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# Semisymmetrization and Mendelsohn quasigroups

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*Abstract.* The semisymmetrization of an arbitrary quasigroup builds a semisymmetric quasigroup structure on the cube of the underlying set of the quasigroup. It serves to reduce homotopies to homomorphisms. An alternative semisymmetrization on the square of the underlying set was recently introduced by A. Krapež and Z. Petrić. Their construction in fact yields a Mendelsohn quasigroup, which is idempotent as well as semisymmetric. We describe it as the Mendelsohnization of the original quasigroup. For quasigroups isotopic to an abelian group, the relation between the semisymmetrization and the Mendelsohnization is studied. It is shown that the semisymmetrization is the total space for an action of the Mendelsohnization on the abelian group. The Mendelsohnization of an abelian group isotope is then identified as the idempotent replica of its semisymmetrization, with fibers isomorphic to the abelian group.

*Keywords:* semisymmetric; quasigroup; Mendelsohn triple system

*Classification:* 20N05

## 1. Introduction

Semisymmetric quasigroups, defined as magmas $(Q,\cdot)$ that satisfy the identity $x(yx)=y$, are one of the most important varieties of quasigroups. Firstly, semisymmetric quasigroups form one of the basic quasigroup classes defined by a triality symmetry property, namely equality of the right and left divisions with the opposite of the multiplication, see [28, Example 9]. Secondly, a semisymmetrization functor $\Delta$ reduces homotopies of quasigroups to homomorphisms between their semisymmetrizations, which are semisymmetric quasigroups, see [29]. Thirdly, idempotent semisymmetric quasigroups, also known as *Mendelsohn quasigroups*, are coexistent with certain well-known designs, the "3-cyclic" or Mendelsohn triple systems [2, Chapter 25]. Fourthly, semisymmetric quasigroups are very common, for instance being modeled by the operation $-x-y$ on any abelian group or commutative Moufang loop. They also appear as subquasigroups of the recently identified para-Paige and Okubo quasigroups, which are related by $\mathsf{D}_4$-triality to the Paige loop $\mathsf{PSL}_{1+3}(2)$ of order 120, the smallest finite non-associative simple Moufang loop, see Section 3.3.2 and [31].

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The semisymmetrization $Q^{\Delta}$ of a quasigroup $Q$ is built on the direct cube $Q^3$ of its underlying set. The semisymmetrization functor from the category **Qtp** of quasigroup homotopies to the category **P** of homomorphisms of semisymmetric quasigroups is a right adjoint to the forgetful functor from **P** to **Qtp** sending homomorphisms $f\colon P \to P'$ between semisymmetric quasigroups to homotopies $(f, f, f)\colon P \to P'$, see [25]. Incidentally, this is one of the rare natural cases in category theory where the left adjoint is forgetful and the right adjoint is constructive (compare the situations in [13, Section IV.2], for example).

Recently, A. Krapež and Z. Petrić defined an alternative semisymmetrization $Q^{\Gamma}$ of a quasigroup $Q$, built on the direct square $Q^2$ of the underlying set of the quasigroup, see [12]. They showed their construction was functorial, but were unable to place it in an adjoint situation. They used a language of "twisted quasigroups" and "biquasigroups", chosen for its convenient symmetry properties, to define $Q^{\Gamma}$. However, rewriting the definition in terms of the three basic quasigroup operations (3.2), it becomes apparent that $Q^{\Gamma}$ is actually idempotent, and thus forms a Mendelsohn quasigroup, see [9, Remark 7.14]. For this reason, we refer to $Q^{\Gamma}$ as the *Mendelsohnization* of the quasigroup $Q$.

The goal of the paper is to initiate a study of the relationship between the semisymmetrization $Q^{\Delta}$ and Mendelsohnization $Q^{\Gamma}$ of a quasigroup $Q$. In general, this problem is quite difficult to handle (compare Problems 5.3 and 5.5), so in the current paper we focus on the linear case, namely quasigroups that are isotopic to abelian groups. Section 2 provides the relevant background on semisymmetric quasigroups and their semisymmetrization. Section 3 deals with Mendelsohn quasigroups, Mendelsohn triple systems, and the Mendelsohnization construction. For a quasigroup $Q$ that is an isotope of an abelian group $A$, Section 4 shows how the semisymmetrization $Q^{\Delta}$ may be recovered from the Mendelsohnization $Q^{\Gamma}$, and an action of $Q^{\Gamma}$ on the abelian group $A$, by the short exact sequence (4.4). In particular, the characteristic congruence on $Q^{\Delta}$, originally introduced in [8], is now recognized as the kernel of the surjection in the short exact sequence.

