

$$b_{n,k}^{(p,q)}(x) = p^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k (1-x)_{p,q}^{n-k}.$$

Bernstein polynomials, their Durrmeyer variants and Szász operators, which are a generalization of Bernstein polynomials, have been studied intensively by many researchers; for details one may refer to [1], [5], [22], [25].

In 2016, Sharma [23] introduced the (p,q)-Bernstein-Durrmeyer operators for $0 < q < p \leqslant 1$, $n \in \mathbb{N}$ and $f \in C[0,1]$ as:

$$D_{n,p,q}(f;x) = [n+1]_{p,q} p^{-n^2} \sum_{k=0}^{n} b_{n,k}^{(p,q)}(x) \Big(\frac{q}{p}\Big)^{-k} \int_0^1 b_{n,k}^{(p,q)}(qt) f(t)\,\mathrm{d}_{p,q} t. \tag{1.4}$$

2. Construction of operators and auxiliary results

We consider $0 < q < p \leqslant 1$ and for any $n \in \mathbb{N}$, $f \in C[0,1]$ we define the modified (p,q)-Bernstein-Durrmeyer operators for $x \in [0,1]$ as:

$$\text{(2.1)} \qquad D^*_{n,p,q}(f;x) = \frac{[n+1]^2_{p,q}}{[n]_{p,q}} p^{-n^2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x)\Big(\frac{q}{p}\Big)^{-k} \times \int_0^{[n]_{p,q}/[n+1]_{p,q}} b^*_{n,k}(p,q;qt) f(t)\, \mathrm{d}_{p,q}t,$$

where $r_n(p,q;x) = [n+1]_{p,q}/[n]_{p,q}x$. In case of $p = 1$ these operators turn out to be the modified q-Bernstein-Durrmeyer operators defined in [22] and if we replace $r_n(q;x)$ by x, we get (1.2). Moreover, if we take $r_n(p,q;x) = x$, we get (1.4).

For $0 < q < p \leqslant 1$, $n \in \mathbb{N}$ and $f \in C[0,1]$, modified (p,q)-Bernstein operators are defined as

$$\text{(2.2)} \qquad B^*_{n,p,q}(f;x) = p^{-n(n-1)/2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x) f\Big(\frac{p^{-k}[k]_{p,q}}{[n]_{p,q}}\Big),$$

where

$$b^*_{n,k}(p,q;x) = p^{k(k-1)/2} \frac{[n+1]^n_{p,q}}{[n]^n_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \Big(\frac{[n]_{p,q}}{[n+1]_{p,q}} - x\Big)^{n-k}_{p,q}$$
$$= p^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \Big(\frac{[n+1]_{p,q}}{[n]_{p,q}} x\Big)^k \Big(1 - \frac{[n+1]_{p,q}}{[n]_{p,q}} x\Big)^{n-k}_{p,q}.$$

Lemma 2.1. *For $0 < q < p \leqslant 1$ and $n \in \mathbb{N}$ we have*

(i) $\quad B^*_{n,p,q}(1;x) = 1,$

(ii) $\quad B^*_{n,p,q}(t;x) = p^{-n} \dfrac{[n+1]_{p,q}}{[n]_{p,q}} x,$

(iii) $\quad B^*_{n,p,q}(t^2;x) = p^{-n-1} \dfrac{[n+1]_{p,q}}{[n]^2_{p,q}} x + p^{-2n} q \dfrac{[n-1]_{p,q}[n+1]^2_{p,q}}{[n]^3_{p,q}} x^2.$

Lemma 2.2. *For $0 < q < p \leqslant 1$ and $s, t \in \mathbb{R}^+$ we have*

$$\beta_{p,q}(t,s) = p^{(s-1)(s-2)/2-(t-1)} \beta_{q/p}(t,s),$$

where $\beta_{q/p}(t,s)$ is q/p-analogue of beta function.

