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A NEW APPROACH TO HOM-LEFT-SYMMETRIC BIALGEBRAS

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Abstract. The main purpose of this paper is to consider a new definition of Hom-left-symmetric bialgebra. The coboundary Hom-left-symmetric bialgebra is also studied. In particular, we give a necessary and sufficient condition that s-matrix is a solution of the Hom-S-equation by a cocycle condition.

Keywords: Hom-left-symmetric algebra; Hom-S-equation; Hom-left-symmetric bialgebra

MSC 2020: 17B60, 17A30, 81R12

1. Introduction

Hom-left-symmetric algebras were first introduced by Makhlouf and Silvestrov in [5]. And then it was further investigated in [10] and [13]. Recently, Sheng and Chen constructed strict Hom-Lie 2-algebras from Hom-left-symmetric algebras in [7].

Yau developed a generalization of the classical Yang-Baxter equation (CYBE), a twisted generalization of the CYBE and the closely related object of Hom-Lie bialgebra in [9], [11], [12]. Sheng and Bai introduced a new definition of a Hom-Lie bialgebra and investigated their properties in [6]. Bimodule theory of Hom-left-symmetric algebra and Hom-left symmetric bialgebra were first considered in [8], but the related Hom-S-equation was not studied.

Inspired by the work of Sheng and Bai (see [6]), we give a new definition of Hom-left symmetric bialgebra. The coboundary Hom-left symmetric bialgebra and Hom-S-equation are also considered.

The paper is organized as follows. In Section 2, we review some necessary results on Hom-Lie algebras and Hom-left symmetric algebras. In Section 3, we introduce the

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new definition of Hom-left symmetric bialgebras and list some equivalent conditions. Finally, we consider the s-matrices and Hom-S-equations. In particular, we describe the condition for r being a solution of the Hom-S-equation using a cocycle condition.

2. Preliminaries

Let us begin with some results on Hom-Lie algebras and Hom-left-symmetric algebras. For more details, one can refer to [3], [5], [6], [7], [8] and the references therein.

A Hom-Lie algebra is a triple $(\mathcal{G}, [\,,\,], \varphi)$ consisting of a linear space $\mathcal{G}$, a skew-symmetric bilinear map $[\,,\,]\colon \wedge^2\mathcal{G} \to \mathcal{G}$ and an algebra morphism φ satisfying

$$[\varphi(x), [y, z]] + [\varphi(y), [z, x]] + [\varphi(z), [x, y]] = 0$$

for any $x, y, z \in \mathcal{G}$. The Hom-Lie algebra $(\mathcal{G}, [\,,\,], \varphi)$ is said to be *regular* (involutive) if φ is nondegenerate (satisfies $\varphi^2 = I$).

There is a more general notion of Hom-Lie algebras introduced by Makhlouf and Silvestrov in [5], in which φ is only a homomorphism of linear spaces. A Hom-Lie algebra in this paper is called a *multiplicative Hom-Lie algebra* in [2].

A regular Hom-Lie algebra $(\mathcal{G}, [\,,\,], \varphi)$ is called a *symplectic Hom-Lie algebra* if there is a nondegenerate skew-symmetric 2-Hom-cocycle ω (the symplectic form) on $\mathcal{G}$, that is for any $x, y, z \in \mathcal{G}$

$$\omega([x, y], \varphi(z)) + \omega([y, z], \varphi(x)) + \omega([z, x], \varphi(y)) = 0.$$

We denote it by $(\mathcal{G}, [\,,\,], \varphi, \omega)$.

A Hom-left-symmetric algebra is a triple $(A, \cdot, \varphi)$, where A is a vector space, $\cdot\colon A \times A \to A$ is a bilinear map, and $\varphi \in \mathrm{gl}(A, A)$ satisfies

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \quad (x, y, z)_\varphi = (y, x, z)_\varphi$$

for any $x, y, z \in A$. We denote $(x, y, z)_\varphi = (x \cdot y) \cdot \varphi(z) - \varphi(x) \cdot (y \cdot z)$ and call it *φ-associator*.

Let (A, φ) be a Hom-left-symmetric algebra and V a vector space. Let $S, T\colon A \to \mathrm{gl}(V)$ be two linear maps and $\psi \in \mathrm{gl}(V)$. The quadruple (S, T, V, ψ) is called a *bimodule* of (A, φ) if for any $x, y \in A$, $v \in V$,

$$\psi(S(x)v) = S(\varphi(x))\psi(v), \quad \psi(T(x)v) = T(\varphi(x))\psi(v), \tag{2.1}$$

$$S(\varphi(x))S(y)v - S(xy)\psi(v) = S(\varphi(y))S(x)v - S(yx)\psi(v), \tag{2.2}$$

$$S(\varphi(x))T(y)v - T(\varphi(y))S(x)v = T(xy)\psi(v) - T(\varphi(y))T(x)v. \tag{2.3}$$

Let L (or R) be the left (or right) multiplication operator associated to $(A,\cdot,\varphi)$, i.e.

$$L(x_0)x_1 = R(x_1)x_0 = x_0x_1 \quad \forall\, x_0, x_1 \in A.$$

Then (A, L, R, φ) is a representation of (A,φ) called the *regular representation*.

Suppose that (S, T, V, ψ) is a bimodule on Hom-left-symmetric algebra $(A,\cdot,\varphi)$. Let $S^*, T^*\colon A \to \mathrm{gl}(V^*)$, $\varphi^*\colon A^* \to A^*$, $\psi^*\colon V^* \to V^*$ be the dual maps of φ and ψ, respectively, given by

$$\begin{aligned} \langle S^*(x)u^*, v\rangle &= -\langle S(x)v, u^*\rangle, & \langle T^*(x)u^*, v\rangle &= -\langle T(x)v, u^*\rangle,\\ \varphi^*(x^*)(y) &= x^*(\varphi(y)), & \psi^*(u^*)(v) &= u^*(\psi(v)). \end{aligned}$$

If in addition for all $x, y \in A$, $x^* \in A^*$, $u^* \in V^*$, $v \in V$, the following hold:

$$\psi(S(\varphi(x))v) = S(x)\psi(v), \quad \psi(T(\varphi(x))v) = T(x)\psi(v), \tag{2.4}$$

$$S(x)S(\varphi(y))v - \psi(S(xy)v) = S(y)S(\varphi(x))v - \psi(S(yx)v), \tag{2.5}$$

$$S(x)T(\varphi(y))v - T(y)S(\varphi(x))v = \psi(T(xy)v) - T(y)T(\varphi(x))v, \tag{2.6}$$

then $(S^* - T^*, -T^*, V^*, \psi^*)$ becomes a bimodule of (A,φ).