Staying in the context of abelian group isotopes, Section 5.1 gives a new interpretation of the Mendelsohnization construction, as the idempotent replica (largest idempotent quotient) of the semisymmetrization. In other words, the Mendelsohnization functor $\Gamma$ is factorized as the composite of the semisymmetrization functor $\Delta$ with the idempotent replication functor $V$ (Corollary 5.2). Problem 5.3 asks whether a similar factorization is still available in the general case. Bearing in mind that $\Delta$ is a right adjoint, while $V$ is a left adjoint, it becomes apparent why the Mendelsohnization functor $\Gamma$ does not appear in any immediately obvious adjoint situation.

Section 5.2 investigates the fibers of the projection $p\colon Q^\Delta \to Q^\Gamma$ from the semisymmetrization to the Mendelsohnization, still in the context of an isotope $Q$ of an abelian group $A$. Each fiber is identified as an isomorphic copy of $A$ – Proposition 5.4 (a). In terms of the model-theoretic notion of a Mal'cev product, see [14], this means that the semisymmetrization of an abelian group isotope lies in the Mal'cev product $\mathbf{A} \circ \mathbf{M}$ of the variety $\mathbf{A}$ of abelian groups with the variety $\mathbf{M}$ of Mendelsohn quasigroups – Proposition 5.4 (b). Problem 5.3 asks for the smallest class $\mathbf{K}$ such that the semisymmetrization of an arbitrary quasigroup lies in the Mal'cev product $\mathbf{K} \circ \mathbf{M}$.

In general, the paper follows the notational conventions of [30]. Thus we default to algebraic notation with functions following their arguments, sometimes as a superfix, and composed in natural reading order from left to right. This convention avoids the proliferation of brackets in non-associative situations, or complicating twists and back-tracking in category theory.

## 2. Background and notation

**2.1 Quasigroups.** For the purposes of this paper, it is most convenient to define a *quasigroup* equationally as an algebra $(Q, \cdot, /, \backslash)$, with respective binary operations of *multiplication*, *right division*, and *left division*, such that the identities $y\backslash(y \cdot x) = x = (x \cdot y)/y$ and $y \cdot (y\backslash x) = x = (x/y) \cdot y$ are satisfied. For an element $q$ of $Q$, there is a *right multiplication*

$$R(q)\colon Q \to Q; \qquad x \mapsto x \cdot q$$

and a *left multiplication*

$$L(q)\colon Q \to Q; \qquad x \mapsto q \cdot x,$$

both of which are permutations of $Q$. The quasigroup multiplication $x \cdot y$ may also be written by juxtaposition as $xy$, which binds more strongly than $x \cdot y$. Thus the associative law may be written as $xy \cdot z = x \cdot yz$, for example.

**2.2 The homotopy category.** A *homotopy* $(f_1, f_2, f_3)\colon Q \to Q'$ from a quasigroup $Q$ to a quasigroup $Q'$ is a triple of functions from $Q$ to $Q'$ such that

$$xf_1 \cdot yf_2 = (x \cdot y)f_3$$

for all $x, y$ in $Q$. Write $\mathbf{Q}$ for the category of homomorphisms between quasigroups, and $\mathbf{Qtp}$ for the category of homotopies between quasigroups. Then there is a forgetful functor

$$\Sigma\colon \mathbf{Q} \to \mathbf{Qtp} \tag{2.1}$$

preserving objects, sending a quasigroup homomorphism $f\colon Q \to Q'$ to the homotopy $(f,f,f)\colon Q \to Q'$. A function $f\colon Q \to Q'$ connecting the underlying sets of equational quasigroups $(Q,\cdot,/,\backslash)$ and $(Q',\cdot,/,\backslash)$ is a quasigroup homomorphism if it is a homomorphism $f\colon (Q,\cdot) \to (Q',\cdot)$ for the multiplications. Thus a homotopy $(f_1,f_2,f_3)$ having equal components $f_1 = f_2 = f_3$ is an element of the image of the morphism part of the forgetful functor (2.1).

**2.3 Semisymmetric quasigroups.** A quasigroup is *semisymmetric* if it satisfies the identity $x \cdot yx = y$. (Compare [28, Example 9] for an interpretation of this identity in terms of the semantic triality of quasigroups.) Using right and left multiplications, semisymmetry amounts to the equality

$$R(x) = L(x)^{-1}, \tag{2.2}$$

and may thus be expressed in equivalent form as $xy{\cdot}x = y$. The simplest models of semisymmetric quasigroups are abelian groups with $-x-y$ as the multiplication operation.

