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INEQUALITIES FOR THE ARITHMETICAL FUNCTIONS OF EULER AND DEDEKIND

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Abstract. For positive integers n, Euler's phi function and Dedekind's psi function are given by

$$\varphi(n) = n \prod_{\substack{p \mid n \\ p \text{ prime}}} \Big(1 - \frac{1}{p}\Big) \quad \text{and} \quad \psi(n) = n \prod_{\substack{p \mid n \\ p \text{ prime}}} \Big(1 + \frac{1}{p}\Big),$$

respectively. We prove that for all $n \geqslant 2$ we have

$$\Big(1 - \frac{1}{n}\Big)^{n-1} \Big(1 + \frac{1}{n}\Big)^{n+1} \leqslant \Big(\frac{\varphi(n)}{n}\Big)^{\varphi(n)} \Big(\frac{\psi(n)}{n}\Big)^{\psi(n)}$$

and

$$\Big(\frac{\varphi(n)}{n}\Big)^{\psi(n)} \Big(\frac{\psi(n)}{n}\Big)^{\varphi(n)} \leqslant \Big(1 - \frac{1}{n}\Big)^{n+1} \Big(1 + \frac{1}{n}\Big)^{n-1}.$$

The sign of equality holds if and only if n is a prime. The first inequality refines results due to Atanassov (2011) and Kannan & Srikanth (2013).

Keywords: Euler's phi function; Dedekind's psi function; inequalities

MSC 2020: 11A25

1. Introduction

In this paper, we are concerned with two classical arithmetical functions: Euler's phi (or totient) function φ and Dedekind's psi function ψ. If $n \in \mathbb{N}$, then $\varphi(n)$ is the number of positive integers up to n which are relatively prime to n. The product formula

$$\varphi(n) = n \prod_{\substack{p \mid n \\ p \text{ prime}}} \Big(1 - \frac{1}{p}\Big)$$

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is an important tool for calculating $\varphi(n)$. Closely related to the φ-function is Dedekind's psi function given by

$$\psi(n) = n \prod_{\substack{p \mid n \\ p \text{ prime}}} \Big(1 + \frac{1}{p}\Big).$$

Both functions can be expressed in terms of the Möbius function μ which is defined for $n \in \mathbb{N}$ by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is square-free and } k \text{ is the number of prime factors of } n, \\ 0 & \text{if } n \text{ is not square-free.} \end{cases}$$

We have the representations

$$\varphi(n) = \sum_{d \mid n} \frac{n}{d} \mu(d) \quad \text{and} \quad \psi(n) = \sum_{d \mid n} \frac{n}{d} |\mu(d)|.$$

These functions have remarkable applications in various mathematical and physical problems. Recently, Solé and Planat in [8] proved that a certain inequality involving the Dedekind psi function is equivalent to the famous Riemann hypothesis. For more information on this subject we refer the reader to Apostol, see [1], Mitrinović et al., see [4], Sándor, see [5], Sándor and Crstici, see [7] and the references cited therein.

Our work is motivated by two interesting papers published by Atanassov, see [2] and Kannan and Srikanth, see [3]. Atanassov proved that for $n \geqslant 2$,

$$n^{2n} < \varphi(n)^{\varphi(n)} \psi(n)^{\psi(n)}. \tag{1.1}$$

Kannan and Srikanth claimed that if

$$\Theta(n) = \frac{\varphi(n) + \psi(n)}{2n},$$

then for $n \geqslant 2$,

$$n^{2n\Theta(n)} \leqslant \varphi(n)^{\varphi(n)} \psi(n)^{\psi(n)}. \tag{1.2}$$

Since $\Theta(n) \geqslant 1$ $(n \geqslant 2)$, it follows that (1.2) refines (1.1). Unfortunately, the proof given in [3] is incorrect. For instance, the inequality in the last line of page 20 is false.

