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ON KNESER SOLUTIONS OF THE n-TH ORDER NONLINEAR DIFFERENTIAL INCLUSIONS

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Abstract. The paper deals with the existence of a Kneser solution of the n-th order nonlinear differential inclusion

$$x^{(n)}(t) \in -A_1(t, x(t), \ldots, x^{(n-1)}(t))x^{(n-1)}(t) - \ldots - A_n(t, x(t), \ldots, x^{(n-1)}(t))x(t) \quad \text{for a.a. } t \in [a, \infty),$$

where $a \in (0, \infty)$, and $A_i\colon [a, \infty) \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$, are upper-Carathéodory mappings. The derived result is finally illustrated by the third order Kneser problem.

Keywords: asymptotic n-th order vector problems; R_δ-set; inverse limit technique; Kneser problem

MSC 2010: 34A60, 34B15, 34B40

1. INTRODUCTION

The problem of the existence of Kneser solutions has been widely studied since the 1800's when the pioneering work about monotone solutions for the second-order differential equations on the half-line was published by Kneser [22]. The Kneser-type results were afterwards followed e.g. by Thomas [28], Fermi [13] who investigated the distribution of electrons in heavy atoms, and by many others (cf. e.g. [9], [18], [20], [21], [23] and the references quoted therein). The Kneser-type problems belong to boundary value problems on infinite intervals that appear in many practical problems, for example in linear elasticity, nonlinear fluid flow, and foundation engineering (see e.g. [1], [17] and the references therein).

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The Kneser-type problems have been studied during last 120 years in detail, from the recent papers dealing with this topic, let us mention e.g. [10], [26], [27]. In the mentioned publications various generalizations of classical results have been obtained, including delay differential equations or differential equations with regularly varying coefficients. In the present paper, one of the first attempts of studying Kneser-type problems for differential inclusions is presented.

The stimulation for studying Kneser-type problems for differential inclusions comes e.g. from asymptotic control problems

$$
\begin{gathered}
x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t), u(t)), \quad t \in [a, \infty),\ u \in U, \\
x(a) = c_0,\ (-1)^i x^{(i)}(t) \geqslant 0 \quad \forall\, i = 0, \ldots, n-1,\ \text{and}\ t \in [a, \infty),
\end{gathered}
$$

where $u = u(t) \in U$ are control parameters. Defining the multivalued mapping

$$
F(t, x_1, \ldots, x_n) := \{f(t, x_1, \ldots, x_n, u)\}_{u \in U},
$$

the solutions of the original problem coincide with those of

$$
\begin{gathered}
x^{(n)}(t) \in F(t, x(t), \ldots, x^{(n-1)}(t)), \quad t \in [a, \infty),\ u \in U, \\
x(a) = c_0,\ (-1)^i x^{(i)}(t) \geqslant 0 \quad \forall\, i = 0, \ldots, n-1,\ \text{and}\ t \in [a, \infty).
\end{gathered}
$$

The n-th order differential inclusions (and their associated boundary value problems) are also generated by the single-valued problems with discontinuous right-hand side (cf. e.g. [14]). Such problems also arise when dealing with functions satisfying a differential equation to within required accuracy, i.e. when

$$
\|x^{(n)}(t) - f(t, x(t), \ldots, x^{(n-1)}(t))\| \leqslant \varepsilon,
$$

or when solving problems including differential inequalities.

The paper is organized as follows. First, the basic properties of multivalued mappings and the continuation principle for the n-th order asymptotic boundary value problems developed in [6] are recalled. The principle is afterwards applied in order to obtain the existence of a solution of the n-th order nonlinear Kneser-type problem with multivalued r.h.s.

$$
\begin{cases}
x^{(n)}(t) \in -A_1(t, x(t), \ldots, x^{(n-1)}(t))x^{(n-1)}(t) - \ldots \\
\qquad -A_n(t, x(t), \ldots, x^{(n-1)}(t))x(t) \quad \text{for a.a. } t \in [a, \infty), \\
x(a) = c_0, \\
(-1)^i x^{(i)}(t) \geqslant 0 \quad \forall\, i = 0, \ldots, n-1,\ \text{and}\ t \in [a, \infty),
\end{cases}
$$

where $a \in (0, \infty)$, and $A_i\colon [a, \infty) \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$, are upper-Carathéodory mappings. Finally, the obtained result is illustrated by the third order Kneser problem.

2. Preliminaries

First, let us recall some geometric notions of subsets of metric spaces; in particular, of compact absolute retracts, compact contractible sets and R_δ-sets. For more details, see, e.g., [4], [11], [16].

For a subset $A \subset X$ of a metric space $X = (X, d)$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X\colon \exists a \in A\colon d(x, a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X. A subset $A \subset X$ is called a *retract* of X if there exists a retraction $r\colon X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$.

We say that a metric space X is an *absolute retract* (AR-space) if, for each metric space Y and every closed $A \subset Y$, each continuous mapping $f\colon A \to X$ is extendable over Y. Let us note that X is an AR-space if and only if it is a retract of some normed space. Moreover, if X is a retract of a convex set in a Fréchet space, then it is an AR-space, too. So, in particular, for an arbitrary interval $J \subset \mathbb{R}$ and $k, n \in \mathbb{N}$, the spaces $C(J, \mathbb{R}^k)$, $C^n(J, \mathbb{R}^k)$, $AC^n_{\mathrm{loc}}(J, \mathbb{R}^k)$ are AR-spaces as well as their convex subsets. The foregoing symbols denote, as usual, the spaces of functions $f\colon J \to \mathbb{R}^k$ which are continuous, have continuous n-th derivatives, and locally absolutely continuous n-th derivatives, respectively, endowed with the respective topologies.

We say that a nonempty subset A of a metric space X is *contractible* if there exist a point $x_0 \in A$ and a homotopy $h\colon A \times [0, 1] \to A$ such that $h(x, 0) = x$ and $h(x, 1) = x_0$ for every $x \in A$. A nonempty set $A \subset X$ is called an *R_δ-set* if there exists a decreasing sequence $\{A_n\}_{n=1}^\infty$ of compact AR-spaces (or, despite of the hierarchy (2.1) below, compact, contractible sets) such that

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Note that any R_δ-set is nonempty, compact and connected. The following hierarchy holds for nonempty compact subsets of a metric space:

$$\begin{aligned} \text{compact+convex} \subset \text{compact } AR\text{-space} \subset \text{compact+contractible} \subset R_\delta\text{-set} \\ \subset \text{compact+acyclic} \subset \text{compact+connected}, \end{aligned} \tag{2.1}$$

and all the above inclusions are proper.