**2.4 Semisymmetrization.** Let $\mathbf{P}$ denote the category of homomorphisms between semisymmetric quasigroups. Then each quasigroup $Q$ or $(Q,\cdot,/,\backslash)$ defines a semisymmetric quasigroup structure $Q\Delta$ on the direct cube $Q^3$ with multiplication as follows:

$$(x_1,x_2,x_3)\cdot(y_1,y_2,y_3) = (x_2/\!/y_3, x_3\backslash\!\backslash y_1, x_1\cdot y_2) \tag{2.3}$$

– writing $x/\!/y = y/x$ and $x\backslash\!\backslash y = y\backslash x$, see [25]. If $(f_1,f_2,f_3)\colon (Q,\cdot) \to (Q',\cdot)$ is a quasigroup homotopy, define

$$(f_1,f_2,f_3)^\Delta\colon Q\Delta \to Q'\Delta; \qquad (x_1,x_2,x_3) \mapsto (x_1f_1, x_2f_2, x_3f_3). \tag{2.4}$$

This map is a quasigroup homomorphism. Indeed, for $(x_1,x_2,x_3)$ and $(y_1,y_2,y_3)$ in $Q\Delta$, one has

$$\begin{aligned}
&(x_1f_1, x_2f_2, x_3f_3)\cdot(y_1f_1, y_2f_2, y_3f_3)\\
&\qquad = (x_2f_2/\!/y_3f_3, x_3f_3\backslash\!\backslash y_1f_1, x_1f_1\cdot y_2f_2)\\
&\qquad = \big((x_2/\!/y_3)f_1, (x_3\backslash\!\backslash y_1)f_2, (x_1\cdot y_2)f_3\big)\\
&\qquad = \big((x_1,x_2,x_3)\cdot(y_1,y_2,y_3)\big)(f_1,f_2,f_3)^\Delta.
\end{aligned}$$

Consider the functor

$$\Delta\colon \mathbf{Qtp} \to \mathbf{P}, \tag{2.5}$$

known as the *semisymmetrization functor*, which has object part (2.3) and morphism part (2.4). This functor has a left adjoint, namely the restriction $\Sigma:$

$\mathbf{P} \to \mathbf{Qtp}$ of the forgetful functor (2.1), see [25, Theorem 5.2]. The unit of the adjunction at a semisymmetric quasigroup $P$ is the homomorphism

$$\eta_P \colon P \to P\Sigma\Delta; \qquad x \mapsto (x, x, x), \tag{2.6}$$

see [25, (5.3)]. The count $\varepsilon_Q$ at a quasigroup $Q$ is the homotopy

$$(\pi_1, \pi_2, \pi_3) \colon Q\Delta\Sigma \to Q \tag{2.7}$$

with $(x_1, x_2, x_3)\pi_i = x_i$ for $1 \leq i \leq 3$, see [25, (5.4)].

## 3. Mendelsohnization

### 3.1 Mendelsohn triple systems and quasigroups.

**3.1.1 Mendelsohn quasigroups.** A quasigroup which is both idempotent and semisymmetric is described as a *Mendelsohn quasigroup*. The category of homomorphisms between Mendelsohn quasigroups is denoted by $\mathbf{M}$.

**3.1.2 Mendelsohn triple systems.** Just as totally symmetric idempotent quasigroups are coexistent with Steiner triple systems (compare [27, Section 1.3], for example), so Mendelsohn quasigroups are coexistent with Mendelsohn triple systems (as introduced in [15]). A *Mendelsohn triple system* $(M, \mathcal{C})$ is a set $M$ with a set $\mathcal{C}$ of 3-cycles

$$(x\ y\ z) = (z\ x\ y) = (y\ z\ x) \tag{3.1}$$

such that each ordered pair $(x, y)$ of distinct elements from $M$ lies in a unique 3-cycle (3.1) (compare e.g. [2, Chapter 25], [4]). In the language of [6], Mendelsohn triple systems are $(v, 3)$-*Mendelsohn designs*. In Mendelsohn's original paper [15], they were described as *cyclic triple systems*.

**3.1.3 Quasigroups and triple systems.** A Mendelsohn triple system $(M, \mathcal{C})$ corresponds to a Mendelsohn quasigroup $(M, \cdot)$ with

$$x \cdot y = z \;\Leftrightarrow (x\ y\ z) \in \mathcal{C} \qquad \text{or} \qquad x = y = z$$

for $x, y, z \in M$. Note that commutative Mendelsohn quasigroups are totally symmetric, so a Mendelsohn triple system $(M, \mathcal{C})$ with

$$\forall\, x \neq y \in M \;\; \{(x\ y\ z), (y\ x\ z)\} \subseteq \mathcal{C}$$

is a Steiner triple system, with a Steiner block $\{x, y, z\}$ corresponding to each pair $\{(x\ y\ z), (y\ x\ z)\}$ of mutually reversed 3-cycles.