A correct proof of (1.2) was given by Sándor, see [6], page 51, who actually offered an improvement of (1.2):

$$n^{2n\Theta(n)} \leqslant \Big(\frac{\varphi(n) + \psi(n)}{2}\Big)^{2n\Theta(n)} < \varphi(n)^{\varphi(n)} \psi(n)^{\psi(n)} \quad (n \geqslant 2). \tag{1.3}$$

The referee pointed out that a slight modification of the proof of (1.3) leads to the following chain of inequalities:

$$\varphi(n)^{\psi(n)}\psi(n)^{\varphi(n)} < \Big(\frac{1}{2}\Big(\frac{1}{\varphi(n)}+\frac{1}{\psi(n)}\Big)\Big)^{-2n\Theta(n)} < (\varphi(n)\psi(n))^{n\Theta(n)} < n^{2n\Theta(n)} \quad (n \geqslant 2). \tag{1.4}$$

From (1.3) and (1.4) we conclude that the elegant inequalities

$$1 < \Big(\frac{\varphi(n)}{n}\Big)^{\varphi(n)}\Big(\frac{\psi(n)}{n}\Big)^{\psi(n)} \tag{1.5}$$

and

$$\Big(\frac{\varphi(n)}{n}\Big)^{\psi(n)}\Big(\frac{\psi(n)}{n}\Big)^{\varphi(n)} < 1 \tag{1.6}$$

are valid for $n \geqslant 2$. It is the aim of this paper to present refinements of (1.5) and (1.6). Moreover, we show that the constant bounds given in (1.5) and (1.6) are best possible.

In Section 2, we collect several lemmas. Our refinements of (1.5) and (1.6) (as stated in the Abstract) are presented in Section 3.

2. Lemmas

The following seven lemmas are helpful to prove the two theorems given in the next section.

Lemma 2.1. *Let*

$$h(x) = (1-x)^{1-x}(1+x)^{1+x} \tag{2.1}$$

and

$$g(x) = (1-x)^{1+x}(1+x)^{1-x}. \tag{2.2}$$

Then h is strictly increasing on $[0,1]$ and g is strictly decreasing on $[0,1]$.

Proof. We have

$$h'(x) = h(x)\log\Big(\frac{1+x}{1-x}\Big) \quad \text{and} \quad g'(x) = -g(x)\Big(\frac{4x}{1-x^2}+\log\Big(\frac{1+x}{1-x}\Big)\Big).$$

This gives $h'(x) > 0$ and $g'(x) < 0$ for $x \in (0,1)$. $\square$

Lemma 2.2. *Let $0 < s,\, t < 1$. Then*

$$(2.3)\qquad (1-st)^{1-st}(1+st)^{1+st} < [(1-s)(1-t)]^{(1-s)(1-t)}[(1+s)(1+t)]^{(1+s)(1+t)}.$$

Proof. Let $0 < r < 1 < a$ and

$$q(a,r) = (a+r)\log(a+r) - (a-r)\log(a-r).$$

Since

$$\frac{\partial}{\partial a} q(a,r) = \log\Bigl(\frac{a+r}{a-r}\Bigr) > 0, \quad \frac{\mathrm{d}^2}{\mathrm{d}r^2} q(1,r) = -\frac{2r}{1-r^2} < 0,$$
$$q(1,0) = 0, \quad q(1,1) = 2\log(2),$$

we obtain

$$q(a,r) > q(1,r) > 0.$$

Thus,

$$(a+r)^{a+r} > (a-r)^{a-r}.$$

This gives

$$[(1+s)(1+t)]^{(1+s)(1+t)} = (1+s+t+st)^{1+s+t+st} > (1+s+t-st)^{1+s+t-st}.$$

Let $R(s,t)$ be the expression on the right-hand side of (2.3). Then

$$(2.4)\qquad R(s,t) > (1-s-t+st)^{1-s-t+st}(1+s+t-st)^{1+s+t-st} = h(s+t-st),$$

where h is defined in (2.1). Since $0 < st < s+t-st < 1$, we conclude from Lemma 2.1 that

$$(2.5)\qquad h(s+t-st) > h(st).$$

Combining (2.4) and (2.5) gives (2.3). $\square$

Throughout, we use the following notations:

$$(2.6)\qquad \sigma = \prod_{k=1}^{m} s_k, \quad P = \prod_{k=1}^{m}(1-s_k), \quad Q = \prod_{k=1}^{m}(1+s_k).$$