Lemma 2.3. *For $s = 0, 1, 2, 3, \dots$ we have*

$$\int_0^{[n]_{p,q}/[n+1]_{p,q}} b^*_{n,k}(p,q;qt)t^s \,\mathrm{d}_{p,q}t = \frac{[n]^{s+1}_{p,q}}{[n+1]^{s+1}_{p,q}} \Big(\frac{q}{p}\Big)^k p^{-sk} p^{n(n+2s+1)/2} \frac{[n]_{p,q}!\,[k+s]_{p,q}!}{[k]_{p,q}!\,[n+s+1]_{p,q}!}.$$

P r o o f. By Lemma (2.2), we have

$$\begin{aligned}
\int_0^{[n]_{p,q}/[n+1]_{p,q}} & b^*_{n,k}(p,q;qt)t^s \,\mathrm{d}_{p,q}t \\
&= p^{k(k-1)/2} \frac{[n+1]^n_{p,q}}{[n]^n_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^k \\
&\quad \times \int_0^{[n]_{p,q}/[n+1]_{p,q}} t^{k+s} \Big(\frac{[n]_{p,q}}{[n+1]_{p,q}} - qt\Big)^{n-k}_{p,q} \mathrm{d}_{p,q}t \\
&= p^{k(k-1)/2} \frac{[n]^{s+1}_{p,q}}{[n+1]^{s+1}_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^k \int_0^1 t^{k+s}(1-qt)^{n-k}_{p,q} \,\mathrm{d}_{p,q}t \\
&= p^{k(k-1)/2} \frac{[n]^{s+1}_{p,q}}{[n+1]^{s+1}_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^k \beta_{p,q}(s+k+1, n-k+1) \\
&= p^{k(k-1)/2} \frac{[n]^{s+1}_{p,q}}{[n+1]^{s+1}_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^k p^{(n-k)(n-k-1)/2-(s+k)} \\
&\quad \times \beta_{q/p}(s+k+1, n-k+1).
\end{aligned}$$

Using $\beta_q(t+1, s+1) = [t]_{p,q}!\,[s]_{p,q}!/[s+t+1]_{p,q}$ and $[n]_{p,q}! = p^{n(n-1)/2}[n]_{q/p}$, we get

$$\begin{aligned}
\int_0^{[n]_{p,q}/[n+1]_{p,q}} & b^*_{n,k}(p,q;qt)t^s \,\mathrm{d}_{p,q}t \\
&= p^{k(k-1)/2} \frac{[n]^{s+1}_{p,q}}{[n+1]^{s+1}_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \\
&\quad \times q^k p^{(n-k)(n-k-1)/2-(s+k)} \frac{[s+k]_{q/p}!\,[n-k]_{q/p}!}{[n+s+1]_{q/p}!} \\
&= p^{k(k-1)/2} \frac{[n]^{s+1}_{p,q}}{[n+1]^{s+1}_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \\
&\quad \times q^k p^{(n+s)(n+s+1)/2-(s+k)(s+k+1)/2} \frac{[s+k]_{p,q}!\,[n-k]_{p,q}!}{[n+s+1]_{p,q}!} \\
&= \frac{[n]^{s+1}_{p,q}}{[n+1]^{s+1}_{p,q}} \Big(\frac{q}{p}\Big)^k p^{-sk} p^{n(n+2s+1)/2} \frac{[n]_{p,q}!\,[k+s]_{p,q}!}{[k]_{p,q}!\,[n+s+1]_{p,q}!}.
\end{aligned}$$

□

Lemma 2.4. *For $0 < q < p \leqslant 1$ and $n \in \mathbb{N}$ we have*

(i) $$D^*_{n,p,q}(1;x) = 1,$$

(ii) $$D^*_{n,p,q}(t;x) = \frac{[n]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}}(p^n + q[n+1]_{p,q}x),$$

(iii) $$D^*_{n,p,q}(t^2;x) = \frac{p^{2n}(p+q)[n]^2_{p,q} + p^{n-1}q(p+q)^2[n+1]_{p,q}[n]^2_{p,q}x}{[n+1]^2_{p,q}[n+2]_{p,q}[n+3]_{p,q}} + \frac{q^4[n+1]^2_{p,q}[n]_{p,q}[n-1]_{p,q}x^2}{[n+1]^2_{p,q}[n+2]_{p,q}[n+3]_{p,q}}$$

Proof. By Lemma (2.3), we have

(2.3) $$\int_0^{[n]_{p,q}/[n+1]_{p,q}} b^*_{n,k}(p,q;qt)\,\mathrm{d}_{p,q}t = \frac{[n]_{p,q}}{[n+1]_{p,q}}\Big(\frac{q}{p}\Big)^k p^{n(n+1)/2}\frac{1}{[n+1]_{p,q}},$$

(2.4) $$\int_0^{[n]_{p,q}/[n+1]_{p,q}} b^*_{n,k}(p,q;qt)t\,\mathrm{d}_{p,q}t = \frac{[n]^2_{p,q}}{[n+1]^2_{p,q}}\Big(\frac{q}{p}\Big)^k p^{-k}p^{n(n+3)/2}\frac{[k+1]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}},$$