**3.1.4 Eves' equihoops and Eisenstein integers.** For completeness, it is worth recording Eves' term *equihoop* for entropic Mendelsohn quasigroups, see [3], [7], [21, Example 436], [22, Example 6.5B]. As idempotent entropic magmas, these quasigroups are distributive. (In other words, being idempotent, and "medial" in Stein's sense [33], they are "medial" in Soublin's sense, see [32]!)

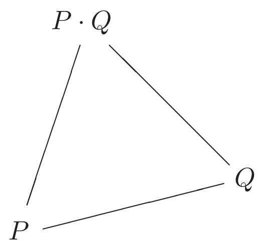


FIGURE 1. A Mendelsohn quasigroup structure in $\mathbb{R}^2$ or $\mathbb{C}^1$.

A well-known model is provided by the set of points in the real plane (or complex line), as in Figure 1. For points $P$ and $Q$, the point $P \cdot Q$ is the third vertex of the equilateral triangle with vertices $P$, $Q$, $P \cdot Q$ in anti-clockwise order. Of course, if $P$ coincides with $Q$, then so does $P \cdot Q$.

Identifying the respective points $P$, $Q$, $P \cdot Q$ with their complex coordinates $u, v, u \cdot v$, and taking the primitive sixth root of unity $\zeta = \exp(\pi \mathrm{i}/3)$, one has $u \cdot v - u = \zeta(v - u)$, and thus $u \cdot v = u(1 - \zeta) + v\zeta$, or $u \cdot v = uv\,\underline{\zeta}$ in the notation of [21], [22]. Note that the Eisenstein integers $\mathbb{Z}[\zeta]$ form a subquasigroup of $(\mathbb{C}, \cdot)$. Thus if $J$ is an ideal of the ring $\mathbb{Z}[\zeta]$ of Eisenstein integers, the quotient $\mathbb{Z}[\zeta]/J$ carries an entropic Mendelsohn quasigroup structure (compare [16]).

**3.2 The Mendelsohnization construction.** For a quasigroup $Q$, define a quasigroup $Q^\Gamma = (Q^2, *)$, with a product defined on the underlying set $Q^2$ by

$$(x_1, x_2) * (y_1, y_2) = \big((x_1 y_2)/x_2, y_1\backslash(x_1 y_2)\big) \tag{3.2}$$

(compare [12, page 9]). For a homotopy $(f_1, f_2, f_3)\colon Q \to Q'$, define a map

$$(f_1, f_2, f_3)^\Gamma : Q^2 \to Q'^2; \qquad (x_1, x_2) \mapsto (x_1 f_1, x_2 f_2) \tag{3.3}$$

which is a homomorphism from $(Q^2, *)$ to $(Q'^2, *)$ (compare [12, page 10]). A. Krapež and Z. Petrić recognized the semisymmetry of the product (3.2). In fact, it is also idempotent, see [9, Remark 7.14], so we obtain a functor

$$\Gamma\colon \mathbf{Qtp} \to \mathbf{M}, \tag{3.4}$$

known as the *Mendelsohnization functor*, defined by object part (3.2) and morphism part (3.3). (In [12, page 9], $\Gamma$ was identified as a functor with codomain $\mathbf{P}$.)

**3.3 A minimal Mendelsohn, but not Steiner, quasigroup.** Taking the quasigroup $Q$ to be the abelian group $\mathbb{Z}/_2 = \{0,1\}$ of residues modulo 2 under addition, the Mendelsohnization $Q^\Gamma = (Q^2, *)$ is

(3.5)

| $*$ | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | $00_0$ | $11_0$ | $01_1$ | $10_2$ |
| 01 | $10_0$ | $01_0$ | $11_2$ | $00_1$ |
| 10 | $11_2$ | $00_1$ | $10_0$ | $01_3$ |
| 11 | $01_1$ | $10_2$ | $00_3$ | $11_0$ |

writing ordered pairs as bit strings. When the bit strings are interpreted as binary representations of the numbers $0,1,2,3$, we have the opposite of the minimal Mendelsohn, but not Steiner, quasigroup presented in [23, Example 2.193].