Let $(A,\cdot,\varphi)$ and $(B,\circ,\psi)$ be two Hom-left-symmetric algebras. Suppose that there are linear maps $l_A, r_A\colon A \to \mathrm{gl}(B)$ and $l_B, r_B\colon B \to \mathrm{gl}(A)$ such that (l_A, r_A, ψ) is a bimodule of A and (l_B, r_B, φ) is a bimodule of B and for any $x, y \in A$, $a, b \in B$ they satisfy the following conditions:

$$\begin{aligned} r_A(\varphi(x))[a,b] &= r_A(l_B(b)x)\psi(a) - r_A(l_B(a)x)\psi(b) + \psi(a)\circ(r_A(x)b)\\ &\quad - \psi(b)\circ(r_A(x)a),\\ l_A(\varphi(x))(a\circ b) &= -l_A(l_B(a)x - r_B(a)x)\psi(b) + (l_A(x)a - r_A(x)a)\circ\psi(b)\\ &\quad + r_A(r_B(b)x)\psi(a) + \psi(a)\circ(l_A(x)b),\\ r_B(\psi(a))[x,y] &= r_B(l_A(y)a)\varphi(x) - r_B(l_A(x)a)\varphi(y) + \varphi(x)\cdot(r_B(a)y)\\ &\quad - \varphi(y)\cdot(r_B(a)x),\\ l_B(a)(x\cdot y) &= -l_B(l_A(x)a - r_A(x)a)\varphi(y) + (l_B(a)x - r_B(a)x)\cdot\varphi(y)\\ &\quad + r_B(r_A(y)a)\varphi(x) + \varphi(x)\cdot(l_B(a)y). \end{aligned}$$

Then there is a Hom-left-symmetric algebra structure on the vector space $A \oplus B$ given by

$$(x+a)*(y+b) = (x\cdot y + l_B(a)y + r_B(b)x) + (a\circ b + l_A(x)b + r_A(y)a).$$

We denote this Hom-left-symmetric algebra by $\left(A \bowtie_{l_B,r_B}^{l_A,r_A} B, \varphi \oplus \psi\right)$ or simply $(A \bowtie B, \varphi \oplus \psi)$. And $(A, B, l_A, r_A, l_B, r_B, \varphi, \psi)$ satisfying the above conditions is called a *matched pair* of Hom-left-symmetric algebras. On the other hand, every Hom-left-symmetric algebra which is a direct sum of the underlying vector spaces of two subalgebras can be obtained in the above way.

3. Hom-left-symmetric bialgebras

Let (S, T, V, ψ) be a bimodule of Hom-left-symmetric algebra $(A, \cdot, \varphi)$. In view of Section 2, we know that $(S^* - T^*, -T^*, V^*, \psi^*)$ is not a bimodule in general. If $(S^* - T^*, -T^*, V^*, \psi^*)$ is a bimodule of $(A, \cdot, \varphi)$, we say that (S, T, V, ψ) is an admissible bimodule. In this section, we mainly focus on the regular representation.

Proposition 3.1. *Let $(A, \cdot, \varphi)$ be a Hom-left-symmetric algebra. The regular representation (A, φ, L, R) is admissible if and only if the following equations hold:*

$$(\varphi^2(x) - x)\varphi(y) = 0, \quad \varphi(y)(\varphi^2(x) - x) = 0, \tag{3.1}$$

$$x(\varphi(y)z) - \varphi((xy)z) = y(\varphi(x)z) - \varphi((yx)z), \tag{3.2}$$

$$x(z\varphi(y)) - (\varphi(x)z)y = \varphi(z(xy)) - (z\varphi(x))y. \tag{3.3}$$

Proof. Using (2.4)–(2,6), we can get the results. □

Definition 3.2. A Hom-left-symmetric algebra $(A, \cdot, \varphi)$ is *admissible* if its regular representation is admissible.

Clearly, we have the following result:

Proposition 3.3. *If a Hom-left-symmetric algebra $(A, \cdot, \varphi)$ is admissible, then the sub-adjacent Hom-Lie algebra $(\mathcal{G}(A), [\ ,\ ], \varphi)$ is also admissible.*

Corollary 3.4. *Let $(A, \cdot, \varphi)$ be a regular admissible Hom-left-symmetric algebra. Then*

$$L^*(x)\varphi^{*2}(\xi) = L^*(x)(\xi), \quad R^*(x)\varphi^{*2}(\xi) = R^*(x)(\xi).$$

Proof. If $(A, \cdot, \varphi)$ is regular, then we have $\varphi^2(x) \cdot y = x \cdot y$. Hence,

$$\begin{aligned}\langle L^*(x)\varphi^{*2}(\xi), y\rangle &= -\langle \xi, \varphi^2(x \cdot y)\rangle = -\langle \xi, \varphi^2(x) \cdot \varphi^2(y)\rangle \\ &= \langle \xi, x \cdot \varphi^2(y)\rangle = -\langle \xi, x \cdot y\rangle = -\langle L^*(x)\xi, y\rangle.\end{aligned}$$

It follows that $L^*(x)\varphi^{*2}(\xi) = L^*(x)(\xi)$. Similarly, $R^*(x)\varphi^{*2}(\xi) = R^*(x)(\xi)$. □

Proposition 3.5. *Let $(A,\cdot,\varphi)$ and $(A^*,\circ,\varphi^*)$ be two regular admissible Hom-left-symmetric algebras. Then we get*

$$\xi\circ L^*_\cdot(\varphi(x))\eta=\varphi^{*2}(\xi)\circ L^*_\cdot(\varphi(x))\eta,\quad \xi\circ R^*_\cdot(\varphi(x))\eta=\varphi^{*2}(\xi)\circ R^*_\cdot(\varphi(x))\eta$$

for any $x\in A$, $\xi,\eta\in A^*$.