Lemma 2.3. *Let $s_k \in (0,1)$ $(k = 1, \ldots, m)$. Then*

$$(2.7)\qquad \sigma + P \leqslant 1.$$

P r o o f. We use induction. If $m = 1$, then $\sigma + P = 1$. Next, we assume that (2.7) holds. Let $s = s_{m+1}$. Then

$$\sigma s + P(1-s) \leqslant (1-P)s + P(1-s) \leqslant s + 1 - s = 1.$$

This means that (2.7) is valid if σ and P have $m+1$ factors. $\square$

Lemma 2.4. *Let* $s_k \in (0, \frac{1}{2}]$ $(k = 1, \dots, m)$. *Then*

$$P^P \leqslant (1-\sigma)^{1-\sigma}. \tag{2.8}$$

P r o o f. We have $\sigma \leqslant P$ and $\sigma + P \leqslant 1$. Next, we apply Bernoulli's inequality

$$(1+t)^a \geqslant 1 + at \quad (t \geqslant -1,\ a \geqslant 1)$$

with $t = -\sigma$ and $a = (1-\sigma)/P$. Then we have

$$(1-\sigma)^{(1-\sigma)/P} = (1+t)^a \geqslant 1 + at = 1 + \frac{1-\sigma}{P}(-\sigma) = \frac{1}{P}(P-\sigma)(1-P-\sigma) + P \geqslant P.$$

This leads to (2.8). $\square$

Lemma 2.5. *Let* $s_k \in (0, \frac{1}{2}]$ $(k = 1, \dots, m)$ *and* $t \in (0,1)$. *If*

$$(1-\sigma)^{1-\sigma}(1+\sigma)^{1+\sigma} \leqslant P^P Q^Q, \tag{2.9}$$

then

$$1 \leqslant \left(\frac{P^P}{(1-\sigma)^{1-\sigma}}\right)^{1-t} \left(\frac{Q^Q}{(1+\sigma)^{1+\sigma}}\right)^{1+t}. \tag{2.10}$$

P r o o f. Since $0 < a$, $b \leqslant 1$ implies $a^b \geqslant a$, we conclude from (2.8) that

$$\left(\frac{P^P}{(1-\sigma)^{1-\sigma}}\right)^{1-t} \geqslant \frac{P^P}{(1-\sigma)^{1-\sigma}}. \tag{2.11}$$

Using (2.8) and (2.9) gives

$$\frac{Q^Q}{(1+\sigma)^{1+\sigma}} \geqslant \frac{(1-\sigma)^{1-\sigma}}{P^P} \geqslant 1. \tag{2.12}$$

Since $a, b \geqslant 1$ implies $a^b \geqslant a$, we obtain from (2.12)

$$\left(\frac{Q^Q}{(1+\sigma)^{1+\sigma}}\right)^{1+t} \geqslant \frac{Q^Q}{(1+\sigma)^{1+\sigma}}. \tag{2.13}$$

Finally, from (2.9), (2.11) and (2.13) we conclude that (2.10) is valid. $\square$

Let

$$f(t_1,\ldots,t_m)=\prod_{i=1}^{m}(1+t_i)\sum_{i=1}^{m}\log(1-t_i)+\prod_{i=1}^{m}(1-t_i)\sum_{i=1}^{m}\log(1+t_i) \tag{2.14}$$

with $t_i\in(0,1)$ $(i=1,\ldots,m)$.