We also employ the following definitions and statements from the multivalued analysis in the sequel. Let X and Y be arbitrary metric spaces. We say that F is a *multivalued mapping* from X to Y (written $F\colon X \multimap Y$) if for every $x \in X$, a nonempty subset $F(x)$ of Y is prescribed. We associate with F its *graph* Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x, y) \in X \times Y\colon\ y \in F(x)\}.$$

If $X \cap Y \neq \emptyset$ and $F\colon X \multimap Y$, then a point $x \in X \cap Y$ is called a *fixed point* of F if $x \in F(x)$. The set of all fixed points of F will be denoted by $\mathrm{Fix}(F)$, i.e.

$$\mathrm{Fix}(F) := \{x \in X\colon\ x \in F(x)\}.$$

A multivalued mapping $F\colon X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if for each open $U \subset Y$, the set $\{x \in X\colon\ F(x) \subset U\}$ is open in X. Every upper semicontinuous map with closed values has a closed graph.

Let Y be a separable metric space and $(\Omega, \mathcal{U}, \nu)$ a *measurable space,* i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure ν on $\mathcal{U}$. A multivalued mapping $F\colon \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega\colon\ F(\omega) \subset V\} \in \mathcal{U}$ for each open set $V \subset Y$.

We say that a mapping $F\colon J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$, is *upper-Carathéodory* if the map $F(\cdot, x)\colon J \multimap \mathbb{R}^n$ is measurable on every compact subinterval of J for all $x \in \mathbb{R}^m$, the map $F(t, \cdot)\colon \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c. for almost all $t \in J$, and the set $F(t, x)$ is compact and convex for all $(t, x) \in J \times \mathbb{R}^m$.

In the sequel, we will employ the following selection statement and the subsequent convergence result.

Lemma 2.1 (cf., e.g., [7])**.** *Let $F\colon [a, b] \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping satisfying $|y| \leqslant r(t)(1 + |x|)$ for every $(t, x) \in [a, b] \times \mathbb{R}^m$, and every $y \in F(t, x)$, where $r\colon [a, b] \to [0, \infty)$ is an integrable function. Then the composition $F(t, q(t))$ admits for every $q \in C([a, b], \mathbb{R}^m)$, a single-valued measurable selection.*

Lemma 2.2 (cf. [8], Theorem 0.3.4)**.** *Let $[a, b] \subset \mathbb{R}$ be a compact interval. Assume that the sequence of absolutely continuous functions $x_k\colon [a, b] \to \mathbb{R}^n$ satisfies the following conditions:*

(i) *the set $\{x_k(t)\colon\ k \in \mathbb{N}\}$ is bounded for every $t \in [a, b]$,*
(ii) *there exists a function $\alpha\colon [a, b] \to \mathbb{R}$, integrable in the sense of Lebesgue, such that*

$$|\dot{x}_k(t)| \leqslant \alpha(t) \quad \text{for a.a. } t \in [a, b] \text{ and for all } k \in \mathbb{N}.$$

Then there exists a subsequence of $\{x_k\}$ (for the sake of simplicity, denoted in the same way as the sequence) converging to an absolutely continuous function $x\colon [a,b] \to \mathbb{R}^n$ in the following way:

1. *$\{x_k\}$ converges uniformly to x,*
2. *$\{\dot{x}_k\}$ converges weakly in $L^1([a,b],\mathbb{R}^n)$ to $\dot{x}$.*

The following lemma is a slight modification of the well known result.

Lemma 2.3 (cf. [29], page 88)**.** *Let $[a,b] \subset \mathbb{R}$ be a compact interval, let E_1, E_2 be Euclidean spaces and $F\colon [a,b] \times E_1 \multimap E_2$ an upper-Carathéodory mapping.*

Assume in addition that, for every nonempty, bounded set $\mathcal{B} \subset E_1$, there exists $\nu = \nu(\mathcal{B}) \in L^1([a,b],[0,\infty))$ such that

$$|F(t,x)| \leqslant \nu(t)$$

for a.a. $t \in [a,b]$ and every $x \in \mathcal{B}$.

Let us define the Nemytskiĭ operator $N_F\colon C([a,b],E_1) \multimap L^1([a,b],E_2)$ in the following way:

$$N_F(x) := \{f \in L^1([a,b],E_2)\colon\ f(t) \in F(t,x(t)),\ \text{a.e. on } [a,b]\}$$

for every $x \in C([a,b],E_1)$. Then, if sequences $\{x_i\} \subset C([a,b],E_1)$ and $\{f_i\} \subset L^1([a,b],E_2)$, $f_i \in N_F(x_i)$, $i \in \mathbb{N}$, are such that $x_i \to x$ in $C([a,b],E_1)$ and $f_i \to f$ weakly in $L^1([a,b],E_2)$, then $f \in N_F(x)$.

In the sequel, the following special case of the continuation principle, developed recently in [6], Theorem 3.1 and Corollary 4.2 will be employed (especially, for $n = 2$, cf. [5]).

Proposition 2.1. *Let us consider the b.v.p.*

$$\begin{cases} x^{(n)}(t) \in C(t,x(t),\dots,x^{(n-1)}(t)) & \text{for a.a. } t \in J, \\ x \in S, \end{cases} \tag{2.2}$$

where J is a given (possibly noncompact) interval, $C\colon J \times \mathbb{R}^{kn} \multimap \mathbb{R}^k$ is an upper-Carathéodory mapping and $S \subset AC^{n-1}_{\mathrm{loc}}(J,\mathbb{R}^k)$.