Proof. For any $x,y\in A$, $\xi,\eta\in A^*$,

$$\begin{aligned}\langle\xi\circ R^*_\cdot(\varphi(x))\eta,y\rangle &= -\langle R^*_\cdot(\varphi(x))\eta,L^*_\circ(\xi)y\rangle\\ &=\langle\eta,\varphi^2(L^*_\circ(\xi)y)\varphi(x)\rangle=\langle\xi\varphi^{*2}R^*_\cdot(\varphi(x))\eta,y\rangle.\end{aligned}$$

It follows that

$$\xi\circ R^*_\cdot(\varphi(x))\eta=\xi\circ\varphi^{*2}R^*_\cdot(\varphi(x))\eta.$$

In view of $(A^*,\circ,\varphi^*)$ also being admissible, we have

$$\xi\circ\varphi^{*2}R^*_\cdot(\varphi(x))\eta=\varphi^{*2}(\xi)\circ\varphi^{*2}R^*_\cdot(\varphi(x))\eta=\varphi^{*2}(\xi)\circ R^*_\cdot(\varphi(x))\eta.$$

Therefore, we get

$$\xi\circ R^*_\cdot(\varphi(x))\eta=\varphi^{*2}(\xi)\circ R^*_\cdot(\varphi(x))\eta.$$

Analogously,

$$\xi\circ L^*_\cdot(\varphi(x))\eta=\varphi^{*2}(\xi)\circ L^*_\cdot(\varphi(x))\eta.$$

□

Definition 3.6. A pair of admissible Hom-left-symmetric algebras $(A,\cdot,\varphi)$ and $(A^*,\circ,\varphi^*)$ is called a *Hom-left-symmetric bialgebra* if

$$\begin{aligned}(3.4)\quad \langle\Delta[x,y],\varphi^*(\xi)\otimes\eta\rangle &=\langle(L_\cdot(\varphi(x))\otimes\varphi+\varphi\otimes\mathrm{ad}_{\varphi(x)})\Delta(y),\varphi^*(\xi)\otimes\eta\rangle\\ &\quad-\langle(L_\cdot(\varphi(y))\otimes\varphi+\varphi\otimes\mathrm{ad}_{\varphi(y)})\Delta(x),\varphi^*(\xi)\otimes\eta\rangle,\\ (3.5)\quad \langle\Delta^*[x,y],\varphi(x)\otimes y\rangle &=\langle(L_\circ(\varphi^*(\xi))\otimes\varphi^*+\varphi^*\otimes\mathrm{ad}_{\varphi^*(\xi)})\Delta^*(\eta),\varphi(x)\otimes y\rangle\\ &\quad-\langle(L_\circ(\varphi^*(\eta))\otimes\varphi^*+\varphi^*\otimes\mathrm{ad}_{\varphi^*(\eta)})\Delta^*(\xi),\varphi(x)\otimes y\rangle.\end{aligned}$$

We denote a Hom-left-symmetric bialgebra by $(A,A^*,\varphi,\varphi^*)$.

Theorem 3.7. *A pair of admissible Hom-left-symmetric algebras $(A,\cdot,\varphi)$ and $(A^*,\circ,\varphi^*)$ is a Hom-left-symmetric bialgebra if and only if $(\mathcal{G}(A),\mathcal{G}(A^*),L^*_\cdot,L^*_\circ)$ is a matched pair of Hom-Lie algebras.*

P r o o f. $(\mathcal{G}(A), \mathcal{G}(A^*), L^*_\cdot, L^*_\circ)$ is a matched pair of Hom-Lie algebras if and only if

$$L^*_\circ(\varphi^*(\xi))[x,y] = [L^*_\circ(\xi)x, \varphi(y)] + [\varphi(x), L^*_\circ(\xi)y] + L^*_\circ(L_\cdot(y)\xi)\varphi(x) - L^*_\circ(L_\cdot(x)\xi)\varphi(y) \tag{3.6}$$

and

$$L^*_\cdot(\varphi(x))[\xi,\eta] = [L^*_\cdot(x)\xi, \varphi^*(\eta)] + [\varphi^*(\xi), L^*_\cdot(x)\eta] + L^*_\cdot(L_\circ(\eta)x)\varphi^*(\xi) - L^*_\cdot(L_\circ(\xi)x)\varphi^*(\eta). \tag{3.7}$$

According to (3.6), we get

$$\begin{aligned}
&\langle -L^*_\circ(\varphi^*(\xi))[x,y] + [L^*_\circ(\xi)x, \varphi(y)] + [\varphi(x), L^*_\circ(\xi)y] \\
&\qquad + L^*_\circ(L_\cdot(y)\xi)\varphi(x) - L^*_\circ(L_\cdot(x)\xi)\varphi(y), \eta\rangle \\
&= \langle [x,y], \varphi^*(\xi)\circ\eta\rangle - \langle \mathrm{ad}_{\varphi(y)} L^*_\circ(\xi)x, \eta\rangle + \langle \mathrm{ad}_{\varphi(x)} L^*_\circ(\xi)y, \eta\rangle \\
&\quad - \langle \varphi(x), (L_\cdot(y)\xi)\circ\eta\rangle + \langle \varphi(y), (L_\cdot(x)\xi)\circ\eta\rangle \\
&= \langle [x,y], \varphi^*(\xi)\circ\eta\rangle - \langle x, \xi\circ \mathrm{ad}^*_{\varphi(y)}\eta\rangle + \langle y, \xi\circ \mathrm{ad}^*_{\varphi(x)}\eta\rangle \\
&\quad - \langle x, \varphi^*((L_\cdot(y)\xi))\circ\varphi^*(\eta)\rangle + \langle y, \varphi^*((L_\cdot(x)\xi))\circ\varphi^*(\eta)\rangle \\
&= \langle [x,y], \varphi^*(\xi)\circ\eta\rangle - \langle x, \varphi^{*2}(\xi)\circ \mathrm{ad}^*_{\varphi(y)}\eta\rangle + \langle y, \varphi^{*2}(\xi)\circ \mathrm{ad}^*_{\varphi(x)}\eta\rangle \\
&\quad - \langle x, L^*_\cdot(\varphi(y))\varphi^*(\xi)\circ\varphi^*(\eta)\rangle + \langle y, L^*_\cdot(\varphi(x))\varphi^*(\xi)\circ\varphi^*(\eta)\rangle \\
&= \langle \Delta[x,y], \varphi^*(\xi)\otimes\eta\rangle - \langle \Delta(x), \varphi^{*2}(\xi)\otimes \mathrm{ad}^*_{\varphi(y)}\eta\rangle \\
&\quad + \langle \Delta(y), \varphi^{*2}(\xi)\otimes \mathrm{ad}^*_{\varphi(x)}\eta\rangle - \langle \Delta(x), L^*_\cdot(\varphi(y))\varphi^*(\xi)\otimes\varphi^*(\eta)\rangle \\
&\quad + \langle \Delta(y), L^*_\cdot(\varphi(x))\varphi^*(\xi)\otimes\varphi^*(\eta)\rangle \\
&= \langle \Delta[x,y], \varphi^*(\xi)\otimes\eta\rangle - \langle (\varphi\otimes \mathrm{ad}_{\varphi(y)})\Delta(x), \varphi^*(\xi)\otimes\eta\rangle \\
&\quad + \langle (\varphi\otimes \mathrm{ad}_{\varphi(x)})\Delta(y), \varphi^*(\xi)\otimes\eta\rangle \\
&\quad - \langle (L_\cdot(\varphi(y))\otimes\varphi)\Delta(x), \varphi^*(\xi)\otimes\eta\rangle \\
&\quad + \langle (L_\cdot(\varphi(x))\otimes\varphi)\Delta(y), \varphi^*(\xi)\otimes\eta\rangle \\
&= 0,
\end{aligned}$$