Lemma 2.6. *Let $j\in\{1,\ldots,m\}$ and*

$$F_j(x)=f(t_1,\ldots,t_{j-1},x,t_{j+1},\ldots,t_m),$$

where f is defined in (2.14). *If $t_i\in(0,\frac12]$ $(i=1,\ldots,m,\ i\neq j)$, then F_j is strictly decreasing on $(0,\frac12]$.*

Proof. Let $x\in(0,\frac12]$. We have

$$F_j(x)=D_j(1+x)\log(C_j(1-x))+C_j(1-x)\log(D_j(1+x))$$

with

$$C_j=\prod_{\substack{i=1\\ i\neq j}}^{m}(1-t_i)\quad\text{and}\quad D_j=\prod_{\substack{i=1\\ i\neq j}}^{m}(1+t_i).$$

Since $0<C_j<1<D_j$, we obtain

$$F_j''(x)=-\left(C_j\frac{3+x}{(1+x)^2}+D_j\frac{3-x}{(1-x)^2}\right)<0.$$

This yields

$$F_j'(x)<F_j'(0)=C_j-D_j+D_j\log(C_j)-C_j\log(D_j)<0,$$

which implies that F_j is strictly decreasing on $(0,\frac12]$. □

Lemma 2.7. *Let f be the function defined in* (2.14) *and $s\in(0,\frac12]$. Then*

$$f(s,\ldots,s)\leqslant\log((1-s)^{1+s}(1+s)^{1-s}).$$

Proof. We define for $\alpha\in\mathbb{R}$, $\alpha\geqslant1$ and $s\in(0,\frac12]$:

$$K(\alpha,s)=\alpha(1+s)^{\alpha}\log(1-s)+\alpha(1-s)^{\alpha}\log(1+s).$$

Partial differentiation yields

$$\frac{\partial}{\partial\alpha}K(\alpha,s)=Y(\alpha,s)+Z(\alpha,s)$$

with

$$Y(\alpha,s)=(1+s)^{\alpha}\log(1-s)+(1-s)^{\alpha}\log(1+s)$$

and

$$Z(\alpha,s)=\alpha[(1+s)^{\alpha}+(1-s)^{\alpha}]\log(1-s)\log(1+s).$$

We have $Z(\alpha,s)<0$ and

$$\alpha\frac{\partial}{\partial\alpha}Y(\alpha,s)=Z(\alpha,s).$$

It follows that

$$Y(\alpha,s)\leqslant Y(1,s)=\log(g(s)), \tag{2.15}$$

where g is defined in (2.2). Since $g(0)=1$, we conclude from Lemma 2.1 and (2.15) that $Y(\alpha,s)<0$. Thus,

$$\frac{\partial}{\partial\alpha}K(\alpha,s)<0 \quad\text{and}\quad K(\alpha,s)\leqslant K(1,s).$$

In particular, for $m\in\mathbb{N}$,

$$f(s,\ldots,s)=K(m,s)\leqslant K(1,s)=\log\big((1-s)^{1+s}(1+s)^{1-s}\big).$$

$\square$

3. Main results

We use the following notations:

$$S(n)=\Big(\frac{\varphi(n)}{n}\Big)^{\varphi(n)}\Big(\frac{\psi(n)}{n}\Big)^{\psi(n)} \quad\text{and}\quad T(n)=\Big(\frac{\varphi(n)}{n}\Big)^{\psi(n)}\Big(\frac{\psi(n)}{n}\Big)^{\varphi(n)}.$$

Our first theorem provides a refinement of (1.5).

Theorem 3.1. *For all integers $n\geqslant 2$ we have*

$$\Big(1-\frac{1}{n}\Big)^{n-1}\Big(1+\frac{1}{n}\Big)^{n+1}\leqslant S(n). \tag{3.1}$$

The sign of equality holds if and only if n is a prime.

P r o o f. Let

$$n = p_1^{r_1} \dots p_m^{r_m},$$

where $p_1, \dots, p_m$ are prime numbers and $r_1, \dots, r_m$ are positive integers. We distinguish two cases.

Case 1: n is square-free. Then we have $n = p_1 \dots p_m$. We prove (3.1) by induction on m. If $m = 1$, then

$$\varphi(n) = p_1 - 1 \quad \text{and} \quad \psi(n) = p_1 + 1.$$

It follows that the sign of equality holds in (3.1).

Next, we assume that (3.1) is valid if n has m distinct prime factors. This gives

$$(1-\sigma)^{1-\sigma}(1+\sigma)^{1+\sigma} \leqslant P^P Q^Q, \tag{3.2}$$

where σ, P and Q are defined in (2.6) with $s_k = 1/p_k \in (0, \frac{1}{2}]$ $(k = 1, \dots, m)$.