(2.5) $$\int_0^{[n]_{p,q}/[n+1]_{p,q}} b^*_{n,k}(p,q;qt)t^2\,\mathrm{d}_{p,q}t = \frac{[n]^3_{p,q}}{[n+1]^3_{p,q}}\Big(\frac{q}{p}\Big)^k p^{-2k}p^{n(n+5)/2}\frac{[k+1]_{p,q}[k+2]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}}.$$

(i) For $0 < q < p \leqslant 1$ we use the known identity from [17]:

$$\sum_{k=0}^{n} p^{k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}(r_n(p,q;x))^k(1-r_n(p,q;x))^{n-k}_{p,q} = p^{n(n-1)/2}.$$

Using equality (2.3) and Lemma (2.1) (i), we get

$$D^*_{n,p,q}(1;x) = \frac{[n+1]^2_{p,q}}{[n]_{p,q}}p^{-n^2} \times \sum_{k=0}^{n} b^*_{n,k}(p,q;x)\Big(\frac{q}{p}\Big)^{-k}\frac{[n]_{p,q}}{[n+1]_{p,q}}\Big(\frac{q}{p}\Big)^k p^{n(n+1)/2}\frac{1}{[n+1]_{p,q}},$$

Consequently, this implies $D^*_{n,p,q}(1;x) = 1$.

(ii) Using equality (2.4), Lemma (2.1) (ii) and $[k+1]_{p,q} = p^k + q[k]_{p,q}$, we get

$$
\begin{aligned}
D^*_{n,p,q}(t;x) &= \frac{[n+1]^2_{p,q}}{[n]_{p,q}} p^{-n^2} \\
&\quad \times \sum_{k=0}^{n} b^*_{n,k}(p,q;x)\Big(\frac{q}{p}\Big)^{-k} \frac{[n]^2_{p,q}}{[n+1]^2_{p,q}} \Big(\frac{q}{p}\Big)^{k} p^{-k} p^{n(n+3)/2} \frac{[k+1]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} \\
&= \frac{[n]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} p^{-n^2} p^{n(n+3)/2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x) p^{-k}(p^k + q[k]_{p,q}) \\
&= \frac{[n]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} p^{(-n^2+3n)/2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x) \\
&\quad + \frac{[n]^2_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} p^{(-n^2+3n)/2} q \sum_{k=0}^{n} b^*_{n,k}(p,q;x) \frac{p^{-k}[k]_{p,q}}{[n]_{p,q}} \\
&= \frac{[n]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} p^n B^*_{n,p,q}(1;x) + \frac{[n]^2_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} p^n q B^*_{n,p,q}(t;x) \\
&= \frac{[n]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} p^n + \frac{[n]^2_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} p^n q \Big(p^{-n} \frac{[n+1]_{p,q}}{[n]_{p,q}} x\Big) \\
&= \frac{[n]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} (p^n + q[n+1]_{p,q} x).
\end{aligned}
$$

(iii) Using equality (2.5), Lemma (2.1) (iii), we have

$$
\begin{aligned}
&D^*_{n,p,q}(t^2;x) \\
&\quad = \frac{[n+1]^2_{p,q}}{[n]_{p,q}} p^{-n^2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x)\Big(\frac{q}{p}\Big)^{-k} \frac{[n]^3_{p,q}}{[n+1]^3_{p,q}} \\
&\qquad \times \Big(\frac{q}{p}\Big)^{k} p^{-2k} p^{n(n+5)/2} \frac{[k+1]_{p,q}[k+2]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}} \\
&\quad = \frac{[n]^2_{p,q}}{[n+1]^2_{p,q}[n+2]_{p,q}[n+3]_{p,q}} p^{(-n^2+5n)/2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x) p^{-2k} \\
&\qquad \times (p^k + q[k]_{p,q})((p+q)p^k + q^2[k]_{p,q}) \\
&\quad = \frac{(p+q)[n]^2_{p,q}}{[n+1]^2_{p,q}[n+2]_{p,q}[n+3]_{p,q}} p^{(-n^2+5n)/2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x) \\
&\qquad + \frac{q(p+2q)[n]^3_{p,q}}{[n+1]^2_{p,q}[n+2]_{p,q}[n+3]_{p,q}} p^{(-n^2+5n)/2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x) \frac{p^{-k}[k]_{p,q}}{[n]_{p,q}} \\
&\qquad + \frac{q^3[n]^4_{p,q}}{[n+1]^2_{p,q}[n+2]_{p,q}[n+3]_{p,q}} p^{(-n^2+5n)/2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x) \Big(\frac{p^{-k}[k]_{p,q}}{[n]_{p,q}}\Big)^2
\end{aligned}
$$