which implies that (3.4) is equivalent to (3.6). Similarly, we can verify that (3.5) is equivalent to (3.7). □

Theorem 3.8. *Let $(A, \cdot, \varphi)$ and $(A^*, \circ, \varphi^*)$ be two admissible Hom-left-symmetric algebras. Then the following conditions are equivalent.*

(i) *$(A, A^*, \varphi, \varphi^*)$ is a Hom-left-symmetric bialgebra.*

(ii) $(\mathcal{G}(A)\bowtie\mathcal{G}(A)^*,\mathcal{G}(A),\mathcal{G}(A^*),\omega_p)$ *is a parakähler Hom-Lie algebra, where* ω_p *is given by*

$$\omega_p(x+\xi,y+\eta)=\langle\xi,y\rangle-\langle x,\eta\rangle$$

for any $x,y\in A$, $\xi,\eta\in A^*$.

(iii) $(\mathcal{G}(A),\mathcal{G}(A^*),L^*_\cdot,L^*_\circ,\varphi,\varphi^*)$ *is a matched pair of Hom-Lie algebras.*

(iv) $(A,A^*,L^*_\cdot-R^*_\cdot,-R^*_\cdot,L^*_\circ-R^*_\circ,-R^*_\circ,\varphi,\varphi^*)$ *is a matched pair of Hom-left-symmetric algebras.*

Proof. According to Theorem 3.7, (i) $\Leftrightarrow$ (iii). Due to Theorem 3.9 in [8], (iv) $\Leftrightarrow$ (iii). According to Theorem 2.13 in [8], (ii) $\Leftrightarrow$ (iii). □

Example 3.9. Let $(A,A^*,\varphi,\varphi^*)$ be a Hom-left-symmetric bialgebra. Then its dual $(A^*,A,\varphi^*,\varphi)$ is also a Hom-left-symmetric bialgebra.

Example 3.10. Let $(\mathcal{G},\varphi,\omega)$ be a symplectic Hom-Lie algebra. Suppose that $r\in\wedge^2\mathcal{G}$ is a nondegenerate classical r-matrix satisfying $(\varphi\otimes\varphi)r=r$, and $r^\sharp\colon\ \mathcal{G}^*\to\mathcal{G}$ is an induced linear map given by

$$\langle r^\sharp(\xi),\eta\rangle=\langle r,\xi\wedge\eta\rangle$$

and satisfying

$$\omega(x,y)=\langle {r^\sharp}^{-1}(x),y\rangle.$$

According to [6], $(\mathcal{G},\mathcal{G}^*)$ is a Hom-Lie bialgebra with

$$\langle\Delta(x),\varphi^*(\xi)\otimes\eta\rangle=\langle(\mathrm{ad}_x\otimes\varphi+\varphi\otimes\mathrm{ad}_x)r,\varphi^*(\xi)\otimes\eta\rangle$$

for any $\xi,\eta\in\mathcal{G}^*$.

On the other hand, there exists a Hom-left-symmetric algebra structure $\cdot$ on $\mathcal{G}$ given by $\omega(x\cdot y,\varphi(z))=\omega(\varphi(y),[x,z])$ for all $x,y,z\in\mathcal{G}$. Furthermore, there is a compatible left-symmetric algebra structure on the Lie algebra $\mathcal{G}^*$ given by $a\circ b={r^\sharp}^{-1}(r^\sharp(a)\cdot r^\sharp(b))$ for any $a,b\in\mathcal{G}^*$.

Moreover, by direct calculation, we have

$$L^*_\cdot(x)a={r^\sharp}^{-1}[x,r^\sharp(a)],\quad R^*_\cdot(x)a=-{r^\sharp}^{-1}[r^\sharp(a)\cdot x]$$

and

$$L^*_\circ(a)x=[r^\sharp(a),x],\quad R^*_\circ(a)x=-x\cdot r^\sharp(a)$$

for all $x\in\mathcal{G}$, $a\in\mathcal{G}^*$. Hence, according to Theorem 3.8, as left-symmetric algebras, $(\mathcal{G},\mathcal{G}^*)$ is a left-symmetric bialgebra if and only if $[[x,y],\varphi(z)]=0$ for any $x,y,z\in\mathcal{G}$.

For any $r \in \mathrm{Sym}^2(A)$, the induced linear map $r^\sharp\colon A^* \to A$ is given by $\langle r^\sharp(\xi), \eta\rangle = \langle r, \xi \otimes \eta\rangle$ for any $\xi, \eta \in A^*$. We say that r is invertible if the linear map $r^\sharp$ is an isomorphism.