Moreover, let $H\colon J \times \mathbb{R}^{2kn} \multimap \mathbb{R}^k$ be an upper-Carathéodory map such that

$$H(t,c_1,\dots,c_n,c_1,\dots,c_n) \subset C(t,c_1,\dots,c_n) \quad \text{for all } (t,c_1,\dots,c_n) \in J \times \mathbb{R}^{kn}. \tag{2.3}$$

Assume that

(i) *there exists a retract Q of $C^{n-1}(J,\mathbb{R}^k)$ such that the associated problem*

$$\tag{2.4} \begin{cases} x^{(n)}(t) \in H(t,x(t),\dots,x^{(n-1)}(t),q(t),\dots,q^{(n-1)}(t)) & \text{for a.a. } t\in J,\\ x\in S\cap Q \end{cases}$$

is solvable with an R_δ-set of solutions for each $q\in Q$,

(ii) *there exists a non-negative, locally integrable function $\alpha\colon J\to\mathbb{R}$ such that*

$$|H(t,x(t),\dots,x^{(n-1)}(t),q(t),\dots,q^{(n-1)}(t))| \leqslant \alpha(t)(1+|x(t)|+\ldots+|x^{(n-1)}(t)|),$$

a.e. in J for any $(q,x)\in\Gamma_{\mathfrak{T}}$, where $\mathfrak{T}$ denotes the multivalued map which assigns to any $q\in Q$ the set of solutions of (2.4),

(iii) $\mathfrak{T}(Q)\subset Q$,

(iv) $\mathfrak{T}(Q)$ *is bounded in $C(J,\mathbb{R}^k)$.*

Then problem (2.2) *admits a solution in $S\cap Q$.*

One of the efficient methods which can be used for studying b.v.p.s on noncompact intervals is an *inverse limit method*. Let us recall that by the *inverse system*, we mean a family $\mathcal{S}=\{X_\alpha,\pi_\alpha^\beta,\Sigma\}$, where Σ is a set directed by the relation $\leqslant$, X_α is, for all $\alpha\in\Sigma$, a metric space and $\pi_\alpha^\beta\colon X_\beta\to X_\alpha$ is a continuous function for all $\alpha,\beta\in\Sigma$ such that $\alpha\leqslant\beta$. Moreover, $\pi_\alpha^\alpha=\mathrm{id}_{X_\alpha}$ and $\pi_\alpha^\beta\pi_\beta^\gamma=\pi_\alpha^\gamma$ for all $\alpha\leqslant\beta\leqslant\gamma$. The *limit of the inverse system* $\mathcal{S}$ is denoted by $\varprojlim\mathcal{S}$ and is defined by

$$\varprojlim\mathcal{S} := \{(x_\alpha)\in\Pi_{\alpha\in\Sigma}X_\alpha \pi_\alpha^\beta(x_\beta)=x_\alpha \quad \forall\,\alpha\leqslant\beta\}.$$

If we denote by $\pi_\alpha\colon \varprojlim\mathcal{S}\to X_\alpha$ the restriction of the projection $p_\alpha\colon \Pi_{\alpha\in\Sigma}X_\alpha\to X_\alpha$ onto the α-th axis, then $\pi_\alpha=\pi_\alpha^\beta\pi_\beta$ for all $\alpha\leqslant\beta$.

Let us now consider two inverse systems $\mathcal{S}=\{X_\alpha,\pi_\alpha^\beta,\Sigma\}$ and $\mathcal{S}'=\{Y_{\alpha'},\pi_{\alpha'}^{\beta'},\Sigma'\}$. By a *multivalued mapping of the system $\mathcal{S}$ into the system $\mathcal{S}'$*, we mean a family $\{\sigma,\varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma\colon\Sigma'\to\Sigma$ and multivalued mappings $\varphi_{\sigma(\alpha')}\colon X_{\sigma(\alpha')}\multimap Y_{\alpha'}$ such that for all $\alpha'\leqslant\beta'$,

$$\pi_{\alpha'}^{\beta'}\varphi_{\sigma(\beta')}=\varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')}^{\sigma(\beta')}.$$

The mapping $\{\sigma,\varphi_{\sigma(\alpha')}\}$ induces a *limit mapping* $\varphi\colon\varprojlim\mathcal{S}\multimap\varprojlim\mathcal{S}'$ satisfying for all $\alpha'\in\Sigma'$,

$$\pi_{\alpha'}\varphi=\varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')}.$$

We will make use of the following result.

Proposition 2.2 (cf. [2], [3], [15]). *Let $\mathcal{S} = \{X_m, \pi_m^p, \mathbb{N}\}$ and $\mathcal{S}' = \{Y_m, \pi_m^p, \mathbb{N}\}$ be two inverse systems such that $X_m \subset Y_m$. If $\varphi\colon \varprojlim \mathcal{S} \multimap \varprojlim \mathcal{S}'$ is a limit map induced by a mapping $\{\mathrm{id}, \varphi_m\}$, where $\varphi_m\colon X_m \multimap Y_m$, and if $\mathrm{Fix}(\varphi_m)$ are for all $m \in \mathbb{N}$, R_δ-sets, then the fixed point set $\mathrm{Fix}(\varphi)$ of φ is an R_δ-set, too.*

For more details about the inverse limit method, see, e.g., [2], [3], [4], [15], [19], [25].

3. Kneser solutions

Let us consider the n-th order nonlinear (Kneser-type) multivalued b.v.p.

$$\text{(3.1)}\qquad \begin{cases} x^{(n)}(t) \in -A_1(t, x(t), \ldots, x^{(n-1)}(t))x^{(n-1)}(t) - \ldots \\ \qquad -A_n(t, \ldots, x^{(n-1)}(t))x(t) \quad \text{for a.a. } t \in [a, \infty), \\ x(a) = c_0, \\ (-1)^i x^{(i)}(t) \geqslant 0 \quad \forall\, i = 0, \ldots, n-1, \text{ and } t \in [a, \infty), \end{cases}$$

where

(i) $a \in (0, \infty)$,
(ii) $A_i\colon [a, \infty) \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$, are upper-Carathéodory mappings with

$$|A_i(t, x_1, x_2, \ldots, x_n)| \leqslant \beta(t)(1 + |x_1|)$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $t \in [a, \infty)$, where $\beta \in L^1_{\mathrm{loc}}([a, \infty), \mathbb{R})$,
(iii) $0 \notin A_n(t, x_1, x_2, \ldots, x_n)$ for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and for t in a right neighbourhood of a.