$$= \frac{p^{2n}((p+q)[n]_{p,q}^2 B_{n,p,q}^*(1;x) + q(p+2q)[n]_{p,q}^3 B_{n,p,q}^*(t;x))}{[n+1]_{p,q}^2[n+2]_{p,q}[n+3]_{p,q}}$$
$$+ \frac{p^{2n}q^3[n]_{p,q}^4 B_{n,p,q}^*(t^2;x)}{[n+1]_{p,q}^2[n+2]_{p,q}[n+3]_{p,q}}$$
$$= \frac{p^{2n}(p+q)[n]_{p,q}^2 + p^n q(p+2q)[n+1]_{p,q}[n]_{p,q}^2 x}{[n+1]_{p,q}^2[n+2]_{p,q}[n+3]_{p,q}}$$
$$+ \frac{p^{n-1}q^3[n+1]_{p,q}[n]_{p,q}^2 x + q^4[n+1]_{p,q}^2[n]_{p,q}[n-1]_{p,q}x^2}{[n+1]_{p,q}^2[n+2]_{p,q}[n+3]_{p,q}}$$
$$= \frac{p^{2n}(p+q)[n]_{p,q}^2 + p^{n-1}q(p+q)^2[n+1]_{p,q}[n]_{p,q}^2 x}{[n+1]_{p,q}^2[n+2]_{p,q}[n+3]_{p,q}}$$
$$+ \frac{q^4[n+1]_{p,q}^2[n]_{p,q}[n-1]_{p,q}x^2}{[n+1]_{p,q}^2[n+2]_{p,q}[n+3]_{p,q}}.$$

□

Lemma 2.5. *For $0 < q < p \leqslant 1$ and $n \in \mathbb{N}$ we have*

(i) $D_{n,p,q}^*(t-x;x) = \dfrac{[n]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}}(p^n + q[n+1]_{p,q}x) - x,$

(ii)
$$D_{n,p,q}^*((t-x)^2;x) = x^2\Big(1 + \frac{q^4[n]_{p,q}[n-1]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} - \frac{2q[n]_{p,q}}{[n+2]_{p,q}}\Big)$$
$$+ x\Big(\frac{p^{n-1}q(p+q)^2[n]_{p,q}^2}{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}} - \frac{2p^n[n]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}}\Big)$$
$$+ \frac{p^{2n}(p+q)[n]_{p,q}^2}{[n+1]_{p,q}^2[n+2]_{p,q}[n+3]_{p,q}}.$$

R e m a r k 2.1. For $q \in (0,1)$, $p \in (q,1]$ we easily see that $\lim_{n\to\infty} [n]_{p,q} = 1/(p-q)$. In order to study convergence properties of the operators, we take $q_n \in (0,1)$, $p_n \in (q_n, 1]$ such that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$, so $\lim_{n\to\infty} 1/[n]_{p_n,q_n} = 0$. Such a sequence can always be constructed; for example, we can take $q_n = 1 - n^{-1}$ and $p_n = 1 - \frac{1}{2}n^{-1}$, clearly $\lim_{n\to\infty} p_n^n = \mathrm{e}^{-1/2}$, $\lim_{n\to\infty} q_n^n = \mathrm{e}^{-1}$ and $\lim_{n\to\infty} 1/[n]_{p_n,q_n} = 0$.

3. Korovkin type theorem

Let $C[0,1]$ be the linear space of all real valued continuous functions f on $[0,1]$ and let T be a linear operator defined on $C[0,1]$. We say that T is positive if for every nonnegative $f \in C[0,1]$ we have $T(f,x) \geqslant 0$ for all $x \in [0,1]$. Classical Korovkin's approximation theorem (see [6], [12]) states as follows:

Let (T_n) be a sequence of positive linear operators from $C[0,1]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_{C[0,1]} = 0$ for all $f \in C[0,1]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_{C[0,1]} = 0$ for $i = 0,1,2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

Theorem 3.1. *Let $0 < q_n < p_n \leqslant 1$ such that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$. Then for each $f \in C[0,1]$, $D^*_{n,p_n,q_n}(f;x)$ converges uniformly to f on $C[0,1]$.*