Definition 4.1. A *coboundary Hom-left-symmetric bialgebra* is a Hom-left-symmetric bialgebra (A, A^*) such that

$$\langle \Delta(x), \varphi^*(\xi) \otimes \eta\rangle = \langle (L(x) \otimes \varphi + \varphi \otimes \mathrm{ad}_x) r, \varphi^*(\xi) \otimes \eta\rangle, \tag{4.1}$$

where $r \in \mathrm{Sym}^2(A)$ satisfies

$$\varphi r^\sharp \varphi^* = r^\sharp. \tag{4.2}$$

Obviously, $\varphi r^\sharp \varphi^* = r^\sharp$ is equivalent to $(\varphi \otimes \varphi) r = r$.

Proposition 4.2. *Let (A, A^*) be a coboundary Hom-left-symmetric bialgebra. Then for any $\xi \in \mathrm{Im}(\varphi^*)$ and $\eta \in A^*$ we get*

$$\xi \circ \eta = \mathrm{ad}^*_{r^\sharp \varphi^*(\xi)} \eta - R^*_\cdot(r^\sharp \varphi^*(\eta))\xi. \tag{4.3}$$

Proof. Suppose $r = X \otimes Y$, we obtain

$$\begin{aligned}
\langle x, \xi \circ \eta\rangle &= \langle \Delta(x), \xi \otimes \eta\rangle \\
&= \langle (L_\cdot(x) \otimes \varphi + \varphi \otimes \mathrm{ad}_x)(X \otimes Y), \xi \otimes \eta\rangle \\
&= \langle xX \otimes \varphi(Y), \xi \otimes \eta\rangle + \langle \varphi(X) \otimes [x, Y], \xi \otimes \eta\rangle \\
&= \langle xX, \xi\rangle\langle \varphi(Y), \eta\rangle + \langle xX, \eta\rangle\langle \varphi(Y), \xi\rangle + \langle \varphi(X), \xi\rangle\langle [x, Y], \eta\rangle \\
&\quad + \langle \varphi(X), \eta\rangle\langle [x, Y], \xi\rangle \\
&= \langle x\langle Y, \varphi^*(\eta)\rangle X, \xi\rangle + \langle x\langle Y, \varphi^*(\xi) X\rangle, \eta\rangle + \langle [x, \langle X, \varphi^*(\xi)\rangle Y, \eta\rangle \\
&\quad + \langle [x, \langle X, \varphi^*(\eta)\rangle Y, \xi\rangle \\
&= \langle x \cdot r^\sharp \varphi^*(\eta), \xi\rangle + \langle [x, r^\sharp \varphi^*(\xi)], \eta\rangle \\
&= -\langle x, R^*_\cdot(r^\sharp \varphi^*(\eta))\xi\rangle + \langle x, \mathrm{ad}^*_{r^\sharp \varphi^*(\xi)} \eta\rangle.
\end{aligned}$$

It follows that

$$\xi \circ \eta = \mathrm{ad}^*_{r^\sharp \varphi^*(\xi)} \eta - R^*_\cdot(r^\sharp \varphi^*(\eta))\xi.$$

□

Corollary 4.3. *Let (A, A^*) be a coboundary Hom-left-symmetric bialgebra. Then we get*

$$\varphi^*(\xi \circ \eta) = \varphi^*(\mathrm{ad}^*_{r^\sharp\varphi^*(\xi)}\eta - R^*_\cdot(r^\sharp\varphi^*(\eta))\xi) \tag{4.4}$$

for any $\xi, \eta \in A^*$.

Proof. For any $x \in A$, since A is admissible, using (3.1) and (4.2),

$$\begin{aligned}\langle R^*_\cdot(r^\sharp\varphi^{*2}(\xi))\varphi^*(\eta), x\rangle &= -\langle\varphi^*(\eta), x \cdot r^\sharp\varphi^{*2}(\xi)\rangle = -\langle\eta, \varphi(x) \cdot \varphi r^\sharp\varphi^{*2}(\xi)\rangle\\ &= -\langle\eta, \varphi(x) \cdot r^\sharp\varphi(\xi)\rangle = -\langle\eta, \varphi(x) \cdot \varphi^2 r^\sharp\varphi(\xi)\rangle\\ &= -\langle\varphi^*\eta, x \cdot \varphi r^\sharp\varphi(\xi)\rangle = \langle R^*_\cdot(\varphi r^\sharp\varphi(\xi))\varphi^*\eta, x\rangle\\ &= \langle\varphi^* R^*_\cdot(r^\sharp\varphi(\xi))\eta, x\rangle.\end{aligned}$$

Hence,

$$\varphi^* R^*_\cdot(r^\sharp\varphi(\xi))\eta = R^*_\cdot(r^\sharp\varphi^{*2}(\xi))\varphi^*(\eta). \tag{4.5}$$

Analogously,

$$\varphi^* L^*_\cdot(r^\sharp\varphi(\xi))\eta = L^*_\cdot(r^\sharp\varphi^{*2}(\xi))\varphi^*(\eta).$$

Therefore,

$$\varphi^*\mathrm{ad}^*_{r^\sharp\varphi(\xi)}\eta = \mathrm{ad}^*_{r^\sharp\varphi^{*2}(\xi)}\varphi^*(\eta). \tag{4.6}$$

By Proposition 4.2, we obtain

$$\varphi^*(\xi \circ \eta) = \varphi^*(\xi) \circ \varphi^*(\eta) = \mathrm{ad}^*_{r^\sharp\varphi^{*2}(\xi)}\varphi^*(\eta) - R^*_\cdot(r^\sharp\varphi^{*2}(\eta))\varphi^*(\xi). \tag{4.7}$$

Combining this with (4.5)–(4.7), we get the conclusion. □

Corollary 4.4. *Let (A, A^*) be a coboundary Hom-left-symmetric bialgebra. If A is regular, then for any $\xi \in A^*$, we have*

$$R^*_\cdot(r^\sharp\varphi^*(\xi)) = R^*_\cdot(\varphi r^\sharp(\xi), \quad \mathrm{ad}^*_{r^\sharp\varphi^*(\xi)} = \mathrm{ad}_{\varphi r^\sharp\xi}, \quad L^*_\cdot(r^\sharp\varphi^*(\xi)) = L^*_\cdot(\varphi r^\sharp(\xi)). \tag{4.8}$$

