

Antonio Boccuto; Xenofon Dimitriou

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DIEUDONNÉ-TYPE THEOREMS FOR LATTICE GROUP-VALUED K -TRIANGULAR SET FUNCTIONS

ANTONIO BOCCUTO AND XENOFON DIMITRIOU

Some versions of Dieudonné-type convergence and uniform boundedness theorems are proved, for k -triangular and regular lattice group-valued set functions. We use sliding hump techniques and direct methods. We extend earlier results, proved in the real case. Furthermore, we pose some open problems.

Keywords: lattice group, (D) -convergence, k -triangular set function, (s) -bounded set function, Fremlin lemma, limit theorem, Brooks–Jewett theorem, Dieudonné theorem, Nikodým boundedness theorem

Classification: 28A12, 28A33, 28B10, 28B15, 40A35, 46G10

1. INTRODUCTION

Dieudonné-type theorems (see [34]) are the object of several studies about convergence and uniform boundedness theorems for regular set functions and related topics about (weak) compactness of measures. A historical comprehensive survey can be found in [16]. Among the most important developments existing in the literature about these subjects, see for instance [2, 3, 30, 31, 32, 33, 38, 45], and in particular, concerning the setting of lattice group-valued measures, we quote [6, 9, 10, 12, 13]. In [14, 24] some Dieudonné-type theorems were proved for lattice group-valued finitely additive regular measures in the context of filter convergence, while some versions of uniform boundedness theorems in this setting are proved in [11, 25]. In [39, 40, 41, 47] some Dieudonné-type theorems were proved for k -triangular and non-additive regular set functions. Some examples of k -triangular set functions are the M -measures, that is monotone set functions m with $m(\emptyset) = 0$, continuous from above and from below and compatible with respect to supremum and infimum, which have several applications in various branches, among which intuitionistic fuzzy sets and observables (see also [1, 17, 27, 35, 42]). Some examples of non-monotone 1-triangular set functions are the Saeki mesuroids (see [43]). In [17, 19, 20, 21, 22, 23] some limit theorems were proved for lattice group-valued k -subadditive capacities and k -triangular set functions.

In this paper we prove some Dieudonné convergence theorems and a version of Nikodým boundedness theorem for regular and k -triangular lattice group-valued set functions, extending earlier results proved in the real case in [39, 40, 41] using some

diagonal matrix theorems. Our techniques are direct and inspired by sliding hump-type methods. We use the tool of (D) -convergence, because we can apply the powerful Fremlin lemma (see also [37, 42]), which replaces the $\frac{\varepsilon}{2^n}$ -technique and allows to replace a sequence of regulators with a single (D) -sequence. Observe that, in the lattice group context, in the Nikodým boundedness theorem we assume the existence of a single increasing sequence of positive elements of the involved lattice group, with respect to which the set functions are supposed to be pointwise bounded on a suitable sublattice, playing a role similar to that of the class of all open subsets of a topological space. We see that in general this condition cannot be replaced by a simple setwise boundedness (see also [11, 25, 46]). Finally, some open problems are posed.

2. PRELIMINARIES

We begin with recalling the following basic facts on lattice groups (see also [16, 28]).

- Definition 2.1.** (a) A lattice group R is said to be *Dedekind complete* if every nonempty subset of R , bounded from above, has supremum in R .
- (b) A Dedekind complete lattice group R is *super Dedekind complete* iff for every nonempty set $A \subset R$, bounded from above, there is a countable subset A' , with $\bigvee A' = \bigvee A$.
- (c) A nonempty subset S of a lattice group R is *bounded* iff there exists an element $u \in R$ with $|x| \leq u$ for each $x \in S$.
- (d) Let $(t_n)_n$ be an increasing sequence of positive elements of R , and let $\emptyset \neq S \subset R$. We say that S is *bounded by* $(t_n)_n$ iff for every $x \in S$ there is $n_* \in \mathbb{N}$ such that $|x| \leq t_{n_*}$.
- (e) A sequence $(\sigma_p)_p$ in a lattice group R is called an (O) -*sequence* iff it is decreasing and $\bigwedge_{p=1}^{\infty} \sigma_p = 0$.
- (f) A bounded double sequence $(a_{t,l})_{t,l}$ in R is a (D) -*sequence* or a *regulator* iff $(a_{t,l})_l$ is an (O) -sequence for any $t \in \mathbb{N}$.
- (g) A lattice group R is *weakly σ -distributive* iff $\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \right) = 0$ for every (D) -sequence $(a_{t,l})_{t,l}$ in R .
- (h) A sequence $(x_n)_n$ in R is said to be *order convergent* (or (O) -*convergent*) to x iff there exists an (O) -sequence $(\sigma_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x| \leq \sigma_p$ for each $n \geq n_0$, and in this case we write $(O)\lim_n x_n = x$.
- (i) We say that $(x_n)_n$ is (O) -*Cauchy* iff there is an (O) -sequence $(\tau_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x_q| \leq \tau_p$ for each $n, q \geq n_0$.
- (j) A sequence $(x_n)_n$ in R is (D) -*convergent* to x iff there is a (D) -sequence $(a_{t,l})_{t,l}$ in R such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $n_0 \in \mathbb{N}$ with $|x_n - x| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ whenever $n \geq n_0$, and we write $(D)\lim_n x_n = x$.

- (k) We say that $(x_n)_n$ is (D) -Cauchy iff there exists a (D) -sequence $(b_{t,l})_{t,l}$ in R such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $n_0 \in \mathbb{N}$ with $|x_n - x_q| \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$ whenever $n, q \geq n_0$.
- (l) A lattice group R is said to be (O) -complete (resp. (D) -complete) iff every (O) -Cauchy (resp. (D) -Cauchy) sequence is (O) -convergent (resp. (D) -convergent).
- (m) We call *sum* of a series $\sum_{n=1}^{\infty} x_n$ in R the limit $(O) \lim_n \sum_{r=1}^n x_r$, if it exists in R .
- (n) If R is a vector lattice, then we say that $(x_n)_n$ (r) -converges to x iff there exists $u \in R, u \geq 0$, such that for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with $|x_n - x| \leq \varepsilon u$ whenever $n \geq n_0$.
- (o) A vector lattice R satisfies property (σ) iff for every sequence $(u_n)_n$ of positive elements of R there are a sequence $(a_n)_n$ of positive real numbers and an element $u \in R$ with $a_n u_n \leq u$ for each $n \in \mathbb{N}$.
- (p) A lattice \mathcal{E} of subsets of an infinite set G satisfies property (E) iff for each disjoint sequence $(C_h)_h$ in \mathcal{E} there is a subsequence $(C_{h_r})_r$, such that \mathcal{E} contains the σ -algebra generated by the sets $C_{h_r}, r \in \mathbb{N}$ (see also [44]).