**3.3.1 The greedy construction.** The suffices appearing on the bit strings in the body of the multiplication table (3.5) refer to a greedy construction of an idempotent, but not commutative, quasigroup on the ordered 4-element set $\{00 < 01 < 10 < 11\}$. The suffices give the time at which a body entry may be entered into the Latin square, starting with time 0 for the initial population of the diagonal according to the idempotence, and the respective greedy not-commutative choices for $01 * 00$ and $00 * 01$. Thus at discrete (integer) time $t$ for $1 \leq t \leq 3$, the Latin square property forces the entries with suffix $t$, and the square is completed at time $t = 3$.

**3.3.2 Para-Paige and Okubo quasigroups.** The split octonion algebra over a finite field of order $q$, realized by the algebra $\mathsf{Zorn}(q)$ of Zorn vector-matrices, see [10], [26, Section 1.7], [34], carries three algebra structures under which the norm or Zorn determinant, see [26, (1.24)], is multiplicative:

- the original Zorn vector-matrix multiplication, giving rise to the split octonion algebra $(\mathsf{Zorn}(q), \cdot)$;
- the multiplication $x \circ y = \overline{x} \cdot \overline{y}$ (with $\overline{a}$ denoting the conjugation [26, (1.23)] of a Zorn vector-matrix $a$), which gives rise to the *para-Zorn* algebra $\mathsf{PZorn}(q) = (\mathsf{Zorn}(q), \circ)$ whose properties are closely related to those of the split octonion algebra, see [1], [20];

- ∘ considering a particular order-3 automorphism $\varrho$ of $\mathsf{Zorn}(q)$ which is derived from a $\mathsf{D}_4$-graph automorphism, the multiplication $x * y = \overline{x\varrho} \cdot \overline{x\varrho^2}$, giving rise to the *Okubo algebra* $\mathsf{Okubo}(q) = (\mathsf{Zorn}(q), *)$, see [1], [17], [18].

In each of these algebras, the set of elements of norm 1 forms a quasigroup under the multiplication. Quotients of these quasigroups, under the quasigroup congruence identifying elements with their negations, then form simple quasigroups, see [31]. These respective simple quasigroups are the Paige loop $\mathsf{PSL}_{1+3}(q)$ from $\mathsf{Zorn}(q)$, see [19], the *para-Paige quasigroup* $\mathsf{PP}(q)$ from $\mathsf{PZorn}(q)$, and the *Okubo quasigroup* $\mathsf{OQ}(q)$ from $\mathsf{Okubo}(q)$.

It then turns out that the Mendelsohnization (3.5) of the abelian group $\mathbb{Z}/_2$ appears both within the para-Paige quasigroup $\mathsf{PP}(2)$ and the Okubo quasigroup $\mathsf{OQ}(2)$. Indeed, the 120-element para-Paige quasigroup $\mathsf{PP}(2)$ has 126 subquasigroups isomorphic to the Mendelsohnization of the abelian group $\mathbb{Z}/_2$ [31, Proposition 12.2], while the 120-element Okubo quasigroup $\mathsf{OQ}(2)$ has 9 subquasigroups isomorphic to the semisymmetrization of the abelian group $\mathbb{Z}/_2$, and 9 subquasigroups isomorphic to its Mendelsohnization, see [31, Proposition 12.3].

## 4. Abelian group isotopes

This section studies the relationship between semisymmetrizations and Mendelsohnizations of abelian group isotopes, as a preparation for the more complicated study of the relationship in the general case (cf. Problem 5.3, for example). The class **AGI** of abelian group isotopes forms a variety, see [11]. Let $xyzP_1 = xy^{-1}z$ and $xyzP_2 = zy^{-1}x$ be the shortest Mal'cev operations for groups (in the sense of [24]).[1] Then a defining identity for abelian group isotopes within the variety of quasigroups may be written as

$$R(x)R(y)R(z)P_1 = R(x)R(y)R(z)P_2,$$

i.e., as $(tx/y)z = (tz/y)x$ in the language of quasigroups (compare [5, Proposition 1.8 (iii)]).

In the next two sections, we will be considering the situation where there is an isotopy $(f_1, f_2, f_3)\colon Q \to A$ from a quasigroup $Q$ to an abelian group $(A, +, 0)$, or just $A$. Since semisymmetrization $\Delta$ and Mendelsohnization $\Gamma$ are functors, there are quasigroup isomorphisms

$$(f_1, f_2, f_3)^\Delta\colon Q^\Delta \to A^\Delta \qquad \text{and} \qquad (f_1, f_2, f_3)^\Gamma\colon Q^\Gamma \to A^\Gamma. \tag{4.1}$$

[1] For longer Mal'cev operations, one may take $xy^{-1}z[[x,y],[y,z]]$, etc.