Proof. By Korovkin's theorem it is sufficient to show that

$$\lim_{n\to\infty} \|D^*_{n,p_n,q_n}(t^m;x) - x^m\|_{C[0,1]} = 0, \quad m = 0,1,2.$$

By Lemma (2.4) (i) it is clear that

$$\lim_{n\to\infty} \|D^*_{n,p_n,q_n}(1;x) - 1\|_{C[0,1]} = 0.$$

Now, by Lemma (2.4) (ii)

$$\begin{aligned}|D^*_{n,p_n,q_n}(t;x) - x|_{C[0,1]} &= \Big|\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}[n+2]_{p_n,q_n}}(p_n^n + q_n[n+1]_{p_n,q_n}x) - x\Big| \\ &\leqslant \Big|\frac{p_n^n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}[n+2]_{p_n,q_n}} + \Big(\frac{[n]_{p_n,q_n}}{[n+2]_{p_n,q_n}} - 1\Big)x\Big|.\end{aligned}$$

Taking supremum on both sides of the above inequality, we get

$$\|D^*_{n,p_n,q_n}(t;x) - x\|_{C[0,1]} \leqslant \Big|\frac{p_n^n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}[n+2]_{p_n,q_n}}\Big| + \Big|\frac{[n]_{p_n,q_n}}{[n+2]_{p_n,q_n}} - 1\Big|$$

$$\begin{aligned}&\lim_{n\to\infty} \|D^*_{n,p_n,q_n}(t;x) - x\|_{C[0,1]} \\ &\qquad \leqslant \lim_{n\to\infty}\Big|\frac{p_n^n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}[n+2]_{p_n,q_n}}\Big| + \lim_{n\to\infty}\Big|\frac{[n]_{p_n,q_n}}{[n+2]_{p_n,q_n}} - 1\Big|,\end{aligned}$$

which yields

$$\lim_{n\to\infty} \|D^*_{n,p_n,q_n}(t;x) - x\|_{C[0,1]} = 0.$$

Finally, using Lemma (2.4) (iii), we get

$$\begin{aligned}\lim_{n\to\infty} \|D^*_{n,p_n,q_n}(t^2;x) - x^2\|_{C[0,1]} &\leqslant \lim_{n\to\infty}\Big|\frac{p_n^{2n}(p_n+q_n)[n]^2_{p_n,q_n}}{[n+1]^2_{p_n,q_n}[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}\Big| \\ &\quad + \lim_{n\to\infty}\Big|\frac{p_n^{n-1}q_n(p_n+q_n)^2[n]^2_{p_n,q_n}}{[n+1]_{p_n,q_n}[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}\Big| \\ &\quad + \lim_{n\to\infty}\Big|\frac{q_n^4[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}} - 1\Big|,\end{aligned}$$

$$\lim_{n\to\infty} \|D^*_{n,p_n,q_n}(t;x) - x\|_{C[0,1]} = 0.$$

The proof is now complete. □

4. Direct theorems

Now we will compute the rate of convergence in terms of modulus of continuity. The modulus of continuity of $f \in C[0,1]$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by

$$\omega(f,\delta) = \sup_{|x-y|\leqslant\delta;\, x,y\in[0,1]} |f(x) - f(y)|.$$

The modulus of continuity possesses the following property:

$$\omega(f,\lambda\delta) \leqslant (1+\lambda)\omega(f,\delta). \tag{4.6}$$

Further, let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g\in W^2}\{\|f-g\| + \delta\|g''\|\},$$

where $\delta > 0$ and $W^2 = \{g \in C[0,1]\colon\ g', g'' \in C[0,1]\}$. By [17], Theorem 2.4, there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leqslant C w_2(f,\sqrt{\delta}),$$

where

$$w_2(f,\sqrt{\delta}) = \sup_{0<h\leqslant\sqrt{\delta}}\ \sup_{x\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C[0,1]$.

Theorem 4.1. *Let $0 < q_n < p_n \leqslant 1$ such that $\lim\limits_{n\to\infty} p_n = 1$, $\lim\limits_{n\to\infty} q_n = 1$ and $\lim\limits_{n\to\infty} [n]_{p_n,q_n} = \infty$. Then for each $f \in C[0,1]$,*

$$|D^*_{n,p_n,q_n}(f;x) - f(x)| \leqslant 2\omega(f,\sqrt{\delta_n(x)}),$$

where $\delta_n(x) = D^*_{n,p_n,q_n}((t-x)^2;x)$.