Remark 2.2. Note that every Dedekind complete lattice group is both (O) - and (D) -complete. Moreover, observe that every (O) -convergent sequence is also (D) -convergent to the same limit in any lattice group, while the converse is true if and only if the involved (ℓ) -group is weakly σ -distributive. Furthermore, it is known that every (r) -convergent sequence in any vector lattice is (O) -convergent too (see also [28, 48]). The converse, in general, is not true. For example, let \mathcal{B} be the σ -algebra of all Borel subsets of $[0, 1]$, λ be the Lebesgue measure on $[0, 1]$, $L^0 := L^0([0, 1], \mathcal{B}, \lambda)$ be the space of all measurable real-valued functions defined on $[0, 1]$, with the identification of λ -null sets, and $R := \{f \in L^0([0, 1], \mathcal{B}, \lambda) : f \text{ is bounded}\}$. If $(u_n)_n$ is any sequence of positive elements of R , then there exists a sequence $(L_n)_n$ of positive real numbers such that $u_n \leq \underline{L}_n$ for every $n \in \mathbb{N}$, where \underline{L}_n denotes the function which assumes the constant value L_n . Since \mathbb{R} fulfils property (σ) , there are a sequence $(a_n)_n$ of positive real numbers and a positive real number v with $a_n L_n \leq v$, and hence $a_n u_n \leq a_n \underline{L}_n \leq \underline{v}$, for every $n \in \mathbb{N}$. Hence, R satisfies property (σ) . It is known that in L^0 order and (r) -convergence coincide with almost everywhere convergence, while in R , order convergence coincides with the almost everywhere convergence dominated by a constant function, and (r) -convergence coincides with uniform convergence (see also [48]). Moreover, since L^0 is weakly σ -distributive (see also [8]), then in L^0 (O) - and (D) -convergence coincide in L^0 , and so they coincide also in R . Hence, R is weakly σ -distributive too. Finally, observe that, in the space L^0 , order, (D) - and (r) -convergence are equivalent (see also [8, 48]).

We now recall the following property of convergence in lattice groups (see also [22, Proposition 3.1]).

Proposition 2.3. Let R be a Dedekind complete lattice group, $x \in R$, and $(x_n)_n$ be a sequence in R , such that

2.3.1) for every subsequence $(x_{n_q})_q$ of $(x_n)_n$ there is a sub-subsequence $(x_{n_{q_r}})_{r,}$ convergent to x with respect to a single (D) -sequence $(a_{t,l})_{t,l}$.

Then $(D) \lim_n x_n = x$ with respect to $(a_{t,l})_{t,l}$.

Proof. Suppose by contradiction that there are $\varphi \in \mathbb{N}^{\mathbb{N}}$ and a strictly increasing sequence $(n_q)_q$ with $|x_{n_q} - x| \not\leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for each $q \in \mathbb{N}$. Thus any subsequence of $(x_{n_q})_q$ does not (D) -converge to x with respect to $(a_{t,l})_{t,l}$, obtaining a contradiction with 2.3.1). □

Remark 2.4. An analogous of Proposition 2.3 holds, if (D) -convergence is replaced by (O) -convergence.

We now recall the Fremlin lemma, by means of which it is possible to replace a sequence of regulators with a single (D) -sequence, and which will be fundamental in the sequel, to prove our main results, because it has the same role as the $\frac{\varepsilon}{2^n}$ -argument. This is one of the reasons for which we often prefer to deal with (D) -convergence rather than (O) -convergence.

Lemma 2.5. (see also Fremlin [37, Lemma 1C], Riečan and Neubrunn [42, Theorem 3.2.3]) Let R be any Dedekind complete (ℓ) -group and $(a_{t,l}^{(n)})_{t,l}, n \in \mathbb{N}$, be a sequence of regulators in R . Then for every $u \in R, u \geq 0$ there is a (D) -sequence $(a_{t,l})_{t,l}$ in R with

$$u \wedge \left(\sum_{n=1}^q \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)} \right) \right) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \text{for every } q \in \mathbb{N} \text{ and } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

We now deal with the main properties of k -triangular lattice group-valued set functions. Let R be a Dedekind complete and weakly σ -distributive lattice group, G be an infinite set, $\mathcal{L} \subset \mathcal{P}(G)$ be an algebra, $m : \mathcal{L} \rightarrow R$ be a bounded set function and k be a fixed positive integer.

Definition 2.6. (a) The *semivariation* of m is defined by setting

$$v(m)(A) := \bigvee \{ |m(B)| : B \in \mathcal{L}, B \subset A \}, \quad A \in \mathcal{L}.$$

If $\mathcal{E} \subset \mathcal{L}$ is a lattice, then we put

$$v_{\mathcal{E}}(m)(A) := \bigvee \{ |m(B)| : B \in \mathcal{E}, B \subset A \}, \quad A \in \mathcal{L}.$$

The set function $v_{\mathcal{E}}(m)$ is called the *semivariation of m with respect to \mathcal{E}* .

(b) We say that m is

$$m(A) - k m(B) \leq m(A \cup B) \leq m(A) + k m(B) \text{ whenever } A, B \in \Sigma, A \cap B = \emptyset \quad (1)$$

and

$$0 = m(\emptyset) \leq m(A) \text{ for each } A \in \Sigma. \quad (2)$$

- (c) Let $\mathcal{E} \subset \mathcal{L}$ be a sublattice of \mathcal{L} . We say that a set function $m : \mathcal{L} \rightarrow R$ is \mathcal{E} - (s) -bounded iff there exists a (D) -sequence $(a_{t,l})_{t,l}$ such that, for every disjoint sequence $(C_h)_h$ in \mathcal{E} , $(D) \lim_h v_{\mathcal{E}}(m)(C_h) = 0$ with respect to $(a_{t,l})_{t,l}$. A set function m is (s) -bounded iff it is \mathcal{L} - (s) -bounded.
- (d) We say that the set functions $m_j : \mathcal{L} \rightarrow R$ are \mathcal{E} -uniformly (s) -bounded iff there exists a (D) -sequence $(a_{t,l})_{t,l}$ such that, for every disjoint sequence $(C_h)_h$ in \mathcal{E} ,

$$(D) \lim_h \left(\bigvee_j v_{\mathcal{E}}(m_j)(C_h) \right) = 0$$

with respect to $(a_{t,l})_{t,l}$. The m_j 's are uniformly (s) -bounded iff they are \mathcal{L} -uniformly (s) -bounded.

- (e) We say that the set functions $m_j : \mathcal{L} \rightarrow R$, $j \in \mathbb{N}$, are equibounded on \mathcal{L} iff there is $u \in R$ with $|m_j(A)| \leq u$ for every $j \in \mathbb{N}$ and $A \subset \mathcal{L}$.

Now we recall the following

Proposition 2.7. (see also Boccuto and Dimitriou [22, Proposition 2.6]) If $m : \mathcal{L} \rightarrow R$ is k -triangular, then $v(m)$ is k -triangular too.