Thus while the various constructions and theorems are formulated in their full generality using $Q^\Delta$ and $Q^\Gamma$, the explicit calculations and proofs will work with the notationally more convenient isomorphic copies $A^\Delta$ and $A^\Gamma$, respectively.

**4.1 The Mendelsohn extension.** Let $Q$ be an isotope of an abelian group $A$. To relate the semisymmetrization $Q^\Delta$ and Mendelsohnization $Q^\Gamma$ of $Q$, we will recognize $Q^\Delta$ as the "total space" $E$ for a module action of $Q^\Gamma$ on $A$, along the lines of [26, Section 10.3]. Thus in an adaptation of [26, (10.21)], we define actions

$$R\colon Q^\Gamma \to \mathrm{Mlt}\, A; \qquad [y_1 \;\; y_2] \mapsto R_+(y_2) \tag{4.2}$$

and

$$L\colon Q^\Gamma \to \mathrm{Mlt}\, A; \qquad [x_1 \;\; x_2] \mapsto L_+(x_1) \tag{4.3}$$

of $Q^\Gamma$ on $A$. The quasigroup $Q^\Gamma \ltimes A$, known as the *Mendelsohn extension*, is then built on the underlying set $Q^\Gamma \times A$ with

$$\begin{aligned}&\big([x_1 \;\; x_2], a\big) \cdot \big([y_1 \;\; y_2], b\big)\\ &\qquad = \big([x_1 \;\; x_2] * [y_1 \;\; y_2], aR\big([y_1 \;\; y_2]\big) + bL\big([x_1 \;\; x_2]\big)\big)\end{aligned}$$

as its multiplication.

**Remark 4.1.** The definitions (4.2) and (4.3) of the actions of $Q^\Gamma$ on $A$ are respectively reminiscent of the operators $R_{\hat{y}}$ and $L_{\hat{x}}$ introduced by A. Krapež and Z. Petrić in [12, Section 4].

**Remark 4.2.** For the benefit of readers, say from a narrowly combinatorial background, who might be less familiar with the full use of the isomorphism concept, it may be helpful to see the way that the isomorphism

$$D\colon Q^\Gamma \ltimes A \to A^\Gamma \ltimes A; \qquad \big((q_1, q_2), a\big) \mapsto \big([q_1 f_1 \;\; q_2 f_2], a\big)$$

arises. Thus the actions (4.2) and (4.3) will take the explicit forms

$$Q^\Gamma \to \mathrm{Mlt}\, A; \qquad (q_1', q_2') \mapsto R_+(q_2' f_2)$$

and

$$Q^\Gamma \to \mathrm{Mlt}\, A; \qquad (q_1, q_2) \mapsto L_+(q_1 f_1),$$

respectively. The product

$$\big((q_1, q_2), a\big) \cdot \big((q_1', q_2'), b\big) = \big((q_1, q_2) * (q_1', q_2'), aR_+(q_2' f_2) + bL_+(q_1 f_1)\big)$$

in $Q^\Gamma \ltimes A$ is then mapped under $D$ to the product

$$\begin{aligned}\big([q_1f_1 \;\; q_2f_2] * [q_1'f_1 \;\; q_2'f_2], aR\big([q_1'f_1 \;\; q_2'f_2]\big) + bL\big([q_1f_1 \;\; q_2f_2]\big)\big) \\ = \big((q_1,q_2),a\big)^D \cdot \big((q_1',q_2'),b\big)^D\end{aligned}$$

in $A^\Gamma \ltimes A$.

### 4.2 Semisymmetrization and Mendelsohnization.

**Theorem 4.3.** *For an isotope $Q$ of an abelian group $A$, the Mendelsohn extension $Q^\Gamma \ltimes A$ is isomorphic to the semisymmetrization $Q^\Delta$ of $Q$.*

Proof: Define a map

$$\Theta\colon Q^\Gamma \ltimes A \to Q^\Delta; \qquad \big([x_1 \;\; x_2], a\big) \mapsto [x_1 + a \;\; x_2 + a \;\; -a].$$