Moreover, let there exist $r > 0$ such that

(iv)

$$\text{(3.2)}\qquad c_0 \in \Big(0, \Big(\frac{\delta}{a + \delta}\Big)^{n-1} \frac{r}{2n!}\Big),$$

where $\delta \in (0, 1/(a + 1))$ is so small that

$$\text{(3.3)}\qquad 2(a + 1)^{n-1} \int_a^{a+\delta} f^*(\tau)\, \mathrm{d}\tau \leqslant r$$

with f^* defined by

$$\text{(3.4)}\qquad \begin{aligned} f^*(t) := \max\{|-A_1(t, x_1, \ldots, x_n)x_n - \ldots - A_n(t, x_1, \ldots, x_n)x_1|\colon \\ 0 \leqslant (-1)^{i-1} x_i \leqslant r t^{1-i},\ i = 1, \ldots, n\} \end{aligned}$$

Theorem 3.1. *Let us consider the n-th order Kneser-type b.v.p.* (3.1) *and let the conditions* (i)–(iv) *be satisfied. Moreover, let for all $q \in Q$, where*

$$Q := \{x \in C^{n-1}([a,\infty),\mathbb{R})\colon\ x(a) = c_0,\ (-1)^i x^{(i)}(t) \geqslant 0,\ i = 0,\dots,n-1,\ t \in [a,\infty)\},$$

the following condition hold:

(v)

$$(-1)^n(-a_1(t,q(t),\dots,q^{(n-1)}(t))x_n - \dots - a_n(t,q(t),\dots,q^{(n-1)}(t))x_1) \geqslant 0$$

for all $t \geqslant a$, all measurable selections a_i of A_i, $i = 1,\dots,n$, and all x_i satisfying

$$0 \leqslant (-1)^{i-1}x_i \leqslant rt^{1-i}, \quad i = 1,\dots,n.$$

Then the b.v.p. (3.1) *has a solution in Q.*

P r o o f. Let us still consider the associated problems

$$(\mathrm{P}_q) \qquad \begin{cases} x^{(n)}(t) \in -A_1(t,q(t),\dots,q^{(n-1)}(t))x^{(n-1)}(t) - \dots \\ \qquad -A_n(t,\dots,q^{(n-1)}(t))x(t) \quad \text{for a.a. } t \in [a,\infty), \\ x(a) = c_0, \\ (-1)^i x^{(i)}(t) \geqslant 0 \quad \text{for all } i = 0,\dots,n-1, \text{ and } t \in [a,\infty) \end{cases}$$

and let us verify that the b.v.p. (P_q) satisfies for all $q \in Q$, all assumptions of Proposition 2.1.

ad (i) First, let us show that the b.v.p. (P_q) has for each $q \in Q$, an R_δ-set of solutions.

For this purpose, let us consider, together with the b.v.p.s (P_q), the family of associated problems on compact intervals

$$(\mathrm{P}_q^m) \qquad \begin{cases} x^{(n)}(t) \in -A_1(t,q(t),\dots,q^{(n-1)}(t))x^{(n-1)}(t) - \dots \\ \qquad -A_n(t,\dots,q^{(n-1)}(t))x(t) \quad \text{for a.a. } t \in [a,m], \\ x(a) = c_0 \\ (-1)^i x^{(i)}(t) \geqslant 0 \quad \text{for all } t \in [a,m],\ i = 0,\dots,n-1, \end{cases}$$

where $m \in \mathbb{N}$, $m > a$. Let us first study problems (P_q^m)-more concretely, let us show that the set of solutions of problem (P_q^m) for an arbitrary $q \in Q$, $m \in \mathbb{N}$, is a nonempty, compact and convex, i.e. in particular an R_δ-set.

Let $v_i(\cdot)$ be a measurable selection of $A_i(\cdot,q(\cdot),\dots,q^{(n-1)}(\cdot))$, $i = 1,\dots,n$. It was shown in [12] (see Lemma 2.1 in [12] and the remarks below) that, under the above assumptions imposed on A_i, the following two norms in $AC^{n-1}([a,m],\mathbb{R})$, where $m > a$ is arbitrary, are equivalent:

$$\|x\| := \sup_{t\in[a,m]} |x(t)| + \sup_{t\in[a,m]} |\dot{x}(t)| + \ldots + \sup_{t\in[a,m]} |x^{(n-1)}(t)| + \int_a^m |x^{(n)}(t)|\,\mathrm{d}t,$$

$$\|x\|_* := \sup_{t\in[a,m]} |x(t)| + \int_a^m |x^{(n)}(t) + v_n(t)x(t) + v_{n-1}(t)\dot{x}(t) + \ldots + v_1(t)x^{(n-1)}(t)|\,\mathrm{d}t.$$

If $x(\cdot)$ is a solution of the b.v.p. (P_q^m) for some $q \in Q$, $m \in \mathbb{N}$, $m > a$, then

$$\|x\|_* = c_0.$$

Since $\sup\limits_{t\in[a,m]} |x^{(i)}(t)| \leqslant \|x\|$, $i = 1,\ldots,n-1$, and the norms $\|x\|_*$ and $\|x\|$ are equivalent, there exists $M > 0$ such that $\sup\limits_{t\in[a,m]} |x^{(i)}(t)| \leqslant Mc_0$, $i = 1,\ldots,n-1$.

Moreover, the sets

$$\begin{aligned} S_m := \{&(x,\dot{x},\ddot{x},\ldots,x^{(n-1)}) \in AC^{n-1}([a,m],\mathbb{R}) \times \ldots \times AC([a,m],\mathbb{R}),\\ &x(0) = c_0,\ (-1)^i x^{(i)}(t) \geqslant 0 \quad \forall\, t \in [a,m],\ i = 0,\ldots,n-1\},\\ S := \{&(x,\dot{x},\ddot{x},\ldots,x^{(n-1)}) \in AC_{\mathrm{loc}}^{(n-1)}([a,\infty),\mathbb{R}) \times \ldots \times AC_{\mathrm{loc}}([a,\infty),\mathbb{R}),\\ &x(0) = c_0,\ (-1)^i x^{(i)}(t) \geqslant 0 \quad \forall\, t \in [a,\infty),\ i = 0,\ldots,n-1\} \end{aligned}$$

are closed and convex.

Let us prove now that the set of solutions and their derivatives of the b.v.p. (P_q^m) is convex and compact.