Proposition 2.8. (see also Boccuto and Dimitriou [22, Proposition 2.7]) Let $m : \mathcal{L} \rightarrow R$ be a k -triangular set function. Then for every $n \in \mathbb{N}$, $n \geq 2$, and for every pairwise disjoint sets $E_1, E_2, \dots, E_n \in \mathcal{L}$ we have

$$m(E_1) - k \sum_{q=2}^n m(E_q) \leq m\left(\bigcup_{q=1}^n E_q\right) \leq m(E_1) + k \sum_{q=2}^n m(E_q), \tag{3}$$

and in particular

$$m(E_1) \leq m\left(\bigcup_{q=1}^n E_q\right) + k \sum_{q=2}^n m(E_q). \tag{4}$$

We now turn to regular lattice group-valued set functions.

Definition 2.9. Let \mathcal{G}, \mathcal{H} be two sublattices of \mathcal{L} , such that \mathcal{G} is closed under countable unions, and the complement of every element of \mathcal{H} belongs to \mathcal{G} . A set function $m : \mathcal{L} \rightarrow R$ is said to be regular iff there exists a (D) -sequence $(a_{t,l})_{t,l}$ such that

- 2.9.1) for every $E \in \mathcal{L}$ there are two sequences $(V_n)_n$ in \mathcal{G} and $(K_n)_n$ in \mathcal{H} with $V_n \supset E \supset K_n$ for each $n \in \mathbb{N}$ and such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ with

$$v(m)(V_n \setminus K_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever $n \geq n_0$, and

2.9.2) for every $W \in \mathcal{H}$ there are two sequences $(G_n)_n$ in \mathcal{G} and $(F_n)_n$ in \mathcal{H} with $W \subset F_{n+1} \subset G_n \subset F_n$ for every $n \in \mathbb{N}$, and such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $n^* \in \mathbb{N}$ with

$$v(m)(G_n \setminus W) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever $n \geq n^*$.

We now prove the following property of regular set functions.

Proposition 2.10. (see also Boccuto and Dimitriou [17, Theorem 3.10]) If G is a compact Hausdorff topological space, $\mathcal{L}, \mathcal{G}, \mathcal{H}$ are the classes of all Borel, open and compact subsets of G , respectively, and $m : \mathcal{L} \rightarrow R$ is a k -triangular, increasing and regular set function, then

$$(O) \lim_n m(I_n) = 0 \tag{5}$$

whenever $(I_n)_n$ is a decreasing sequence in \mathcal{L} with $\bigcap_{n=1}^{\infty} I_n = \emptyset$, with respect to a single regulator independent of the choice of $(I_n)_n$.

Proof. Let $(I_n)_n$ be as in (5). Let $(a_{t,l})_{t,l}$ be a (D) -sequence satisfying 2.9.1). For every $n \in \mathbb{N}$ there is $K_n \in \mathcal{H}$ with $K_n \subset I_n$ and $m(I_n \setminus K_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}$. By virtue of Lemma 2.5, there is a (D) -sequence $(\alpha_{t,l})_{t,l}$ with

$$m(G) \wedge \left(\sum_{n=1}^q \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)} \right) \right) \leq \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)} \text{ for each } q \in \mathbb{N} \text{ and } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

Let $O_n := G \setminus K_n$, $n \in \mathbb{N}$. Note that $O_n \in \mathcal{G}$ for every n and $G = \bigcup_{n=1}^{\infty} O_n$, since $\bigcap_{n=1}^{\infty} K_n = \emptyset$. As G is compact, there is $n_0 \in \mathbb{N}$ with $G = \bigcup_{i=1}^{n_0} O_i$, and hence $\bigcap_{i=1}^{n_0} K_i = \emptyset$, whenever $n \geq n_0$. For such n 's, taking into account (3), we have

$$\begin{aligned} m(I_n) &\leq m(G) \wedge \left(m(I_n \setminus \left(\bigcap_{i=1}^n K_i \right)) \right) \\ &\leq m(G) \wedge \left(m \left(\bigcup_{i=1}^n (I_i \setminus K_i) \right) \right) \\ &\leq m(G) \wedge \left(k \sum_{i=1}^n m(I_i \setminus K_i) \right) \leq k \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)} \end{aligned} \tag{6}$$

(see also [39, Lemma 1]). Thus the assertion follows. □

Remark 2.11. Observe that, if \mathcal{L} is an algebra with property (E) and $m : \mathcal{L} \rightarrow R$ is positive, increasing and satisfies (5), then m is also (s) -bounded (with respect to a single regulator). To prove this, let $(A_n)_n$ be any disjoint sequence in \mathcal{L} and $(B_n)_n$ be any subsequence of $(A_n)_n$. By property (E) , there is a subsequence $(C_n)_n$ of $(B_n)_n$, such that $\bigcup_{n \in P} C_n \in \mathcal{L}$ for every $P \subset \mathbb{N}$. Since m is increasing and $m(\emptyset) = 0$, we get

$$0 \leq m(C_n) \leq m \left(\bigcup_{i=n}^{\infty} C_i \right).$$

From (5) and (7) we get $(O)\lim_n m(C_n) = 0$ with respect to a single regulator (independent of $(A_n)_n$, $(B_n)_n$ and $(C_n)_n$). By arbitrariness of the sequence $(B_n)_n$ and Proposition 2.3 it follows that $(D)\lim_n m(C_n) = 0$ with respect to a single regulator, and this proves the claim.

The converse, in general, is not true (see also [22, Remark 2.12]). □

Proposition 2.12. (see also Boccutto and Dimitriou [17, Proposition 3.4]) If $m : \mathcal{L} \rightarrow R$ is a k -triangular and increasing set function satisfying (5), then we get

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq m(E_1) + k \sum_{n=2}^{\infty} m(E_n) \tag{7}$$

for every sequence $(E_n)_n$ in \mathcal{L} , such that $\bigcup_{n \in A} E_n \in \mathcal{L}$ whenever $A \subset \mathbb{N}$.

The following proposition will be useful in proving our Dieudonné convergence theorem (see also [10, Lemma 3.1]).

Proposition 2.13. With the same notations and assumptions as above, let $m : \mathcal{L} \rightarrow R$ be a regular and k -triangular set function. Then for each $V \in \mathcal{G}$ we get

$$v_{\mathcal{L}}(m)(V) = v_{\mathcal{G}}(m)(V). \tag{8}$$

Proof. Pick arbitrarily $V \in \mathcal{G}$, and let $(\gamma_{t,l})_{t,l}$ be a (D) -sequence related to regularity of m . Choose $B \in \mathcal{L}$ with $B \subset V$, and fix arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$. By regularity of m , there is $O \in \mathcal{G}$, $O \supset B$, with

$$v_{\mathcal{L}}(m)(O \setminus B) \leq \bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)}. \tag{9}$$

Let $U := O \cap V$, then $U \supset B$. From (9) and k -triangularity of m we get

$$\begin{aligned} m(B) &\leq m(U) + k m(U \setminus B) \\ &\leq v_{\mathcal{G}}(m)(V) + k v_{\mathcal{L}}(m)(O \setminus B) \\ &\leq v_{\mathcal{G}}(m)(V) + k \bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)}. \end{aligned} \tag{10}$$