Proof.

$$|D^*_{n,p_n,q_n}(f;x) - f(x)| \leqslant D^*_{n,p_n,q_n}(|f(t)-f(x)|;x).$$

Also, in view of equation (4.6)

$$|f(t)-f(x)| \leqslant \Big(1 + \frac{(t-x)^2}{\delta^2}\Big)\omega(f,\delta). \tag{4.7}$$

Using equations (2.1) and (4.7), we get

$$
\begin{aligned}
|D^*_{n,p_n,q_n}(f;x)-f(x)| &\leqslant \frac{[n+1]^2_{p_n,q_n}}{[n]_{p_n,q_n}} p_n^{-n^2} \sum_{k=0}^{n} b^*_{n,k}(p_n,q_n;x)\Big(\frac{q_n}{p_n}\Big)^{-k} \\
&\quad \times \int_0^{[n]_{p_n,q_n}/[n+1]_{p_n,q_n}} |f(t)-f(x)| b^*_{n,k}(p_n,q_n;qt)\,\mathrm{d}_{p,q}t \\
&= \Big(D^*_{n,p_n,q_n}(1;x)+\frac{1}{\delta^2}D^*_{n,p_n,q_n}((t-x)^2;x)\Big)\omega(f,\delta).
\end{aligned}
$$

But by Lemma (2.5),

$$
\begin{aligned}
D^*_{n,p,q}((t-x)^2;x) &= x^2\Big(1+\frac{q^4[n]_{p,q}[n-1]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}-\frac{2q[n]_{p,q}}{[n+2]_{p,q}}\Big) \\
&\quad + x\Big(\frac{p^{n-1}q(p+q)^2[n]^2_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}}-\frac{2p^n[n]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}}\Big) \\
&\quad + \frac{p^{2n}(p+q)[n]^2_{p,q}}{[n+1]^2_{p,q}[n+2]_{p,q}[n+3]_{p,q}}.
\end{aligned}
$$

Therefore we get

$$
\lim_{n\to\infty} D^*_{n,p,q}((t-x)^2;x)=0
$$

because $[n]_{p_n,q_n}\to\infty$ as $p_n\to 1$, $q_n\to 1$. So, letting $\delta_n(x)=D^*_{n,p_n,q_n}((t-x)^2;x)$ and taking $\delta(x)=\sqrt{\delta_n(x)}$, we finally get

$$
|D^*_{n,p_n,q_n}(f;x)-f(x)| \leqslant 2\omega(f,\sqrt{\delta_n(x)}).
$$

□

Theorem 4.2. *Let $f\in C[0,1]$ and $0<q<p\leqslant 1$ such that $\lim_{n\to\infty}[n]_{p,q}=\infty$. Then for all $n\in\mathbb{N}$ there exists an absolute constant $C>0$ such that*

$$
|D^*_{n,p,q}(f;x)-f(x)| \leqslant C\omega_2(f,\delta_n(x))+\omega(f,\alpha_n(x)),
$$

where

$$
\delta_n(x)=\sqrt{D^*_{n,p,q}((t-x)^2;x)+(\alpha_n(x))^2}, \quad \alpha_n(x)=\frac{[n]_{p,q}(p^n+q[n+1]_{p,q}x)}{[n+1]_{p,q}[n+2]_{p,q}}-x.
$$

Proof. For $x \in [0,1]$ we consider the auxiliary operators $\overline{D}_n^*$ defined by

$$\overline{D}_n^*(f;x) = D_{n,p,q}^*(f;x) + f(x) - f\Big(\frac{[n]_{p,q}(p^n + q[n+1]_{p,q}x)}{[n+1]_{p,q}[n+2]_{p,q}}\Big).$$

From Lemma (2.4) (i), (ii) and Lemma (2.5) (i), we observe that the operators $\overline{D}_n^*(f;x)$ are linear and reproduce the linear functions. Hence,

$$\begin{aligned}
&\overline{D}_n^*(1;x) = D_{n,p,q}^*(1;x) + 1 - 1 = 1,\\
&\overline{D}_n^*(t;x) = D_{n,p,q}^*(t;x) + x - \Big(\frac{[n]_{p,q}(p^n + q[n+1]_{p,q}x)}{[n+1]_{p,q}[n+2]_{p,q}}\Big) = x,\\
&\overline{D}_n^*((t-x);x) = \overline{D}_n^*(t;x) - x\overline{D}_n^*(1;x) = 0.
\end{aligned}$$