Proof. Due to (2.1) and (4.5),

$$R^*_\cdot(r^\sharp\varphi^{*2}(\xi))\varphi^*(\eta) = \varphi^* R^*_\cdot(r^\sharp\varphi(\xi))\eta = R^*_\cdot(\varphi r^\sharp\varphi(\xi))\varphi^*(\eta).$$

If A is regular, then we have

$$R^*_\cdot(r^\sharp\varphi^*(\xi)) = R^*_\cdot(\varphi r^\sharp(\xi)).$$

Similarly,

$$L^*_\cdot(r^\sharp\varphi^*(\xi)) = L^*_\cdot(\varphi r^\sharp(\xi)) \quad \text{and} \quad \mathrm{ad}^*_{r^\sharp\varphi^*(\xi)} = \mathrm{ad}^*_{\varphi r^\sharp(\xi)}.$$

□

Definition 4.5. Let $(A,\cdot,\varphi)$ be a Hom-left-symmetric algebra. For any $r = \sum\limits_i a_i \otimes b_i \in \operatorname{Sym}^2(A)$, $[[r,r]]$ is called *Hom-S-equation* in $(A,\cdot,\varphi)$, where

$$[[r,r]] = \sum_{i,j} -a_i\cdot a_j \otimes \varphi(b_i)\otimes\varphi(b_j) + \varphi(a_i)\otimes b_i\cdot a_j\otimes\varphi(b_j) + \varphi(a_i)\otimes\varphi(a_i)\otimes[b_i,b_j].$$

The Hom-S-equation in a Hom-left-symmetric algebra is an analogue of the Hom-Yang-Baxter equation in a Hom-Lie algebra, see [11]. When $\varphi = I$, Hom-S-equation becomes the S-equation discussed in [1].

Lemma 4.6. *Let $(A,\cdot,\varphi)$ be a regular admissible Hom-left-symmetric algebra. If $r \in \operatorname{Sym}^2(A)$ satisfies $\varphi r^\sharp\varphi^* = r^\sharp$ and $\xi\circ\eta = \operatorname{ad}^*_{r^\sharp\varphi^*(\xi)}\eta - R^*_\cdot(r^\sharp\varphi^*(\eta))\xi$, then we get*

$$[[r,r]](\xi,\eta) = [r^\sharp\varphi^*(\xi), r^\sharp\varphi^*(\eta)] + r^\sharp\varphi^*(\eta\circ\xi) - r^\sharp\varphi^*(\xi\circ\eta). \tag{4.9}$$

Proof. For any $\gamma\in A^*$, using Corollaries 4.3 and 4.4,

$$\begin{aligned}\langle r^\sharp\varphi^*(\xi\circ\eta),\gamma\rangle &= \langle \operatorname{ad}^*_{r^\sharp\varphi^*(\xi)}\eta - R^*_\cdot(r^\sharp\varphi^*(\eta))\xi, \varphi r^\sharp(\gamma)\rangle\\ &= \langle \xi, \varphi r^\sharp(\gamma)\cdot r^\sharp\varphi^*(\eta)\rangle - \langle \eta, [r^\sharp\varphi^*(\xi), \varphi r^\sharp(\gamma)\rangle\\ &= \langle \xi, r^\sharp\varphi^*(\gamma)\cdot r^\sharp\varphi^*(\eta)\rangle - \langle \eta, [r^\sharp\varphi^*(\xi), r^\sharp\varphi^*(\gamma)\rangle.\end{aligned} \tag{4.10}$$

On the other hand,

$$\begin{aligned}[[r,r]](\xi,\eta,\gamma) = &-\langle \xi, r^\sharp\varphi^*(\eta)\cdot r^\sharp\varphi^*(\gamma)\rangle + \langle \eta, r^\sharp\varphi^*(\xi)\cdot r^\sharp\varphi^*(\gamma)\rangle\\ &+\langle \gamma, [r^\sharp\varphi^*(\xi), r^\sharp\varphi^*(\eta)]\rangle.\end{aligned} \tag{4.11}$$

Combining this with (4.10) and (4.11), we obtain

$$[[r,r]](\xi,\eta) = [r^\sharp\varphi^*(\xi), r^\sharp\varphi^*(\eta)] + r^\sharp\varphi^*(\eta\circ\xi) - r^\sharp\varphi^*(\xi\circ\eta).$$

□

If A is regular and (A,A^*) is a coboundary Hom-left-symmetric bialgebra, then (4.1)

$$\Delta(x) = (L_\cdot(x)\otimes\varphi + \varphi\otimes\operatorname{ad}_x)r$$

and for any $\xi,\eta\in A^*$, $\xi\circ\eta$ is given by (4.3).

Theorem 4.7. *Let A be a regular admissible Hom-left-symmetric algebra. Define a bilinear map $\circ\colon A^*\otimes A^*\to A^*$ by (4.3) for some $r\in\operatorname{Sym}^2(A)$ satisfying (4.2). Then $(A^*,\circ,\varphi^*)$ is a Hom-left-symmetric algebra if and only if*

$$(L(x)\otimes\varphi\otimes\varphi + \varphi\otimes L(x)\otimes\varphi + \varphi\otimes\varphi\otimes\operatorname{ad}_x)[[r,r]] = 0.$$

Under this condition, (A,A^*) is a coboundary Hom-left-symmetric bialgebra.