Taking in (10) the supremum as $B \in \mathcal{L}$, $B \subset V$, we obtain

$$v_{\mathcal{L}}(m)(V) \leq v_{\mathcal{G}}(m)(V) + k \bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)}. \tag{11}$$

From (11) and weak σ -distributivity of R we deduce

$$v_{\mathcal{L}}(m)(V) \leq v_{\mathcal{G}}(m)(V) + k \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)} \right) = v_{\mathcal{G}}(m)(V). \tag{12}$$

Since the converse inequality is straightforward, then (8) follows from (12). This ends the proof. □

Definition 2.14. A sequence $m_j : \mathcal{L} \rightarrow R, j \in \mathbb{N}$, of set functions is said to be (RD) -regular on \mathcal{L} iff there is a (D) -sequence $(a_{t,l})_{t,l}$ such that

2.14.1) for every $E \in \mathcal{L}$ there are two sequences $(V_n)_n$ in \mathcal{G} and $(K_n)_n$ in \mathcal{H} such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ with $v(m_j)(V_n \setminus K_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for every $n \geq n_0$, and

2.14.2) for every disjoint sequence $(H_n)_n$ in \mathcal{L} there is a sequence $(O_n)_n$ in \mathcal{G} such that $O_n \supset H_n$ for each $n \in \mathbb{N}$ and $(D) \lim_n v(m_j) \left(\bigcup_{i=n}^{\infty} O_i \right) = 0$ for every $j \in \mathbb{N}$ with respect to $(a_{t,l})_{t,l}$.

We now recall the following

Proposition 2.15. (see also Boccuto and Candeloro [10, Proposition 2.6]) Let R be any Dedekind complete and weakly σ -distributive lattice group, and $m_j : \mathcal{L} \rightarrow R, j \in \mathbb{N}$, be a sequence of regular equibounded set functions. Then they satisfy 2.14.1) and the following property:

2.15.1) there exists a regulator $(\beta_{t,l})_{t,l}$ such that for every $W \in \mathcal{H}$ there are two sequences $(G_n)_n$ in \mathcal{G} and $(F_n)_n$ in \mathcal{H} , with $W \subset F_{n+1} \subset G_n \subset F_n$ for every $n \in \mathbb{N}$ and such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there is $n^* \in \mathbb{N}$ with

$$v_{\mathcal{L}}(m_j)(G_n \setminus W) \leq \bigvee_{t=1}^{\infty} \beta_{t,\varphi(t)}$$

for every $n \geq n^*$.

Definition 2.16. Let $\mathcal{L}, \mathcal{G}, \mathcal{H}$ be as in Definition 2.9. The set functions $m_j : \mathcal{L} \rightarrow R, j \in \mathbb{N}$, are *uniformly regular* iff there exists a (D) -sequence $(a_{t,l})_{t,l}$ such that

2.16.1) for each $E \in \mathcal{L}$ there exist two sequences $(V_n)_n$ in \mathcal{G} and $(K_n)_n$ in \mathcal{H} with $V_n \supset E \supset K_n$ for every $n \in \mathbb{N}$ and such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ with

$$\bigvee_j v(m_j)(V_n \setminus K_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for all $n \geq n_0$, and

2.16.2) for any $W \in \mathcal{H}$ there are two sequences $(G_n)_n$ in \mathcal{G} and $(F_n)_n$ in \mathcal{H} with $W \subset F_{n+1} \subset G_n \subset F_n$ for each $n \in \mathbb{N}$, and such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n^* \in \mathbb{N}$ with

$$\bigvee_j v(m_j)(G_n \setminus W) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

whenever $n \geq n^*$.

3. THE MAIN RESULTS

In this section we prove a Dieudonné convergence-type theorem and a Dieudonné-Nikodým boundedness theorem for regular and k -triangular lattice group-valued set functions. Let R be a Dedekind complete and weakly σ -distributive lattice group. We begin with recalling the following Brooks–Jewett–type theorem for k -triangular set functions.

Theorem 3.1. (see Boccutto and Dimitriou [22, Theorem 3.3]) Let G be any infinite set, $\mathcal{L} \subset \mathcal{P}(G)$ be an algebra, $\mathcal{E} \subset \mathcal{L}$ be a lattice, satisfying property (E), $m_j : \mathcal{L} \rightarrow R$, $j \in \mathbb{N}$, be a sequence of equibounded, k -triangular and \mathcal{E} -(s)-bounded set functions. If the limit $m_0(E) := \lim_j m_j(E)$ exists in R for every $E \in \mathcal{E}$ with respect to a single regulator, then the m_j 's are \mathcal{E} -uniformly (s)-bounded, and m_0 is k -triangular and (s)-bounded.

The following technical lemma will be useful in the sequel.

Lemma 3.2. (see Boccutto and Dimitriou [22, Lemma 3.4]) Let $\mathcal{L} \subset \mathcal{P}(G)$ be an algebra, \mathcal{G} and \mathcal{H} be two sublattices of \mathcal{L} , such that the complement of every element of \mathcal{H} belongs to \mathcal{G} , $m_j : \mathcal{L} \rightarrow R$, $j \in \mathbb{N}$, be a sequence of k -triangular and \mathcal{G} -uniformly (s)-bounded set functions. Fix $W \in \mathcal{H}$ and a decreasing sequence $(H_n)_n$ in \mathcal{G} , with $W \subset H_n$ for each $n \in \mathbb{N}$. If

$$(D) \lim_n \left(\bigvee_{A \in \mathcal{G}, A \subset H_n \setminus W} m_j(A) \right) = \bigwedge_n \left(\bigvee_{A \in \mathcal{G}, A \subset H_n \setminus W} m_j(A) \right) = 0 \text{ for every } j \in \mathbb{N} \quad (13)$$

with respect to a single (D)-sequence $(a_{t,l})_{t,l}$, then

$$(D) \lim_n \left(\bigvee_j \left(\bigvee_{A \in \mathcal{G}, A \subset H_n \setminus W} m_j(A) \right) \right) = \bigwedge_n \left(\bigvee_j \left(\bigvee_{A \in \mathcal{G}, A \subset H_n \setminus W} m_j(A) \right) \right) = 0$$

with respect to $(a_{t,l})_{t,l}$.

The next step is to prove a Dieudonné-type theorem for k -triangular lattice group-valued set functions, which extends [10, Lemma 3.2].

Theorem 3.3. Let $\mathcal{L} \subset \mathcal{P}(G)$ be an algebra, \mathcal{G} and \mathcal{H} be two sublattices of \mathcal{L} , such that \mathcal{G} is closed under countable unions and the complement of every element of \mathcal{H} belongs to \mathcal{G} , $m_j : \mathcal{L} \rightarrow R$, $j \in \mathbb{N}$, be a sequence of equibounded, regular, k -triangular and \mathcal{G} -uniformly (s)-bounded set functions. Then the m_j 's are \mathcal{L} -uniformly (s)-bounded and uniformly regular on \mathcal{L} .