Let $q \in Q$ be arbitrary and let us denote

$$\begin{aligned} &P(t,x(t),\ldots,x^{(n-1)}(t)) :=\\ &\quad -A_1(t,q(t),\ldots,q^{(n-1)}(t))x^{(n-1)}(t) - \ldots - A_n(t,q(t),\ldots,q^{(n-1)}(t))x(t). \end{aligned}$$

If x_1, x_2 are solutions of problem (P_q^m), then it follows from the integral representation of a solution that for a.a. $t \in [a,m]$, we have

$$\begin{aligned} x_1(t) \in x_1(a) &+ \dot{x}_1(a)(t-a) + \frac{1}{2}\ddot{x}_1(a)(t-a)^2 + \ldots + \frac{1}{(n-1)!}x_1^{(n-1)}(a)(t-a)^{n-1}\\ &+ \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} P(s,x_1(s),\dot{x}_1(s),\ldots,x_1^{(n-1)}(s))\,\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} x_2(t) \in x_2(a) &+ \dot{x}_2(a)(t-a) + \frac{1}{2}\ddot{x}_2(a)(t-a)^2 + \ldots + \frac{1}{(n-1)!}x_2^{(n-1)}(a)(t-a)^{n-1}\\ &+ \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} P(s,x_2(s),\dot{x}_2(s),\ldots,x_2^{(n-1)}(s))\,\mathrm{d}s. \end{aligned}$$

Let $\theta \in [0,1]$ be arbitrary. Then

$$\begin{aligned}
\theta x_1(t) + (1-\theta)x_2(t) &\in \theta x_1(a) + (1-\theta)x_2(a) + [\theta \dot{x}_1(a) + (1-\theta)\dot{x}_2(a)](t-a) + \dots \\
&\quad + \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}\theta \cdot P(s, x_1(s), \dot{x}_1(s), \dots, x_1^{(n-1)}(s))\,\mathrm{d}s \\
&\quad + \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}(1-\theta)P(s, x_2(s), \dot{x}_2(s), \dots, x_2^{(n-1)}(s))\,\mathrm{d}s \\
&= \theta x_1(a) + (1-\theta)x_2(a) + [\theta \dot{x}_1(a) + (1-\theta)\dot{x}_2(a)](t-a) + \dots \\
&\quad + \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}P(s, \theta x_1(s) + (1-\theta)x_2(s), \dots, \theta x_1^{(n-1)}(s) \\
&\quad + (1-\theta)x_2^{(n-1)}(s))\,\mathrm{d}s.
\end{aligned}$$

Moreover, for all $k = 1, \dots, n-1$,

$$\begin{aligned}
x_1^{(k)}(t) \in x_1^{(k)}(a) + x_1^{(k+1)}(a)(t-a) + \dots + \frac{1}{(n-1-k)!}x_1^{(n-1-k)}(a)(t-a)^{n-1-k} \\
+ \frac{1}{(n-1-k)!}\int_a^t (t-s)^{n-1-k}P(s, x_1(s), \dot{x}_1(s), \dots, x_1^{(n-1)}(s))\,\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
x_2^{(k)}(t) \in x_2^{(k)}(a) + x_2^{(k+1)}(a)(t-a) + \dots + \frac{1}{(n-1-k)!}x_2^{(n-1-k)}(a)(t-a)^{n-1-k} \\
+ \frac{1}{(n-1-k)!}\int_a^t (t-s)^{n-1-k}P(s, x_2(s), \dot{x}_2(s), \dots, x_2^{(n-1)}(s))\,\mathrm{d}s.
\end{aligned}$$

By similar arguments as before, we can obtain for an arbitrary $\theta \in [0,1]$ and all $k = 1, \dots, n-1$, that

$$\begin{aligned}
&\theta x_1^{(k)}(t) + (1-\theta)x_2^{(k)}(t) \in \theta x_1^{(k)}(a) \\
&\quad + (1-\theta)x_2^{(k)}(a) + [\theta x_1^{(k+1)}(a) + (1-\theta)x_2^{(k)}(a)](t-a) + \dots + \frac{1}{(n-1-k)!} \\
&\quad \times \int_a^t (t-s)^{n-1-k}P\big(s, \theta x_1(s) + (1-\theta)x_2(s), \dots, \theta x_1^{(n-1)}(s) + (1-\theta)x_2^{(n-1)}(s)\big)\,\mathrm{d}s.
\end{aligned}$$

Finally, because of convexity of S_m, we obtain that

$$(\theta x_1 + (1-\theta)x_2,\ \theta \dot{x}_1 + (1-\theta)\dot{x}_2, \dots, \theta x_1^{(n-1)} + (1-\theta)x_2^{(n-1)}) \in S_m$$

and, therefore, the set of solutions of (P_q^m) and their derivatives is convex.

Let us also prove that the set of solutions of (P_q^m) and their derivatives is relatively compact. It follows from the well known Arzelà-Ascoli lemma that the set of solutions is relatively compact in $C^{n-1}([a,m],\mathbb{R})$ if and only if it is bounded and all solutions and their derivatives (up to the $(n-1)$-st order) are equi-continuous.

First, let us show that the set of solutions of (P_q^m) is bounded in $C^{n-1}([a,m],\mathbb{R})$. Let x be a solution of (P_q^m) and let $t\in[a,m]$ be arbitrary.

Since

$$
\begin{aligned}
x^{(n-1)}(t) &= x^{(n-1)}(a) + \int_a^t x^{(n)}(s)\,\mathrm{d}s \quad \text{for a.a. } t\in[a,m],\\
&\ \ \vdots\\
\dot{x}(t) &= \dot{x}(a) + \int_a^t \ddot{x}(s)\,\mathrm{d}s \quad \text{for a.a. } t\in[a,m],\\
x(t) &= x(a) + \int_a^t \dot{x}(s)\,\mathrm{d}s \quad \text{for a.a. } t\in[a,m],
\end{aligned}
$$

it holds that

$$
\begin{aligned}
|x(t)| + |\dot{x}(t)| + \ldots + |x^{(n-1)}(t)| &\leqslant |x(a)| + |\dot{x}(a)| + \ldots + |x^{(n-1)}(a)|\\
&\quad + \int_a^t |\dot{x}(s)| + |\ddot{x}(s)| + \ldots + |x^{(n)}(s)|\,\mathrm{d}s\\
&\leqslant c_0(1+M(n-1)) + \int_a^m |\dot{x}(s)| + |\ddot{x}(s)| + \ldots + |x^{(n-1)}(s)|\\
&\quad + \beta(s)(1+c_0)|x^{(n-1)}(s)| + \ldots + \beta(s)(1+c_0)|x(s)|\,\mathrm{d}s\\
&\leqslant c_0(1+M(n-1)) + \int_a^m \beta(s)(1+c_0)|x(s)| + (1+\beta(s)(1+c_0))|\dot{x}(s)| + \ldots\\
&\quad + (1+\beta(s)(1+c_0))|x^{(n-1)}(s)|\,\mathrm{d}s\\
&\leqslant c_0(1+M(n-1)) + \int_a^m k(s)(|x(s)| + |\dot{x}(s)| + \ldots + |x^{(n-1)}(s)|)\,\mathrm{d}s
\end{aligned}
$$

where for all $s\in[a,m]$, $k(s) := 1+\beta(s)(1+c_0)$. Therefore, by Gronwall's lemma (cf. [24]),

$$
|x(t)| + |\dot{x}(t)| + \ldots + |x^{(n-1)}(t)| \leqslant c_0(1+M(n-1))\exp\left(\int_a^m k(s)\,\mathrm{d}s\right) \quad \text{for a.a. } t\in[a,m]. \tag{3.5}
$$

Therefore, the set of solutions of (P_q^m) and their derivatives (up to the $(n-1)$-st order) is bounded in $C^{n-1}([a,m],\mathbb{R})$.