Here, and throughout the proof, the isomorphisms (4.1) are used to rewrite $Q^\Delta$, $Q^\Gamma$, and $Q^\Gamma \ltimes A$ explicitly in their notationally simpler forms $A^\Delta$, $A^\Gamma$, and $A^\Gamma \ltimes A$, respectively. It will be shown that $\Theta$ is a quasigroup isomorphism. Certainly, it is a bijection, with

$$[z_1 \;\; z_2 \;\; z_3] \mapsto \big([z_1 + z_3 \;\; z_2 + z_3], -z_3\big)$$

as a two-sided inverse. Then for $x_i, y_i, a, b \in A$, one has

$$\begin{aligned}
&\big([x_1 \;\; x_2], a\big)^\Theta \cdot \big([y_1 \;\; y_2], b\big)^\Theta \\
&= \big[x_1 + a \quad x_2 + a \quad -a\big] \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \big[y_1 + b \quad y_2 + b \quad -b\big] \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \big[-x_2 - a \quad -a \quad x_1 + a\big] + \big[-b \quad -y_1 - b \quad y_2 + b\big] \\
&= \big[x_1 + y_2 - x_2 \quad x_1 + y_2 - y_1 \quad 0\big] \\
&\quad + \big[-a - b - x_1 - y_2 \quad -a - b - x_1 - y_2 \quad a + b + x_1 + y_2\big] \\
&= \big(\,\big[x_1 + y_2 - x_2 \quad x_1 + y_2 - y_1\big]\,, a + b + x_1 + y_2\big)^\Theta \\
&= \left([x_1 \;\; x_2] \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + [y_1 \;\; y_2] \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, a + b + x_1 + y_2\right)^\Theta \\
&= \big([x_1 \;\; x_2] * [y_1 \;\; y_2], aR\big([y_1 \;\; y_2]\big) + bL\big([x_1 \;\; x_2]\big)\big)^\Theta \\
&= \big(\big([x_1 \;\; x_2], a\big) \cdot \big([y_1 \;\; y_2],\big)\big)^\Theta
\end{aligned}$$

as required. $\square$

**Corollary 4.4.** *The semisymmetrization $Q^\Delta$ is given by the short exact sequence*

$$\{0\} \to A \xrightarrow{j} Q^\Delta \xrightarrow{p} Q^\Gamma \to \{0\} \tag{4.4}$$

*with* $j\colon a \mapsto \big([0\ \ 0], a\big)$ *and* $p\colon [x_1\ \ x_2\ \ x_3] \mapsto \big[x_1 + x_3\quad x_2 + x_3\big]$.

**4.3 The characteristic congruence.** The *characteristic congruence* on the semisymmetrization $Q^\Delta$ of an isotope $Q$ of an abelian group $A$ is the relation $\nu$ on $A^3$ defined by

$$[x_1\ \ x_2\ \ x_3]\ \nu\ [y_1\ \ y_2\ \ y_3] \ \Leftrightarrow\ x_1 - y_1 = x_2 - y_2 = y_3 - x_3,$$

see [8, Section 3]. In other words, the congruence classes of $\nu$ are the cosets of the subgroup

$$N = [0\ \ 0\ \ 0]^\nu = \big\{[a\ \ a\ \ -a]\colon a \in A\big\}$$

of $A^3$. The subspace $N$ of $A^3$ is invariant under P, since it is the eigenspace of P for the eigenvalue $-1$. Thus $\nu$ is indeed a congruence of $Q^\Delta$.

**Proposition 4.5.** *For an isotope $Q$ of an abelian group $A$, the quotient $Q^{\Delta\nu}$ of the semisymmetrization $Q^\Delta$ by the characteristic congruence $\nu$ is isomorphic to the Mendelsohnization $Q^\Gamma$ of $Q$.*

Proof: The characteristic congruence $\nu$ is the kernel of the homomorphism $p$ in the exact sequence (4.4). □

## 5. Mendelsohnization as the idempotent replica

**5.1 The idempotent replica.** Recall that the *idempotent replica* of a magma is its largest idempotent quotient.

**Theorem 5.1.** *Suppose that $Q$ is an isotope of an abelian group $A$. The Mendelsohnization $Q^\Gamma$ is the idempotent replica of the semisymmetrization $Q^\Delta$.*

Proof: Suppose that $\varrho$ is the idempotent replica congruence of $Q^\Delta$, the smallest congruence on $Q^\Delta$ whose quotient is idempotent. Now by the idempotence of $Q^\Gamma$ and Proposition 4.5, $\varrho \subseteq \nu$. Conversely, consider an element $\mathbf{x} = [x_1\ \ x_2\ \ x_3]$ of $Q^\Delta$. Then $\mathbf{x}^\varrho \cdot \mathbf{x}^\varrho = \mathbf{x}^\varrho$, so

$$\begin{aligned}
&[x_1\ \ x_2\ \ x_3] \cdot [x_1\ \ x_2\ \ x_3] \\
&\quad = [x_1\ \ x_2\ \ x_3] \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + [x_1\ \ x_2\ \ x_3] \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} x_3 - x_2 & x_3 - x_1 & x_1 + x_2 \end{bmatrix}$$
$$= [x_1 \ \ x_2 \ \ x_3] + (-x_1 - x_2 + x_3) \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$

and $(-x_1 - x_2 + x_3) \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \in [0 \ \ 0 \ \ 0]^\varrho$. Thus $N \subseteq [0 \ \ 0 \ \ 0]^\varrho$, and then $\nu \subseteq \varrho$ by the regularity of quasigroup congruences (compare [26, Exercise 2.10 (7)]). □

Theorem 5.1 gives an abstract characterization of the Mendelsohnization functor for abelian group isotopes.