Proof. Let $(H_n)_n$ be a disjoint sequence of elements of \mathcal{L} , $(a_{t,l})_{t,l}$ be a (D)-sequence, satisfying 2.14.1), $u = \bigvee_{j \in \mathbb{N}, A \in \mathcal{L}} m_j(A)$, and according to Lemma 2.5, let $(b_{t,l})_{t,l}$ be a regulator in R , with

$$u \wedge \left(\sum_{h=1}^q \left(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+h)} \right) \right) \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \quad \text{for every } \varphi \in \mathbb{N}^{\mathbb{N}} \text{ and } q \in \mathbb{N}. \quad (14)$$

Let $(c_{t,l})_{t,l}$ be a (D)-sequence associated with \mathcal{G} -uniform (s)-boundedness, and set $d_{t,l} = (k + 1)(b_{t,l} + c_{t,l})$, $e_{t,l} = (k + 1)(a_{t,l} + d_{t,l})$, for every $t, l \in \mathbb{N}$. We prove that the m_j 's

are \mathcal{L} -uniformly (s) -bounded with respect to the regulator $(e_{t,l})_{t,l}$. Otherwise, there is $\varphi \in \mathbb{N}^{\mathbb{N}}$ with the property that for every $h \in \mathbb{N}$ there are $j_h, n_h \in \mathbb{N}$ with $n_h \geq h$ and $B_h \in \mathcal{L}$ with $B_h \subset H_{n_h}$ and

$$m_{j_h}(B_h) \not\leq \bigvee_{t=1}^{\infty} e_{t,\varphi(t)}. \tag{15}$$

By 2.14.1), for every $h \in \mathbb{N}$ there is $A_h \in \mathcal{H}$, $A_h \subset B_h$, with

$$m_{j_h}(B_h \setminus A_h) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{16}$$

From (15) and (16) it follows that

$$m_{j_h}(A_h) \not\leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} : \tag{17}$$

otherwise, thanks to k -triangularity of m_{j_h} , we should get

$$m_{j_h}(B_h) \leq m_{j_h}(A_h) + k m_{j_h}(B_h \setminus A_h) \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)},$$

which contradicts (15). Moreover, observe that from 2.14.1), in correspondence with φ , for every h there are $G_h \in \mathcal{G}$ and $F_h \in \mathcal{H}$, with $A_h \subset G_h \subset F_h$ and

$$[v(m_1) \vee \dots \vee v(m_{j_h})](F_h \setminus A_h) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+h)}.$$

Set now $G_1^* = G_1$, $G_{h+1}^* = G_{h+1} \setminus \left(\bigcup_{r=1}^h F_r\right)$, $h \geq 2$. Since the G_h^* 's are disjoint elements of \mathcal{G} , then, thanks to \mathcal{G} -uniform (s) -boundedness and taking into account Proposition 2.13, we find a positive integer h_0 with

$$\bigvee_j v_{\mathcal{L}}(m_j)(G_h^*) = \bigvee_j v_{\mathcal{G}}(m_j)(G_h^*) \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$$

whenever $h \geq h_0$. Since for every h we get $A_{h+1} \setminus G_{h+1}^* \subset \bigcup_{r=1}^h (F_r \setminus A_r)$, then

$$\begin{aligned} m_{j_h}(A_h) &\leq m_{j_h}(A_h \cap G_h^*) + m_{j_h}(A_h \setminus G_h^*) \\ &\leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)} + k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \text{ for every } h \geq h_0, \end{aligned}$$

which contradicts (17), getting \mathcal{L} -uniform (s) -boundedness of the m_j 's. Conditions 2.16.2) and 2.16.1) on uniform regularity of the m_j 's follow easily from Proposition 2.15 and Lemma 3.2 used with $H_n = G_n \setminus W$, $n \in \mathbb{N}$, and $H_n = V_n \setminus K_n$, $\mathcal{G} = \mathcal{H} = \mathcal{L}$, $W = \emptyset$ respectively, where G_n is as in 2.15.1), V_n and K_n are as in 2.14.1). \square

Now we are in position to prove the following theorem, which extends [10, Theorem 3.3].

Theorem 3.4. Let $G, R, \mathcal{L}, \mathcal{G}, \mathcal{H}$ be as above, and suppose that $m_j : \mathcal{L} \rightarrow R, j \in \mathbb{N}$, is a sequence of equibounded, regular, k -triangular and (s) -bounded set functions, such that there exists

$$m_0(E) := (D) \lim_j m_j(E) \text{ for every } E \in \mathcal{G}$$

with respect to a single regulator. Then,

- 3.4.1) the measures $m_j, j \in \mathbb{N}$, are \mathcal{L} -uniformly (s) -bounded and uniformly regular;
- 3.4.2) there exists in R the limit $m_0(E) = (D) \lim_j m_j(E)$ for each $E \in \mathcal{L}$ with respect to a single regulator;
- 3.4.3) the set function m_0 is regular, k -triangular and (s) -bounded.

Proof. 3.4.1) is a consequence of Theorems 3.1 and 3.3.

3.4.2). Choose arbitrarily $E \in \mathcal{L}$, and let $(y_{t,l})_{t,l}$ be a (D) -sequence associated with uniform regularity. For each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $U \in \mathcal{G}$ with $U \supset E$ and $v_{\mathcal{L}}(m_j)(U \setminus E) \leq \bigvee_{t=1}^{\infty} y_{t,\varphi(t)}$ for every $j \in \mathbb{N}$. Moreover, in correspondence with U there is $j_0 \in \mathbb{N}$ with

$$|m_j(U) - m_{j+p}(U)| \leq \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)}$$

for every $j \geq j_0$ and $p \in \mathbb{N}$, where $(\alpha_{t,l})_{t,l}$ is a regulator related to (D) -convergence on \mathcal{G} . By k -triangularity of m_j and m_{j+p} we get

$$\begin{aligned} m_j(E) - m_{j+p}(E) &\leq m_j(U) - m_{j+p}(U) + k m_j(U \setminus E) + k m_{j+p}(U \setminus E), \\ m_{j+p}(E) - m_j(E) &\leq m_{j+p}(U) - m_j(U) + k m_j(U \setminus E) + k m_{j+p}(U \setminus E), \end{aligned}$$

and hence

$$\begin{aligned} |m_j(E) - m_{j+p}(E)| &\leq |m_j(U) - m_{j+p}(U)| + k m_j(U \setminus E) + k m_{j+p}(U \setminus E) \\ &\leq \bigvee_{i=1}^{\infty} (2k + 1)(y_{i,\varphi(i)} + \alpha_{i,\varphi(i)}) \end{aligned} \tag{18}$$

for every $j \geq j_0$ and $p \in \mathbb{N}$. From (18) it follows that the sequence $(m_j(E))_j$ is (D) -Cauchy in R . Since R is a Dedekind complete lattice group, then the sequence $(m_j(E))_j$ is (D) -convergent, with respect to a regulator independent of E (see also [7, 28]). Thus 3.4.2) is proved.