Now, we suppose that n has one more prime factor p_{m+1}. Then we have to show that

$$(1-\sigma t)^{1-\sigma t}(1+\sigma t)^{1+\sigma t} < [P(1-t)]^{P(1-t)}[Q(1+t)]^{Q(1+t)}, \tag{3.3}$$

where $t = 1/p_{m+1} \in (0, \frac{1}{2}]$. Let

$$A = (1-t)^{(P-1+\sigma)(1-t)} \quad \text{and} \quad B = (1+t)^{(Q-1-\sigma)(1+t)}.$$

Since $P + \sigma \leqslant 1 \leqslant Q - \sigma$, we have $A \geqslant 1$ and $B \geqslant 1$. Using (3.2) we conclude that (2.10) holds. Thus,

$$1 \leqslant AB\left(\frac{P^P}{(1-\sigma)^{1-\sigma}}\right)^{1-t}\left(\frac{Q^Q}{(1+\sigma)^{1+\sigma}}\right)^{1+t}$$

which is equivalent to

$$\begin{aligned}[(1-\sigma)(1-t)]^{(1-\sigma)(1-t)}&[(1+\sigma)(1+t)]^{(1+\sigma)(1+t)} \\ &\leqslant [P(1-t)]^{P(1-t)}[Q(1+t)]^{Q(1+t)}.\end{aligned} \tag{3.4}$$

Combining (3.4) and (2.3) (with $s = \sigma$) we obtain (3.3).

Case 2: n is not square-free. Let $k = p_1 \dots p_m$. Using the result proved in Case 1 gives

$$\left(1-\frac{1}{k}\right)^{1-1/k}\left(1+\frac{1}{k}\right)^{1+1/k} \leqslant \left(\frac{\varphi(k)}{k}\right)^{\varphi(k)/k}\left(\frac{\psi(k)}{k}\right)^{\psi(k)/k}. \tag{3.5}$$

Since $1/n < 1/k$, we obtain from Lemma 2.1

$$(3.6)\quad \Big(1-\frac{1}{n}\Big)^{1-1/n}\Big(1+\frac{1}{n}\Big)^{1+1/n} = h(1/n) < h(1/k) = \Big(1-\frac{1}{k}\Big)^{1-1/k}\Big(1+\frac{1}{k}\Big)^{1+1/k}.$$

Combining (3.5), (3.6) and

$$\frac{\varphi(k)}{k} = \frac{\varphi(n)}{n}, \qquad \frac{\psi(k)}{k} = \frac{\psi(n)}{n}$$

reveals that (3.1) holds with "$<$" instead of "$\leqslant$". This completes the proof of Theorem 3.1. □

The following companion of (3.1) is valid.

Theorem 3.2. *For all integers $n \geqslant 2$ we have*

$$(3.7)\qquad T(n) \leqslant \Big(1-\frac{1}{n}\Big)^{n+1}\Big(1+\frac{1}{n}\Big)^{n-1}.$$

The sign of equality holds if and only if n is a prime.

Proof. Let

$$n = p_1^{r_1}\dots p_m^{r_m},$$

where $p_1,\dots,p_m$ are prime numbers with $p_1 \geqslant \dots \geqslant p_m$ and $r_1,\dots,r_m$ are positive integers. We set $t_i = 1/p_i \in (0,\frac{1}{2}]$ $(i=1,\dots,m)$. Then we have

$$\frac{\varphi(n)}{n} = \prod_{i=1}^{m}(1-t_i) \quad \text{and} \quad \frac{\psi(n)}{n} = \prod_{i=1}^{m}(1+t_i).$$

This leads to

$$(3.8)\qquad \frac{1}{n}\log(T(n)) = f(t_1,\dots,t_m),$$

where f is given in (2.14). Since

$$0 < t_1 \leqslant \dots \leqslant t_m \leqslant \frac{1}{2},$$

we conclude from Lemmas 2.6 and 2.7 that

$$(3.9)\qquad f(t_1,\dots,t_m) \leqslant f(t_1,\dots,t_1) \leqslant \log((1-t_1)^{1+t_1}(1+t_1)^{1-t_1}).$$

Let g be the function defined in (2.2). Since $1/n \leqslant 1/p_1$, Lemma 2.1 implies that

$$(3.10)\qquad (1-t_1)^{1+t_1}(1+t_1)^{1-t_1} = g\Big(\frac{1}{p_1}\Big) \leqslant g\Big(\frac{1}{n}\Big) = U(n)^{1/n},$$

where $U(n)$ denotes the expression on the right-hand side of (3.7). Combining (3.8), (3.9) and (3.10) yields (3.7).