Let us now show that all solutions x of (P_q^m) and their derivatives $\dot{x},\ldots,x^{(n-1)}$ are also equi-continuous. So, let x be a solution of (P_q^m) and $t_1,t_2\in[a,m]$ be arbitrary.

Then we have

$$
\begin{aligned}
(3.6)\qquad & |x(t_1)-x(t_2)| \\
& \leqslant \left|\int_{t_1}^{t_2} |\dot{x}(\tau)|\,\mathrm{d}\tau\right| \leqslant \left|\int_{t_1}^{t_2} (c_0(1+M(n-1)))\exp\Bigl(\int_a^m k(s)\,\mathrm{d}s\Bigr)\,\mathrm{d}\tau\right|.
\end{aligned}
$$

Analogously, we can get for each $k\in\{1,\dots,n-2\}$, that

$$
\begin{aligned}
(3.7)\qquad |x^{(k)}(t_1)-x^{(k)}(t_2)| & \leqslant \left|\int_{t_1}^{t_2} |x^{(k+1)}(\tau)|\,\mathrm{d}\tau\right| \\
& \leqslant \left|\int_{t_1}^{t_2} (c_0(1+M(n-1)))\exp\Bigl(\int_a^m k(s)\,\mathrm{d}s\Bigr)\,\mathrm{d}\tau\right|.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(3.8)\quad & |x^{(n-1)}(t_1)-x^{(n-1)}(t_2)| \\
& \leqslant \left|\int_{t_1}^{t_2} (1+\beta(\tau)(1+c_0))|x^{(n-1)}(\tau)|+\ldots+(1+\beta(\tau)(1+c_0))|x(\tau)|\,\mathrm{d}\tau\right| \\
& \leqslant \left|\int_{t_1}^{t_2} l(\tau)((c_0(1+M(n-1)))\exp\Bigl(\int_a^m k(s)\,\mathrm{d}s\Bigr)\,\mathrm{d}\tau\right|,
\end{aligned}
$$

where for all $\tau\in[a,m]$, $l(\tau):=1+\beta(\tau)(1+c_0)$.

Taking into account estimates (3.6)–(3.8), $x,\dot{x},\dots,x^{(n-1)}$ are equi-continuous, because $c(\cdot),k(\cdot),l(\cdot)\in L^1([a,m],\mathbb{R})$. Thus, the set of solutions of (P_q^m) and their derivatives is relatively compact.

We still have to show that the set of solutions of (P_q^m) and their derivatives (up to the $(n-1)$-st order) is closed. Let $\{x_i\}$ be a sequence of solutions of (P_q^m) such that $\{(x_i,\dot{x}_i,\dots,x_i^{(n-1)})\}\to(x,\dot{x},\dots,x^{(n-1)})$. By estimate (3.5), the sequences $\{x_i\},\{\dot{x}_i\},\dots,\{x_i^{(n-1)}\}$ satisfy the assumptions of Lemma 2.2. Thus, there exists a subsequence of $\{x_i\}$ for the sake of simplicity denoted as the sequence itself, uniformly convergent to x on $[a,m]$, such that $\{\dot{x}_i\},\dots,\{x_i^{(n-1)}\}$ converges uniformly to $\dot{x},\dots,x^{(n-1)}$ on $[a,m]$ and that $\{x_i^{(n)}\}$ converges weakly to $x^{(n)}$ in $L^1([a,m],\,\mathbb{R})$.

If we set $z_i:=(x_i,\dot{x}_i,\dots,x_i^{(n-1)})$, then $\dot{z}_i\to(\dot{x},\ddot{x},\dots,x^{(n)})$ weakly in $L^1([a,m],\ \mathbb{R})$. Let us now consider the system

$$
(3.9)\qquad \dot{z}_i(t)\in G(t,z_i(t))\quad\text{for a.a. } t\in[a,m],
$$

where

$$
G(t,z_i(t))=(\dot{x}_i,\dots,x_i^{(n)},P(t,z_i(t))).
$$

By virtue of Lemma 2.3 for $f_i := \dot{z}_i$, $f := (\dot{x}, \ddot{x}, \dots, x^{(n)})$, $x_i := (z_i)$, it follows that

$$(\dot{x}(t), \ddot{x}(t), \dots, x^{(n)}(t)) \in G(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t))$$

for a.a. $t \in [a, m]$, i.e.

$$x^{(n)}(t) \in P(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t)) \quad \text{for a.a. } t \in [a, m].$$

Moreover, since the set S_m is closed, $(x_i, \dots, x_i^{(n-1)}) \in S_m$ for all $i \in \mathbb{N}$, and

$$(x_i, \dots, x_i^{(n-1)}) \to (x, \dots, x^{(n-1)}),$$

it also holds that $(x, \dot{x}, \dots, x^{(n-1)}) \in S_m$. Altogether, the set of solutions of (P_q^m) and their derivatives is convex and compact, as claimed.

The nonemptiness of the set of solutions of (P_q^m) follows from Theorem 13.1 in [20] and the fact that $A_i(\cdot, q(\cdot), \dots, q^{(n-1)}(\cdot))$, $i = 1, \dots, n$, admit (according to Lemma 2.1) single-valued measurable selections $v_i(\cdot)$, $i = 1, \dots, n$.

Summing up, for all $q \in Q$ and $m \in \mathbb{N}$, it was shown that the set of solutions of problem (P_q^m) on compact interval is nonempty, compact and convex, i.e. in particular an R_δ-set.