3.4.3). Straightforward. □

The next step is to prove a uniform boundedness theorem for k -triangular regular lattice group-valued set functions. We begin with the following result, which extends [11, Proposition 4.5].

Proposition 3.5. Let $m_h : \mathcal{L} \rightarrow R, h \in \mathbb{N}$, be a sequence of k -triangular set functions, and let $(t_n)_n$ be an increasing sequence of positive elements of R . Suppose also that

3.5.1) for every disjoint sequence $(H_j)_j$ in \mathcal{L} , the set $\{m_h(H_j) : h, j \in \mathbb{N}\}$ is bounded by $(t_n)_n$.

Then the set $\{m_h(A) : h \in \mathbb{N}, A \in \mathcal{L}\}$ is bounded in R .

Proof. First of all observe that, thanks to 3.5.1), for every fixed element $A \in \mathcal{L}$ there is $n = n(A) \in \mathbb{N}$ with $0 \leq m_h(A) \leq t_{n(A)}$ for every $h \in \mathbb{N}$. We now prove that the set $\{m_h(A) : h \in \mathbb{N}, A \in \mathcal{L}\}$ is bounded by the sequence $((k + 1)t_n)_n$. Suppose, by contradiction, that this is not true. By hypothesis, there is $n_1 \in \mathbb{N}$ such that $m_h(G) \leq t_{n_1}$ for all $h \in \mathbb{N}$, where G is as in Theorem 3.1. Moreover, there exist $A_1 \in \mathcal{L}$ and $h_1 \in \mathbb{N}$ such that $m_{h_1}(A_1) \not\leq (k + 1)t_{n_1}$. We have also $m_{h_1}(G \setminus A_1) \not\leq t_{n_1}$: otherwise, by k -triangularity of m_{h_1} and (4) used with $q = 2$, $E_1 = A_1$, $E_2 = G \setminus A_1$, we get

$$m_{h_1}(A_1) \leq m_{h_1}(G) + k m_{h_1}(G \setminus A_1) \leq t_{n_1} + k t_{n_1} = (k + 1)t_{n_1}.$$

It is not difficult to check that either $\{m_h(A \cap A_1) : A \in \mathcal{L}, h \in \mathbb{N}\}$, or $\{m_h(A \setminus A_1) : A \in \mathcal{L}, h \in \mathbb{N}\}$ (or both, possibly) is not bounded in R : otherwise, if

$$u_1 = \bigvee \{m_h(A \cap A_1) : A \in \mathcal{L}, h \in \mathbb{N}\},$$

$$u_2 = \bigvee \{m_h(A \setminus A_1) : A \in \mathcal{L}, h \in \mathbb{N}\},$$

then, thanks to triangularity of the m_h 's, we have

$$0 \leq m_h(A) \leq m_h(A \cap A_1) + k m_h(A \setminus A_1) \leq u_1 + k u_2$$

for each $A \in \mathcal{L}$ and $h \in \mathbb{N}$, and hence the set $\{m_h(A) : A \in \mathcal{L}, h \in \mathbb{N}\}$ is bounded in R , getting a contradiction. In the first case, set $C_1 := A_1$, otherwise put $C_1 := G \setminus A_1$. Then, set $D_1 := G \setminus C_1$. Now we use the same argument as above, by replacing G by C_1 : so we find a set $A_2 \subset C_1$, $A_2 \in \mathcal{L}$ and two integers $n_2 > n_1$, $h_2 > h_1$, with $m_{h_2}(A_2) \not\leq (k + 1)t_{n_2}$ and $m_{h_2}(C_1 \setminus A_2) \not\leq t_{n_2}$. Put $C_2 := A_2$ or $C_2 := C_1 \setminus A_2$ according as the $\{m_h(A \cap A_2) : A \in \mathcal{L}, h \in \mathbb{N}\}$ or $\{m_h(A \setminus A_2) : A \in \mathcal{L}, h \in \mathbb{N}\}$ is bounded, set $D_2 := C_1 \setminus C_2$, and let us repeat the same argument as above. Proceeding by induction, we find a disjoint sequence $(D_j)_j$ and two strictly increasing sequences $(n_j)_j, (h_j)_j$ in \mathbb{N} with $m_{h_j}(D_j) \not\leq t_{n_j}$ for every $j \in \mathbb{N}$, obtaining a contradiction with 3.5.1). This ends the proof. \square

We now turn to our main uniform boundedness theorem for regular and k -triangular lattice group-valued set functions, which extends [11, Theorem 4.6].

Theorem 3.6. Let $\mu_j : \mathcal{L} \rightarrow R$, $j \in \mathbb{N}$, be a (RD) -regular sequence of k -triangular set functions, and suppose that there is an increasing sequence $(t_n)_n$ of positive elements of R such that for every $U \in \mathcal{G}$ the set $\{\mu_j(U) : j \in \mathbb{N}\}$ is bounded by $(t_n)_n$.

Then the set $\{\mu_j(E) : j \in \mathbb{N}, E \in \mathcal{L}\}$ is bounded in R .

Proof. Choose arbitrarily $E \in \mathcal{L}$. By 2.14.1), there are a (D) -sequence $(a_{t,l})_{t,l}$, which can be taken independently of E , and a set $U \in \mathcal{G}$, $U \supset E$, with

$$v(\mu_j)(U \setminus E) \leq \bigvee_{t,l=1}^{\infty} a_{t,l} \quad \text{for every } j \in \mathbb{N}. \tag{19}$$

For each $n \in \mathbb{N}$, put $w_n := t_n + \bigvee_{t,l=1}^\infty a_{t,l}$. Taking into account k -triangularity of μ_j , in correspondence with U there is $\bar{n} \in \mathbb{N}$ with

$$\begin{aligned} \mu_j(E) &\leq \mu_j(U) + k v(\mu_j)(U \setminus E) \leq w_{\bar{n}}, \\ -\mu_j(E) &\leq -\mu_j(U) + k v(\mu_j)(U \setminus E) \leq w_{\bar{n}} \end{aligned}$$

for every $j \in \mathbb{N}$. Thus the set $\{\mu_j(E) : j \in \mathbb{N}\}$ is bounded by the sequence $(w_n)_n$.

By virtue of Proposition 3.5, it will be enough to prove that, for every disjoint sequence $(H_n)_n$ in \mathcal{L} , the set $\{\mu_j(H_n) : j, n \in \mathbb{N}\}$ is bounded by the sequence $(y_n)_n$, where $y_n = k n w_n, n \in \mathbb{N}$.