If n is a prime number, then $T(n) = U(n)$. Conversely, if $T(n) = U(n)$, then we conclude from (3.8), (3.9) and (3.10) that $g(1/p_1) = g(1/n)$. Since g is strictly monotonic, we obtain $1/p_1 = 1/n$, that is, n is a prime number. □

Remark 3.1. From (1.3) and (1.4) we obtain for $n \geqslant 2$,

$$\Big(\frac{\varphi(n)+\psi(n)}{2n}\Big)^{\varphi(n)+\psi(n)} < S_n$$

and

$$T_n < \left(\frac{2\varphi(n)\psi(n)}{n(\varphi(n)+\psi(n))}\right)^{\varphi(n)+\psi(n)}.$$

The referee asked whether these bounds for S_n and T_n can be compared with those given in (3.1) and (3.7), respectively. This is not possible. Indeed, using the computer program MAPLE 13 for $n = 2, 3, \ldots, 500$ reveals that the differences of the lower and upper bounds attain positive and negative values.

Application of (3.1) and (3.7) leads to inequality (1.5) and its counterpart (1.6).

Corollary 3.1. *For all integers $n \geqslant 2$ we have*

$$1 < S(n) \quad \textit{and} \quad T(n) < 1. \tag{3.11}$$

The constant bounds are best possible.

P r o o f. Using Lemma 2.1 gives for $n \geqslant 2$,

$$1 = h(0) < h(1/n) = \Big[\Big(1-\frac{1}{n}\Big)^{n-1}\Big(1+\frac{1}{n}\Big)^{n+1}\Big]^{1/n} \tag{3.12}$$

and

$$\Big[\Big(1-\frac{1}{n}\Big)^{n+1}\Big(1+\frac{1}{n}\Big)^{n-1}\Big]^{1/n} = g\Big(\frac{1}{n}\Big) < g(0) = 1. \tag{3.13}$$

From (3.1), (3.12), (3.7) and (3.13) we obtain the two estimates in (3.11).

Let p_n be the nth prime number. Then

$$S(p_n) = \Big(1-\frac{1}{p_n}\Big)^{p_n-1}\Big(1+\frac{1}{p_n}\Big)^{p_n+1}$$

and

$$T(p_n) = \Big(1-\frac{1}{p_n}\Big)^{p_n+1}\Big(1+\frac{1}{p_n}\Big)^{p_n-1}.$$

Since

$$\lim_{n\to\infty} S(p_n) = \lim_{n\to\infty} T(p_n) = 1,$$

we conclude that the constant bounds given in (3.11) are sharp. □

Remark 3.2. In view of the two inequalities in (3.11) it is natural to ask: do there exist a constant upper bound for $S(n)$ and a positive constant lower bound for $T(n)$ which are valid for all $n \geqslant 2$? If p_n denotes the nth prime number, then

$$S(p_n^3) = \Big(1 - \frac{1}{p_n}\Big)^{p_n^2(p_n-1)} \Big(1 + \frac{1}{p_n}\Big)^{p_n^2(p_n+1)}$$

and

$$T(p_n^3) = \Big(1 - \frac{1}{p_n}\Big)^{p_n^2(p_n+1)} \Big(1 + \frac{1}{p_n}\Big)^{p_n^2(p_n-1)}.$$

Since

$$\lim_{n\to\infty} S(p_n^3) = \infty \quad \text{and} \quad \lim_{n\to\infty} T(p_n^3) = 0,$$

we conclude that in both cases the answer is "no".

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