Let $x \in [0,1]$ and $g \in W^2$. Using the Taylor's formula,

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,\mathrm{d}u.$$

Applying $\overline{D}_n^*$ to both sides of the above equation, we have

$$\begin{aligned}
&\overline{D}_n^*(g;x) - g(x)\\
&\quad = g'(x)\overline{D}_n^*((t-x);x) + \overline{D}_n^*\Big(\int_x^t (t-u)g''(u)\,\mathrm{d}u;x\Big)\\
&\quad = D_{n,p,q}^*\Big(\int_x^t (t-u)g''(u)\,\mathrm{d}u;x\Big)\\
&\qquad - \int_x^{[n]_{p,q}(p^n+q[n+1]_{p,q}x)/[n+1]_{p,q}[n+2]_{p,q}} \Big(\frac{[n]_{p,q}(p^n + q[n+1]_{p,q}x)}{[n+1]_{p,q}[n+2]_{p,q}} - u\Big)g''(u)\,\mathrm{d}u.
\end{aligned}$$

On the other hand,

$$\Big|\int_x^t (t-u)g''(u)\,\mathrm{d}u\Big| \leqslant \int_x^t |t-u||g''(u)|\,\mathrm{d}u \leqslant \|g''\| \int_x^t |t-u|\,\mathrm{d}u \leqslant (t-x)^2\|g''\|$$

and

$$\begin{aligned}
&\Big|\int_x^{[n]_{p,q}(p^n+q[n+1]_{p,q}x)/[n+1]_{p,q}[n+2]_{p,q}} \Big(\frac{[n]_{p,q}(p^n + q[n+1]_{p,q}x)}{[n+1]_{p,q}[n+2]_{p,q}} - u\Big)g''(u)\,\mathrm{d}u\Big|\\
&\qquad \leqslant \Big(\frac{[n]_{p,q}(p^n + q[n+1]_{p,q}x)}{[n+1]_{p,q}[n+2]_{p,q}} - x\Big)^2\|g''\|.
\end{aligned}$$

We conclude that

$$
\begin{aligned}
&|\overline{D}_n^*(g;x)-g(x)|\\
&\quad\leqslant \bigg|D_{n,p,q}^*\Big(\int_x^t (t-u)g''(u)\,\mathrm{d}u;x\Big)\\
&\qquad -\int_x^{[n]_{p,q}(p^n+q[n+1]_{p,q}x)/[n+1]_{p,q}[n+2]_{p,q}}\Big(\frac{[n]_{p,q}(p^n+q[n+1]_{p,q}x)}{[n+1]_{p,q}[n+2]_{p,q}}-u\Big)g''(u)\,\mathrm{d}u\bigg|\\
&\quad\leqslant \|g''\|D_{n,p,q}^*((t-x)^2;x)+\|g''\|\Big(\frac{[n]_{p,q}(p^n+q[n+1]_{p,q}x)}{[n+1]_{p,q}[n+2]_{p,q}}-x\Big)^2=\|g''\|\delta_n^2(x).
\end{aligned}
$$

Now, taking into account boundedness of $\overline{D}_n^*$, we have

$$|\overline{D}_n^*(f;x)|\leqslant |D_{n,p,q}^*(f;x)|+2\|f\|\leqslant 3\|f\|.$$

Therefore

$$
\begin{aligned}
&|D_{n,p,q}^*(f;x)-f(x)|\\
&\quad\leqslant |\overline{D}_n^*(f-g;x)-(f-g)(x)|+\Big|f\Big(\frac{[n]_{p,q}(p^n+q[n+1]_{p,q}x)}{[n+1]_{p,q}[n+2]_{p,q}}\Big)-f(x)\Big|\\
&\qquad+|\overline{D}_n^*(g;x)-g(x)|\\
&\quad\leqslant |\overline{D}_n^*(f-g;x)|+|(f-g)(x)|+\Big|f\Big(\frac{[n]_{p,q}(p^n+q[n+1]_{p,q}x)}{[n+1]_{p,q}[n+2]_{p,q}}\Big)-f(x)\Big|\\
&\qquad+|\overline{D}_n^*(g;x)-g(x)|\\
&\quad\leqslant 4\|f-g\|+\omega(f,\alpha_n(x))+\delta_n^2(x)\|g''\|.
\end{aligned}
$$

Hence, taking the infimum on the right-hand side over all $g\in W^2$, we have the following result:

$$|D_{n,p,q}^*(f;x)-f(x)|\leqslant 4K_2(f,\delta_n^2(x))+\omega(f,\alpha_n(x)).$$

In view of the property of K-functional, we get

$$|D_{n,p,q}^*(f;x)-f(x)|\leqslant C\omega_2(f,\delta_n(x))+\omega(f,\alpha_n(x)).$$

This completes the proof of the theorem. □

Now we give the rate of convergence of the operators $D_{n,p,q}^*(f;x)$ in terms of the elements of the usual Lipschitz class $\mathrm{Lip}_M(\gamma)$.