Proceeding by contradiction, assume that there is a disjoint sequence $(H'_n)_n$ in \mathcal{L} , such that the set $\{\mu_j(H'_n) : j, n \in \mathbb{N}\}$ is not bounded by $(y_n)_n$. For every $n \in \mathbb{N}$ there are $i(n), h(n) \in \mathbb{N}$ with

$$\mu_{h(n)}(H'_{i(n)}) \not\leq (k n + 1)w_n. \tag{20}$$

For each $n \in \mathbb{N}$, set $m_n = \mu_{h(n)}, H_n = H'_{i(n)}$. By 2.14.2), the (D) -sequence $(a_{t,l})$ in (19) has the property that for every $n \in \mathbb{N}$ there exists a set $O_n \in \mathcal{G}$ with

$$O_n \supset H_n \text{ for each } n \in \mathbb{N} \text{ and } (D) \lim_n v(m_j) \left(\bigcup_{i=n}^\infty O_i \right) = 0 \text{ for every } j \in \mathbb{N} \tag{21}$$

with respect to $(a_{t,l})_{t,l}$. Hence, there is an integer $n_1 > 1$ with $m_1(E) \leq \bigvee_{t,l=1}^\infty a_{t,l}$ for every $E \in \mathcal{L}, E \subset \bigcup_{i=n_1}^\infty O_i$, and a fortiori for each $E \in \mathcal{L}, E \subset \bigcup_{i=n_1}^\infty H_i$. We get

$$m_1(E \cup H_1) \not\leq w_1 \text{ for each } E \in \mathcal{L}, E \subset \bigcup_{i=n_1}^\infty H_i :$$

otherwise, by k -triangularity of m_1 and (4) used with $q = 2, E_1 = H_1, E_2 = E$, we have

$$m_1(H_1) \leq m_1(E \cup H_1) + k m_1(E) \leq w_1 + k \bigvee_{t,l=1}^\infty a_{t,l} \leq (k + 1)w_1,$$

which contradicts (20). Let $j_2 > n_1$ be an integer such that

$$\bigvee \{m_n(H_1) : n \in \mathbb{N}\} \leq t_{j_2}.$$

By 2.14.2), in correspondence with m_{j_2} there is an integer $n_2 > j_2$ such that $m_{j_2}(E) \leq \bigvee_{t,l=1}^\infty a_{t,l}$ for any $E \in \mathcal{L}, E \subset \bigcup_{i=n_2}^\infty H_i$. For such E 's we have

$$m_{j_2}(E \cup H_1 \cup H_{j_2}) \not\leq w_{j_2} :$$

otherwise, by k -triangularity of m_{j_2} and (4) used with $q = 3, E_1 = H_{j_2}, E_2 = E, E_3 = H_1$, we get

$$\begin{aligned} m_{j_2}(H_{j_2}) &\leq m_{j_2}(E \cup H_1 \cup H_{j_2}) + k m_{j_2}(E) + k m_{j_2}(H_1) \\ &\leq w_{j_2} + k \bigvee_{t,l=1}^\infty a_{t,l} + k w_{j_2} \leq 3 k w_{j_2} \leq (k j_2 + 1)w_{j_2}, \end{aligned}$$

which contradicts (20). Let $j_3 > n_2$ be an integer such that

$$\bigvee \{m_n(H_{j_2}) : n \in \mathbb{N}\} \leq w_{j_3}.$$

By 2.14.2), in correspondence with m_{j_3} there is $n_3 > j_3$ with $m_{j_3}(E) \leq \bigvee_{t,l=1}^\infty a_{t,l}$ for every $E \in \mathcal{L}$, $E \subset \bigcup_{i=n_3}^\infty H_i$. For such E 's we have

$$m_{j_3}(E \cup H_1 \cup H_{j_2} \cup H_{j_3}) \not\leq w_{j_3} :$$

otherwise, by k -triangularity of m_{j_3} and (4) used with $q = 4$, $E_1 = H_{j_3}$, $E_2 = E$, $E_3 = H_1$, $E_4 = H_{j_2}$, we get

$$\begin{aligned} m_{j_3}(H_{j_3}) &\leq m_{j_3}(E \cup H_1 \cup H_{j_2} \cup H_{j_3}) + k m_{j_3}(E) + k m_{j_3}(H_1) + k m_{j_3}(H_{j_2}) \\ &\leq w_{j_3} + k \bigvee_{t,l=1}^\infty a_{t,l} + k w_{j_2} + k w_{j_3} \\ &\leq 4k w_{j_3} \leq (k j_3 + 1)w_{j_3}, \end{aligned}$$

which contradicts (20). Proceeding by induction, it is possible to construct two strictly increasing sequences $(j_h)_h, (n_h)_h$, such that $n_h > j_h \geq h$ for every $h \in \mathbb{N}$, and

$$m_{j_h}(E \cup H_1 \cup H_{j_2} \cup \dots \cup H_{j_h}) \not\leq w_{j_h}$$

whenever $h \in \mathbb{N}$ and $E \in \mathcal{L}$ with $E \subset \bigcup_{i=n_h}^\infty H_i$.

Set $j_1 = 1$ and $H = \bigcup_{h=1}^\infty H_{j_h}$. Note that $H \in \mathcal{G}$ and $m_{j_h}(H) \not\leq w_{j_h}$ for every $h \in \mathbb{N}$. But the set $\{m_h(H) : h \in \mathbb{N}\}$ is bounded by the sequence $(w_n)_n$, and so we get a contradiction. This ends the proof. □

Remark 3.7. Now we show that, under certain hypotheses, regular and k -triangular set functions, with values in spaces of type L^0 as in Remark 2.2, are (RD) -regular too. Note that, by means of techniques analogous to those used below, this result can be proved for finite dimensional space-valued set functions, extending to the k -triangular context some classical theorems proved in the finitely additive setting (see for instance [29, Theorem 2]).

Let $R = L^0 = L^0([0, 1], \mathcal{B}, \lambda)$ be as in Remark 2.2, G be a compact Hausdorff topological space, \mathcal{L} be the σ -algebra of all Borel subsets of G , \mathcal{G} and \mathcal{H} be the classes of all open and of all compact subsets of G , respectively. First of all, observe that 2.9.2) is a consequence of 2.9.1). Indeed, pick arbitrarily $W \in \mathcal{H}$ and let $(V_n)_n$ be a sequence of elements of \mathcal{G} , satisfying 2.9.1). Since G is compact and Hausdorff, G is also normal (see also [36, Theorem XI.1.2]). As G is normal, thanks to [36, Proposition VII.3.2], in correspondence with W and V_1 there is a set $U_1 \in \mathcal{G}$ with $W \subset U_1 \subset \overline{U_1} \subset V_1$, where $\overline{U_1}$ denotes the topological closure of U_1 in G . Analogously, we can associate to W and $U_1 \cap V_2$ a set $U_2 \in \mathcal{G}$ with $W \subset U_2 \subset \overline{U_2} \subset U_1 \cap V_2$. Proceeding by induction, we construct a decreasing sequence $(U_n)_n$ in \mathcal{G} , with $W \subset U_{n+1} \subset \overline{U_{n+1}} \subset U_n \cap V_{n+1}$. Since the sequence $(V_n)_n$ satisfies 2.9.1), it is not difficult to see that the sequences $(U_n)_n$ and $(\overline{U_n})_n$ fulfil 2.9.2).