P r o o f. Clearly, φ^* is an algebra homomorphism. For any $\xi, \eta, \gamma \in A^*$, in view of Lemma 4.6, we have

$$
\begin{aligned}
&(\xi, \eta, \gamma)_{\varphi^*} - (\eta, \xi, \gamma)_{\varphi^*} \\
&\quad = (\xi \circ \eta) \circ \varphi^*(\gamma) - \varphi^*(\xi) \circ (\eta \circ \gamma) - \varphi^*(\xi) \circ (\eta \circ \gamma) - (\eta \circ \xi) \circ \varphi^*(\gamma) \\
&\qquad + \varphi^*(\eta) \circ (\xi \circ \gamma) \\
&\quad = \mathrm{ad}_{r^\sharp \varphi^*(\xi\circ\eta)} \varphi * (\gamma) - R_\cdot^*(r^\sharp \varphi^{*2}(\gamma))(\xi \circ \eta) - \mathrm{ad}_{r^\sharp \varphi^*(\eta\circ\xi)} \varphi * (\gamma) \\
&\qquad + R_\cdot^*(r^\sharp \varphi^{*2}(\gamma))(\eta \circ \xi) - \mathrm{ad}^*_{r^\sharp \varphi^{*2}(\xi)}(\eta \circ \gamma) + R_\cdot^*(r^\sharp \varphi^*(\eta \circ \gamma))\varphi^*(\xi) \\
&\qquad + \mathrm{ad}^*_{r^\sharp \varphi^{*2}(\eta)}(\xi \circ \gamma) - R_\cdot^*(r^\sharp \varphi^*(\xi \circ \gamma))\varphi^*(\eta) \\
&\quad = \mathrm{ad}_{r^\sharp \varphi^*(\xi\circ\eta)} \varphi * (\gamma) - R_\cdot^*(r^\sharp \varphi^{*2}(\gamma))(\mathrm{ad}^*_{r^\sharp \varphi^*(\xi)}\eta - R_\cdot^*(r^\sharp \varphi^*(\eta))\xi) \\
&\qquad - \mathrm{ad}_{r^\sharp \varphi^*(\eta\circ\xi)} \varphi * (\gamma) + R_\cdot^*(r^\sharp \varphi^{*2}(\gamma))(\mathrm{ad}^*_{r^\sharp \varphi^*(\eta)}\xi - R_\cdot^*(r^\sharp \varphi^*(\xi))\eta) \\
&\qquad - \mathrm{ad}^*_{r^\sharp \varphi^{*2}(\xi)}(\mathrm{ad}^*_{r^\sharp \varphi^*(\eta)}\gamma - R_\cdot^*(r^\sharp \varphi^*(\gamma))\eta) + R_\cdot^*(r^\sharp \varphi^*(\eta \circ \gamma))\varphi^*(\xi) \\
&\qquad + \mathrm{ad}^*_{r^\sharp \varphi^{*2}(\eta)}(\mathrm{ad}^*_{r^\sharp \varphi^*(\xi)}\gamma - R_\cdot^*(r^\sharp \varphi^*(\gamma))\xi) - R_\cdot^*(r^\sharp \varphi^*(\xi \circ \gamma))\varphi^*(\eta) \\
&\quad = R_\cdot^*(r^\sharp \varphi^*(\eta \circ \gamma) - r^\sharp \varphi^*(\eta) r^\sharp \varphi^*(\gamma))\varphi^*(\xi) \\
&\qquad + R_\cdot^*(r^\sharp \varphi^*(\xi \circ \gamma) - r^\sharp \varphi^*(\xi) r^\sharp \varphi^*(\gamma))\varphi^*(\eta) \\
&\qquad + \mathrm{ad}^*_{r^\sharp \varphi^*([\xi,\eta]) - [r^\sharp \varphi^*(\xi), r^\sharp \varphi^*(\eta)]} \varphi^*(\gamma) \\
&\quad = R_\cdot^*([[r, r]](\eta, \gamma)\varphi^*(\xi) + R_\cdot^*([[r, r]](\xi, \gamma)\varphi^*(\eta) + \mathrm{ad}^*_{[[r,r]](\xi,\eta)}\varphi^*(\gamma).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\langle (\xi, \eta, \gamma)_{\varphi^*} - (\eta, \xi, \gamma)_{\varphi^*}, x \rangle \\
&\quad = \langle R_\cdot^*([[r, r]](\eta, \gamma))\varphi^*(\xi) + R_\cdot^*([[r, r]](\xi, \gamma))\varphi^*(\eta) - \mathrm{ad}^*_{[[r,r]](\xi,\eta)}\varphi^*(\gamma), x \rangle \\
&\quad = -\langle \varphi^*(\xi), x \cdot [[r, r]](\eta, \gamma) \rangle - \langle \varphi^*(\eta), x \cdot [[r, r]](\xi, \gamma) \rangle \\
&\qquad - \langle \varphi^*(\gamma), [x, [[r, r]](\xi, \eta)] \rangle \\
&\quad = [[r, r]](L_\cdot^*(x)\varphi^*(\xi), \eta, \gamma) + [[r, r]](\xi, L_\cdot^*(x)\varphi^*(\eta), \gamma) + [[r, r]](\xi, \eta, \mathrm{ad}^*_x \varphi^*(\gamma) \\
&\quad = [[r, r]](L_\cdot^*(x)\varphi^{*-1}(\xi), \eta, \gamma) + [[r, r]](\xi, L_\cdot^*(x)\varphi^{*-1}(\eta), \gamma) \\
&\qquad + [[r, r]](\xi, \eta, \mathrm{ad}^*_x \varphi^{*-1}(\gamma) \\
&\quad = \langle [[r, r]], (L_\cdot^*(x) \otimes \varphi \otimes \varphi)(\varphi^{*-1}(\xi) \otimes \varphi^{*-1}(\eta) \otimes \varphi^{*-1}(\gamma)) \rangle \\
&\qquad + \langle [[r, r]], (\varphi \otimes L_\cdot^*(x) \otimes \varphi)(\varphi^{*-1}(\xi) \otimes \varphi^{*-1}(\eta) \otimes \varphi^{*-1}(\gamma)) \rangle \\
&\qquad + \langle [[r, r]], (\varphi \otimes \varphi \otimes \mathrm{ad}^*_x)(\varphi^{*-1}(\xi) \otimes \varphi^{*-1}(\eta) \otimes \varphi^{*-1}(\gamma) \rangle \\
&\quad = \langle (L(x) \otimes \varphi \otimes \varphi + \varphi \otimes L(x) \otimes \varphi + \varphi \otimes \varphi \otimes \mathrm{ad}_x)[[r, r]], \\
&\qquad \varphi^{*-1}(\xi) \otimes \varphi^{*-1}(\eta) \otimes \varphi^{*-1}(\gamma) \rangle.
\end{aligned}
$$

Therefore, $(A^*, \circ)$ is a Hom-left-symmetric algebra if and only if $(L(x) \otimes \varphi \otimes \varphi + \varphi \otimes L(x) \otimes \varphi + \varphi \otimes \varphi \otimes \mathrm{ad}_x)[[r, r]] = 0$. Finally, if $\Delta(x) = (L_\cdot(x) \otimes \varphi + \varphi \otimes \mathrm{ad}_x)r$, obviously, the compatibility conditions in Definition 3.6 hold. $\square$

Let (A,φ) be a regular admissible Hom-left-symmetric algebra and $r \in \mathrm{Sym}^2(A)$ be invertible (that is, $r^\sharp$ is invertible). Define $B \in \mathrm{Sym}^2(A^*)$ by

$$B(x,y) = \langle x, r^{\sharp-1}(y)\rangle$$

for any $x, y \in A$.