Let us prove now using the inverse limit method that the set of solutions of asymptotic problem (P_q^m) is also an R_δ-set. For this purpose, let $q \in Q$ be arbitrary and let us denote (as before)

$$\begin{aligned} &P(t, x(t), \dots, x^{(n-1)}(t)) \\ &\quad := -A_1(t, q(t), \dots, q^{(n-1)}(t))x^{(n-1)}(t) - \dots - A_n(t, q(t), \dots, q^{(n-1)}(t))x(t). \end{aligned}$$

A function $x(\cdot)$ is a solution of (P_q^m) if and only if for a.a. $t \in [a, m]$,

$$\begin{aligned} &x(t) \in x(u) - |x(u)| + c_0 + \dot{x}(a) \cdot t + \dots \\ &\qquad + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} P(s, x(s), \dots, x^{(n-1)}(s)) \, \mathrm{d}s, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\dot{x}(t) \in \dot{x}(u) + |\dot{x}(u)| + \dot{x}(a) + \dots \\ &\qquad + \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} P(s, x(s), \dots, x^{(n-1)}(s)) \, \mathrm{d}s, \end{aligned} \tag{3.11}$$

$$\vdots$$

$$\begin{aligned} x^{(n-1)}(t) \in\ & x^{(n-1)}(u) \pm |x^{(n-1)}(u)| + x^{(n-1)}(a) \\ &+ \int_a^t P(s, x(s), \dots, x^{(n-1)}(s)) \, \mathrm{d}s \end{aligned} \tag{3.12}$$

for each $u \in [a,m]$, provided

$$0 \notin A_n(t, q(t), \dots, q^{(n-1)}(t)), \tag{3.13}$$

on a subset of $[a,m]$ with a nonzero measure.

More concretely, since the constraint in (P_q^m) can be equivalently expressed as

$$\begin{cases} x(a) = c_0, \\ x(u) - |x(u)| = 0,\ \dot{x}(u) + |\dot{x}(u)| = 0,\ \dots,\ x^{(n-1)}(u) \pm |x^{(n-1)}(u)| = 0 \\ \hfill \forall\, u \in [a,m], \end{cases} \tag{3.14}$$

every solution $x(\cdot)$ of (P_q^m) and its derivatives $\dot{x}(\cdot), \dots, x^{(n-1)}(\cdot)$ obviously satisfy (3.10)–(3.12). Reversely, differentiating (3.12), we obtain

$$x^{(n)}(t) \in P(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t)).$$

Moreover, $x(a) \in x(u) - |x(u)| + c_0$, $\dot{x}(a) \in \dot{x}(u) + |\dot{x}(u)| + \dot{x}(a)$, $\ddot{x}(a) \in \ddot{x}(u) - |\ddot{x}(u)| + \ddot{x}(a), \dots, x^{(n-1)}(a) \in x^{(n-1)}(u) \pm |x^{(n-1)}(u)| + x^{(n-1)}(a)$ for each $u \in [a,m]$, i.e. $\dot{x}(u) + |\dot{x}(u)| = 0$, $\ddot{x}(u) - |\ddot{x}(u)| = 0, \dots, x^{(n-1)}(u) \pm |x^{(n-1)}(u)| = 0$ and, in particular, for $u = a$, $|x(a)| = c_0$. Thus, for $x(a) = c_0$, we also have $x(u) - |x(u)| = 0$, by which (3.14) (i.e. the constraint in (P_q^m)) is satisfied. On the other hand, if $x(a) = -c_0$, we arrive at $x(u) - |x(u)| = -2c_0$, i.e. $x(u) = -c_0$ for all $u \in [a,m]$, and subsequently $0 \in A_n(t, q(t), \dots, q^{(n-1)}(t))$ for a.a. $t \in [a,m]$, which is a contradiction with (3.13).

The set of solutions of (P_q^m) and their derivatives is a fixed point set of the map $\varphi_m \colon C^{n-1}([a,m],\mathbb{R}) \times \ldots \times C^1([a,m],\mathbb{R}) \times C([a,m],\mathbb{R}) \multimap C^{n-1}([a,m],\mathbb{R}) \times \ldots \times C^1([a,m],\mathbb{R}) \times C([a,m],\mathbb{R})$, where for all $t \in [a,m]$,

$$\begin{aligned} &\varphi_m(x, \dots, x^{(n-1)})(t) \\ &:= \Bigg\{ \Bigg(\bigcup_{u \in [a,m]} x(u) - |x(u)| + c_0 + \ldots + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\,\mathrm{d}s, \\ &\qquad \bigcup_{u \in [a,m]} \dot{x}(u) + |\dot{x}(u)| + \dot{x}(a) + \ddot{x}(a) \cdot t + \ldots + \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} f(s)\,\mathrm{d}s, \\ &\qquad \vdots \\ &\qquad \bigcup_{u \in [a,m]} x^{(n-1)}(u) \pm |x^{(n-1)}(u)| + x^{(n-1)}(a) + \int_a^t f(s)\,\mathrm{d}s \Bigg) \colon\ f \in L^1([a,m],\mathbb{R}) \\ &\qquad \text{and } f(s) \in P(t, x(s), \dot{x}(s), \dots, x^{(n-1)}(s)) \text{ for a.a. } s \in [a,m] \Bigg\}. \end{aligned}$$

It can be easily seen that $\{\varphi_m\}_{m=1}^{\infty}$ is a map of the inverse system

$$\{C^{n-1}([a,m],\mathbb{R})\times\ldots\times C^1([a,m],\mathbb{R})\times C([a,m],\mathbb{R}),\pi_m^p,\mathbb{N}\}$$

into itself, where for all $p\geqslant m$, $x\in C^{n-1}([a,p],\mathbb{R})\times\ldots\times C^1([a,p],\mathbb{R})\times C([a,p],\mathbb{R})$, $\pi_m^p(x,\dot{x},\ldots,x^{(n-1)}) = (x|_{[a,m]},\dot{x}|_{[a,m]},\ldots,x^{(n-1)}|_{[a,m]})$. Mappings $\{\varphi_m\}_{m=1}^{\infty}$ induce the limit mapping $\varphi\colon C^{n-1}([a,\infty),\mathbb{R})\times\ldots\times C^1([a,\infty),\mathbb{R})\times C([a,\infty),\mathbb{R})\multimap C^{n-1}([a,\infty),\mathbb{R})\times\ldots\times C^1([a,\infty),\mathbb{R})\times C([a,\infty),\mathbb{R})$, where for all $t\geqslant a$,

$$\begin{aligned}
&\varphi(x,\,\ldots,x^{(n-1)})(t)\\
&:=\Bigg\{\Bigg(\bigcup_{u\in[a,\infty)} x(u)-|x(u)|+c_0+\ldots+\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}f(s)\,\mathrm{d}s,\\
&\qquad\bigcup_{u\in[a,\infty)} \dot{x}(u)+|\dot{x}(u)|+\dot{x}(a)+\ddot{x}(a)\cdot t+\ldots+\frac{1}{(n-2)!}\int_a^t(t-s)^{n-2}f(s)\,\mathrm{d}s,\\
&\qquad\vdots\\
&\qquad\bigcup_{u\in[a,\infty)} x^{(n-1)}(u)\pm|x^{(n-1)}(u)|+x^{(n-1)}(a)+\int_a^t f(s)\,\mathrm{d}s\Bigg)\colon\ f\in L^1_{\mathrm{loc}}([a,\infty),\mathbb{R})\\
&\qquad\text{and } f(s)\in P(t,x(s),\dot{x}(s),\ldots,x^{(n-1)}(s)) \text{ for a.a. } s\in[a,\infty)\Bigg\}.
\end{aligned}$$