Let $m_j : \mathcal{L} \rightarrow R$, $j \in \mathbb{N}$, be a sequence of k -triangular and regular set functions. We will prove that $(m_j)_j$ satisfies 2.14.1) and 2.14.2). Since in L^0 the (r) -, (O) - and (D) -convergences coincide (see Remark 2.2), then for every $j \in \mathbb{N}$ there exists $u_j \in R$, $u_j \geq 0$, such that for every $E \in \mathcal{L}$ there are two sequences $(V_n^{(j)})_n$ in \mathcal{G} and $(K_n^{(j)})_n$ in \mathcal{H} , with $V_n^{(j)} \supset E \supset K_n^{(j)}$ for each n and such that for every $\varepsilon > 0$ there is a positive integer $n_0 = n_0(\varepsilon, j, E)$ with

$$v(m_j)(V_n^{(j)} \setminus K_n^{(j)}) \leq \varepsilon u_j \quad \text{whenever } n \geq n_0. \tag{22}$$

For every $n \in \mathbb{N}$, set $V_n := \bigcap_{j=1}^n V_n^{(j)}$, $K_n := \bigcup_{j=1}^n K_n^{(j)}$: note that $V_n \in \mathcal{G}$, $K_n \in \mathcal{H}$ and $V_n \supset E \supset K_n$ for every n . Since R satisfies property (σ) , in correspondence with the sequence $(u_j)_j$ there exist a sequence $(a_j)_j$ of positive real numbers and an element $u \in R$, $u \geq 0$, with $0 \leq a_j u_j \leq u$ for every $j \in \mathbb{N}$. Note that u does not depend on the choice of $E \in \mathcal{L}$. For every $\varepsilon > 0$, $j \in \mathbb{N}$ and $E \in \mathcal{L}$, let $n_* = n_*(\varepsilon, j, E) = n_0(\varepsilon a_j, j, E)$, where n_0 is as in (22). We get

$$v(m_j)(V_n \setminus K_n) \leq v(m_j)(V_n^{(j)} \setminus K_n^{(j)}) \leq \varepsilon a_j u_j \leq \varepsilon u \tag{23}$$

for each $n \geq n_*$. If we take $\sigma_p = \frac{1}{p}u$, $p \in \mathbb{N}$, then it is not difficult to check that 2.14.1) is satisfied.

We now prove 2.14.2). Choose any disjoint sequence $(H_n)_n$ in \mathcal{L} and let u be as in (23). In correspondence with j , $n \in \mathbb{N}$ and $\frac{1}{k 2^{n+j+1}}$ set $O_n^{(j)} = O_n^{(j)}\left(\frac{1}{k 2^{n+j+1}}\right) = V_{n_*\left(\frac{1}{k 2^{n+j+1}}, j, H_n\right)}$ and $F_n^{(j)} = F_n^{(j)}\left(\frac{1}{k 2^{n+j+1}}\right) = K_{n_*\left(\frac{1}{k 2^{n+j+1}}, j, H_n\right)}$, where n_* is as in (23). For each $n \in \mathbb{N}$, put $O_n = \bigcap_{j=1}^n O_n^{(j)}$ and $F_n = \bigcup_{j=1}^n F_n^{(j)}$. Note that $O_n \in \mathcal{G}$, $F_n \in \mathcal{H}$ and $O_n \supset H_n \supset F_n$ for each n . Moreover, from (23) we get

$$v(m_j)(O_n \setminus F_n) \leq v(m_j)(O_n^{(j)} \setminus F_n^{(j)}) \leq \frac{1}{k 2^{n+j+1}} u \quad \text{for every } j, n \in \mathbb{N}. \tag{24}$$

Now, for each $n \in \mathbb{N}$ set $U_n := \bigcup_{i=n}^\infty O_i$, $C_n := \bigcap_{i=n}^\infty F_i$. Since the sequence $(H_n)_n$ is disjoint and $F_n \subset H_n$ for every $n \in \mathbb{N}$, then $C_n = \emptyset$ for every $n \in \mathbb{N}$. Taking into account (7), from (24) we get

$$\begin{aligned} v(m_j)(U_n) &= v(m_j)(U_n \setminus C_n) = v(m_j)\left(\left(\bigcup_{i=n}^\infty O_i\right) \setminus \left(\bigcap_{i=n}^\infty F_i\right)\right) \\ &= v(m_j)\left(\bigcup_{i=n}^\infty (O_i \setminus F_i)\right) \leq k \sum_{i=n}^\infty v(m_j)(O_i \setminus F_i) \leq k \sum_{i=n}^\infty \frac{1}{k 2^{i+j+1}} u = \frac{1}{2^{n+j}} u \end{aligned} \tag{25}$$

(see also [39, Lemma 1]). Thus 2.14.2) is proved. □

The following example shows that, in Theorem 3.5, in general the condition 3.5.1) cannot be replaced by the boundedness of the set $\{m_j(U) : j \in \mathbb{N}\}$.

Example 3.8. (see also Schwartz [46, Example 5]) Let R be the vector lattice c_0 of all real sequences convergent to 0, endowed with the usual ordering, \mathcal{B} be the σ -algebra of all Borel subsets of $[0, 1]$. Note that c_0 is Dedekind complete and weakly σ -distributive, and that in c_0 order, (D) - and (r) -convergence coincide with coordinatewise convergence dominated by an element of c_0 (see also [28, 46, 48]). For every $n \in \mathbb{N}$ and $E \in \mathcal{B}$ set $m_n(E) = (\mu_1(E), \dots, \mu_n(E), 0, \dots, 0, \dots)$, where $\mu_n(E) = \int_E \sin(n\pi x) dx$. It is known (see [46]) that every m_n is a σ -additive measure and the set $\{m_n(E) : n \in \mathbb{N}\}$ is bounded in c_0 for every $E \in \mathcal{B}$. However, it is not possible to find a positive increasing sequence $(t_n)_n$ satisfying the hypothesis of Theorem 3.6, since $\sup\{\mu_n(A) : A \in \mathcal{B}\} = 1$ for each n . Moreover, from this it follows that the set $\{m_n(E) : n \in \mathbb{N}, E \in \mathcal{B}\}$ is not bounded in c_0 .

Open problems:

- (a) Prove similar results with respect to other kinds of (s) -boundedness, boundedness and/or convergence, and relatively to different types of variations in the setting of non-additive lattice-group valued set functions (see also [21, 41]).
- (b) Find some other conditions under which 2.14.1) and/or 2.14.2) hold.

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*Antonio Boccuto, Corresponding author. Dipartimento di Matematica e Informatica, University of Perugia, via Vanvitelli 1, I-06123 Perugia. Italy.
e-mail: antonio.boccuto@unipg.it*

*Xenofon Dimitriou, Department of Mathematics, University of Athens, Panepistimiopolis, GR-15784 Athens. Greece.
e-mail: dxenof@math.uoa.gr*