**Corollary 5.2.** *Let $U\colon \mathbf{M} \to \mathbf{P}$ be the forgetful functor from the category of (homomorphisms between) Mendelsohn quasigroups to the category of (homomorphisms between) semisymmetric quasigroups. Let $V\colon \mathbf{P} \to \mathbf{M}$ be the left adjoint to $U$ (compare* [30, Theorem IV.3.4.4]*). Let $\Delta'$ and $\Gamma'$ be the respective restrictions of $\Delta$ and $\Gamma$ to the category of homotopies between abelian group isotopes. Then $\Gamma' = \Delta' V$.*

It is natural to ask whether Corollary 5.2 holds without the restriction:

**Problem 5.3.** Is $\Gamma = \Delta V$? In other words, is the Mendelsohnization of any quasigroup just the idempotent replica of its semisymmetrization?

**5.2 Fibers of the replication.** For an isotope $Q$ of an abelian group $A$, Theorem 5.1 shows that $Q^\Gamma$ is the idempotent replica of $Q^\Delta$, the quotient of $Q^\Delta$ by its idempotent replica congruence $\varrho$. It is natural to ask for the structure of the fibers of the replication, the $\varrho$-classes within $Q^\Delta$. Using Theorem 4.3, this question is best answered in the Mendelsohn extension $Q^\Gamma \ltimes A$, asking for the structure of a fiber $p^{-1}\{[x_1 \ \ x_2]\}$ for an element $[x_1 \ \ x_2]$ of $Q^\Gamma$, from the exact sequence (4.4).

The fiber $p^{-1}\{[x_1 \ \ x_2]\}$ consists of elements $\big([x_1 \ \ x_2], a\big)$, $\big([x_1 \ \ x_2], b\big)$ with $a, b \in A$. We regard these abelian group elements as parameters for the fiber elements. The product of the fiber elements in the Mendelsohn extension is

$$\Big([x_1 \ \ x_2], aR\big([x_1 \ \ x_2]\big) + bL\big([x_1 \ \ x_2]\big)\Big),$$

parametrized by

$$a \cdot b := aR\big([x_1 \ \ x_2]\big) + bL\big([x_1 \ \ x_2]\big) = a + (x_1 + x_2) + b. \tag{5.1}$$

The first expression of the product exhibits the fiber $\big(p^{-1}\{[x_1 \ \ x_2]\}, \cdot\big)$ as an explicit principal isotope of the abelian group $A$, while the second expression denotes a central shift of $A$ (in the sense of [26, Definition 3.3]). Now (5.1) gives a loop $\big(p^{-1}\{[x_1 \ \ x_2]\}, \cdot, -x_1 - x_2\big)$ which is isotopic to the abelian group $A$, and thus forms a group which is isomorphic to the abelian group $A$ [26, Proposition 1.4]. We may summarize as follows. For the concept of a Mal'cev product, compare [14].

**Proposition 5.4.** *Let $Q$ be an isotope of an abelian group $A$.*

(a) *Each fiber of the idempotent replication $p\colon Q^\Delta \to Q^\Gamma$ of* (4.4) *is isomorphic to $A$.*

(b) *The semisymmetrization $Q^\Delta$ lies in the Mal'cev product $\mathbf{A}\circ\mathbf{M}$ of the variety $\mathbf{A}$ of abelian groups with the variety $\mathbf{M}$ of Mendelsohn quasigroups.*

(c) *The variety $\mathbf{A}$ of abelian groups is the smallest class $\mathbf{K}$ such that the semisymmetrizations of abelian group isotopes lie in the Mal'cev product $\mathbf{K}\circ\mathbf{M}$.*

Proof: For (c), it suffices to note that, by (a), each fiber of the idempotent replication $p\colon A^\Delta \to A^\Gamma$ of (4.4) is isomorphic to $A$. □

**Problem 5.5.** Find the smallest class $\mathbf{K}$ such that the semisymmetrizations of arbitrary quasigroups lie in the Mal'cev product $\mathbf{K}\circ\mathbf{M}$.

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