The fixed point set of the mapping φ is the set of solutions and their derivatives of the problem (P_q). By virtue of Proposition 2.2, the set of solutions and their derivatives of the original problem (P_q) is therefore an R_δ-set.

ad (ii): Assumption (ii) follows immediately from the properties of mappings A_i, $i=1,\ldots,n$, and the definition of (P_q).

ad (iii): Since the set $S:=Q$ is closed and each solution of the b.v.p. (P_q) belongs to Q, it holds that $\overline{\mathfrak{T}(Q)}\subset S$, where the map $\mathfrak{T}$ is the solution mapping that assigns to each $q\in Q$ the set of solutions of (P_q).

ad (iv): It follows directly from the boundary conditions that $\mathfrak{T}(Q)$ is bounded in $C([a,\infty),\mathbb{R})$.

Since all assumptions of Proposition 2.1 are satisfied, the b.v.p. (3.1) admits a solution $x(\cdot)$ such that $0\leqslant x(t)\leqslant c_0$ for all $t\in[a,\infty)$. □

Let us illustrate now the obtained result by a third-order asymptotic b.v.p.

Example 3.1. Let us consider the Kneser-type b.v.p.

$$\text{(3.15)}\qquad \begin{cases} x^{(3)}(t)=f(t,x(t),\dot{x}(t),\ddot{x}(t)) \quad \text{for a.a. } t\in[1,\infty),\\ x(1)=\frac{1}{4},\\ x(t)\geqslant 0,\ \dot{x}(t)\leqslant 0,\ \ddot{x}(t)\geqslant 0 \quad \text{for all } t\in[1,\infty), \end{cases}$$

where

$$f(t,x(t),\dot{x}(t),\ddot{x}(t)) := \frac{\sin(x(t))}{120}(2+\operatorname{sgn}(\dot{x}(t)+1))\ddot{x}(t) + \frac{\operatorname{arctg}(t)}{64}\dot{x}(t) \\ - \frac{\mathrm{e}^t}{1000}(\pi + \operatorname{sgn}(\ddot{x}(t)))x(t).$$

Because of discontinuity at $y = 0$ in $\operatorname{sgn} y$, the Filippov solutions should be considered which can be identified as Carathéodory solutions of the associated multivalued b.v.p.

$$\begin{cases} x^{(3)}(t) \in F(t,x(t),\dot{x}(t),\ddot{x}(t)) & \text{for a.a. } t \in [1,\infty), \\ x(1) = \frac{1}{4}, \\ x(t) \geqslant 0,\ \dot{x}(t) \leqslant 0,\ \ddot{x}(t) \geqslant 0 & \text{for all } t \in [1,\infty), \end{cases} \tag{3.16}$$

where

$$F(t,x(t),\dot{x}(t),\ddot{x}(t)) := \frac{\sin(x(t))}{120}(2+\operatorname{Sgn}(\dot{x}(t)+1))\ddot{x}(t) + \frac{\operatorname{arctg}(t)}{64}\dot{x}(t) \\ - \frac{\mathrm{e}^t}{1000}(\pi + \operatorname{Sgn}(\ddot{x}(t)))x(t)$$

with

$$\operatorname{Sgn} y := \begin{cases} -1, & \text{for } y < 0, \\ [-1,1], & \text{for } y = 0, \\ 1, & \text{for } y > 0. \end{cases}$$

Let us show now that the Kneser b.v.p. problem (3.16) satisfies all assumptions of Theorem 3.1, and so admits a solution. More concretely, the fulfilment of assumptions (i)–(iii) directly follows from the considered r.h.s. and boundary conditions. The assumption (iv) holds as well since, e.g. for $r := 8$,

$$f^*(t) := \max\Big\{\Big|\frac{\sin(x_1)}{120}(2+\operatorname{Sgn}(x_2+1))x_3 + \frac{\operatorname{arctg}(t)}{64}x_2 - \frac{\mathrm{e}^t}{1000}(\pi+\operatorname{Sgn}(x_3))x_1\Big| : \\ x_1 \in [0,8],\ x_2 \in [-8,0],\ x_3 \in [0,8]\Big\} \leqslant \frac{1}{5} + \frac{\pi}{16} + \frac{\mathrm{e}^t(1+\pi)}{125}.$$

Solving the relevant inequality (cf. (3.3))

$$8\int_1^{1+\delta} \frac{1}{5} + \frac{\pi}{16} + \frac{\mathrm{e}^t(1+\pi)}{125}\,\mathrm{d}t \leqslant 8,$$

we obtain that $\delta \leqslant 1.6116$, and subsequently (3.2) that $c_0 = \frac{1}{4}$ should belong to the interval $(0; 0.2538)$, which is true. Thus, the assumption (iv) of Theorem 3.1 holds.

In order to verify assumption (v), let us define the set Q by

$$Q := \left\{x \in C^2([1,\infty),\mathbb{R})\colon\ x(1) = \tfrac{1}{4},\ x(t) \geqslant 0,\ \dot{x}(t) \leqslant 0,\ \ddot{x}(t) \geqslant 0,\ t \in [1,\infty)\right\}.$$

For all $q \in Q$, $t \geqslant 1$ and all $x_1 \in [0,8]$, $x_2 \in [-8,0]$, $x_3 \in [0,8]$, it holds that

$$\frac{\sin(q(t))}{120}(2+\operatorname{Sgn}(\dot{q}(t)+1))x_3 - \frac{\operatorname{arctg}(t)}{64}x_2 + \frac{\mathrm{e}^t}{1000}(\pi+\operatorname{Sgn}(\ddot{q}(t)))x_1 \subseteq [0,\infty),$$

which ensures the validity of assumption (v). Therefore, it is possible to apply Theorem 3.1 and obtain that the b.v.p. (3.16) has a solution in Q. This solution is consequently the Filippov solution of the original problem (3.15).

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