Let $f\in C[0,1]$, $M>0$ and $0<\gamma\leqslant 1$. We recall that f belongs to the class $\mathrm{Lip}_M(\gamma)$ if

$$|f(t)-f(x)|\leqslant M|t-x|^\gamma,\quad t,x\in(0,1].$$

Theorem 4.3. *Let $0 < q < p \leqslant 1$ such that $\lim\limits_{n\to\infty} [n]_{p,q} = \infty$. Then for each $f \in \mathrm{Lip}_M(\gamma)$ we have*

$$|D^*_{n,p,q}(f;x) - f(x)| \leqslant M\delta_n^{\gamma}(x),$$

where $\delta_n(x) = \sqrt{D^*_{n,p,q}((t-x)^2;x)}$.

P r o o f. By the monotonicity of the operators $D^*_{n,p,q}(f;x)$, we can write

$$\begin{aligned}
&|D^*_{n,p,q}(f;x) - f(x)| \leqslant D^*_{n,p,q}(|f(t)-f(x)|;x)\\
&\quad\leqslant \frac{[n+1]^2_{p,q}}{[n]_{p,q}} p^{-n^2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x)\Big(\frac{q}{p}\Big)^{-k}\\
&\qquad\times \int_0^{[n]_{p,q}/[n+1]_{p,q}} |f(t)-f(x)| b^*_{n,k}(p,q;qt)\,\mathrm{d}_{p,q}t\\
&\quad\leqslant M\frac{[n+1]^2_{p,q}}{[n]_{p,q}} p^{-n^2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x)\Big(\frac{q}{p}\Big)^{-k} \int_0^{[n]_{p,q}/[n+1]_{p,q}} |t-x|^{\gamma} b^*_{n,k}(p,q;qt)\,\mathrm{d}_{p,q}t.
\end{aligned}$$

Now, applying the Hölder's inequality and taking into consideration Lemma (2.4) (i) and Lemma (2.5) (ii), we have

$$\begin{aligned}
&|D^*_{n,p,q}(f;x) - f(x)|\\
&\leqslant M\sum_{k=0}^{n}\Big\{\frac{[n+1]^2_{p,q}}{[n]_{p,q}} p^{-n^2} b^*_{n,k}(p,q;x)\Big(\frac{q}{p}\Big)^{-k}\int_0^{[n]_{p,q}/[n+1]_{p,q}} (t-x)^2 b^*_{n,k}(p,q;qt)\,\mathrm{d}_{p,q}t\Big\}^{\gamma/2}\\
&\quad\times\Big\{\frac{[n+1]^2_{p,q}}{[n]_{p,q}} p^{-n^2} b^*_{n,k}(p,q;x)\Big(\frac{q}{p}\Big)^{-k}\int_0^{[n]_{p,q}/[n+1]_{p,q}} b^*_{n,k}(p,q;qt)\,\mathrm{d}_{p,q}t\Big\}^{(2-\gamma)/2}\\
&\leqslant M\Big\{\frac{[n+1]^2_{p,q}}{[n]_{p,q}} p^{-n^2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x)\Big(\frac{q}{p}\Big)^{-k}\int_0^{[n]_{p,q}/[n+1]_{p,q}} (t-x)^2 b^*_{n,k}(p,q;qt)\,\mathrm{d}_{p,q}t\Big\}^{\gamma/2}\\
&\quad\times\Big\{\frac{[n+1]^2_{p,q}}{[n]_{p,q}} p^{-n^2} \sum_{k=0}^{n} b^*_{n,k}(p,q;x)\Big(\frac{q}{p}\Big)^{-k}\int_0^{[n]_{p,q}/[n+1]_{p,q}} b^*_{n,k}(p,q;qt)\,\mathrm{d}_{p,q}t\Big\}^{(2-\gamma)/2}\\
&= M\{D^*_{n,p,q}((t-x)^2;x)\}^{\gamma/2}.
\end{aligned}$$

Choosing $\delta^2(x) = \delta_n^2(x) = D^*_{n,p,q}((t-x)^2;x)$, we arrive at our desired result. $\square$

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