Proposition 4.8. *The s-matrix r satisfies the Hom-S-equation $[[r,r]] = 0$ if and only if*

$$B(\varphi(x\cdot y),z) - B(x,\varphi(y\cdot z)) - B(\varphi(y\cdot x),z) + B(y,\varphi(x\cdot z)) = 0 \tag{4.12}$$

for any $x, y, z \in A$.

P r o o f. If r satisfies the Hom-S-equation $[[r,r]] = 0$, then for any $\xi, \eta \in A^*$, by Lemma 4.5, we obtain

$$[r^\sharp\varphi^*(\xi), r^\sharp\varphi^*(\eta)] + r^\sharp\varphi^*(\eta\circ\xi) - r^\sharp\varphi^*(\xi\circ\eta) = 0. \tag{4.13}$$

For any $x, y, z \in A$, put $x = r^\sharp(\xi)$, $y = r^\sharp(\eta)$ and $z = r^\sharp(\gamma)$. According to Corollary 4.4 and (4.13),

$$\begin{aligned}
&B(\varphi(x\cdot y),z) - B(\varphi(y\cdot x),z)\\
&\quad = B(\varphi(r^\sharp(\xi)\cdot r^\sharp(\eta)), r^\sharp(\gamma)) - B(\varphi(r^\sharp(\eta)\cdot r^\sharp(\xi)), r^\sharp(\gamma))\\
&\quad = \langle \varphi(r^\sharp(\xi)\cdot r^\sharp(\eta)), \gamma\rangle - \langle \varphi(r^\sharp(\eta)\cdot r^\sharp(\xi)), \gamma\rangle\\
&\quad = \langle r^\sharp\varphi^*(\xi)\cdot r^\sharp\varphi^*(\eta), \gamma\rangle - \langle r^\sharp\varphi^*(\eta)\cdot r^\sharp\varphi^*(\xi), \gamma\rangle\\
&\quad = \langle [r^\sharp\varphi^*(\xi), r^\sharp\varphi^*(\eta)], \gamma\rangle\\
&\quad = \langle r^\sharp\varphi^*(\xi\circ\eta) - r^\sharp\varphi^*(\eta\circ\xi), \gamma\rangle\\
&\quad = \langle r^\sharp\varphi^*(\mathrm{ad}^*_{r^\sharp\varphi^*(\xi)}\eta - R^*_\cdot(r^\sharp\varphi^*(\eta))\xi) - r^\sharp\varphi^*(\mathrm{ad}^*_{r^\sharp\varphi^*(\eta)}\xi - R^*_\cdot(r^\sharp\varphi^*(\xi))\eta), \gamma\rangle\\
&\quad = \langle r^\sharp\varphi^*(L^*_\cdot(r^\sharp\varphi^*(\xi))\eta) - r^\sharp\varphi^*(L^*_\cdot(r^\sharp\varphi^*(\eta))\xi), \gamma\rangle\\
&\quad = \langle L^*_\cdot(r^\sharp\varphi^*(\xi))\eta - L^*_\cdot(r^\sharp\varphi^*(\eta))\xi, \varphi r^\sharp(\gamma)\rangle\\
&\quad = \langle L^*_\cdot(\varphi r^\sharp(\xi))\eta - L^*_\cdot(\varphi r^\sharp(\eta))\xi, \varphi r^\sharp(\gamma)\rangle\\
&\quad = -\langle \eta, \varphi r^\sharp(\xi)\cdot\varphi r^\sharp(\gamma)\rangle + \langle \xi, \varphi r^\sharp(\eta)\cdot\varphi r^\sharp(\gamma)\rangle\\
&\quad = -\langle r^\sharp\varphi^*(\xi)\cdot r^\sharp\varphi^*(\gamma), \eta\rangle + \langle r^\sharp\varphi^*(\eta)\cdot r^\sharp\varphi^*(\gamma), \xi\rangle\\
&\quad = -\langle \varphi(r^\sharp(\xi)\cdot r^\sharp(\gamma)), \eta\rangle + \langle \varphi(r^\sharp(\eta)\cdot r^\sharp(\gamma)), \xi\rangle\\
&\quad = -B(\varphi(r^\sharp(\xi)\cdot r^\sharp(\gamma)), r^\sharp(\eta)) + B(\varphi(r^\sharp(\eta)\cdot r^\sharp(\gamma)), r^\sharp(\xi)\rangle\\
&\quad = B(\varphi(y\cdot z),x) - B(\varphi(x\cdot z),y)\\
&\quad = B(x,\varphi(y\cdot z)) - B(y,\varphi(x\cdot z)).
\end{aligned}$$

□

Remark 4.9. If φ is orthogonal or the center of A is zero, by Corollary 4.4, we have $r^\sharp\varphi^* = \varphi r^\sharp$. Thus, we have

$$B(\varphi(x), y) = \langle \varphi(x), r^{\sharp-1}(y)\rangle = \langle x, \varphi^* r^{\sharp-1}(y)\rangle = \langle x, r^{\sharp-1}\varphi(y)\rangle = B(x, \varphi(y)).$$

Therefore, in view of Proposition 4.8, we have

$$B(x\cdot y, \varphi(z)) - B(\varphi(x), y\cdot z)) - B(y\cdot x, \varphi(z)) + B(\varphi(y), x\cdot z) = 0,$$

that is, B is a 2-cocycle on A, see [4].

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