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ALMOST PERIODIC GENERALIZED SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract. The paper aims to study systems of linear ordinary differential equations in the context of an algebra of almost periodic generalized ultradistributions. Conditions on the existence of generalized solutions are given.

Keywords: almost periodicity; generalized solution; ordinary differential equations

MSC 2020: 34C27, 46F30

1. Introduction

Systems of linear ordinary differential equations $u'(t) = A(t)u + f(t)$ with smooth data f and matrix A have only smooth solutions. However, distributional solutions may appear in the case of equations with singular coefficients. Singular linear differential equations or, in general functional differential equations in the setting of distribution theory are important issues for mathematics and its applications, see [17] and references therein. Such equations appeal to the problem of multiplication of distributions. Recently there has been a considerable interest to deal with differential equations in algebras of generalized functions containing the space of distributions, see [15]. These algebras provide a suitable framework where not only nonlinear operations on distributions can be performed but also singular data may be considered. Ordinary differential equations in the context of algebras of generalized function are studied in [9], [13], [14].

Functional and differential equations in the context of periodic and classical almost periodic functions are among the subjects of many works, see [7], [10], [11] and the references therein.

Almost periodic distributions extending the classical Bohr and Stepanoff almost periodic functions are due to Schwartz (see [16]). In view of the role of algebras of

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generalized functions and the relevance of the concept of almost periodicity, an algebra of almost periodic generalized functions containing almost periodic distributions has been introduced and studied in [3]. Let's quote that the only work dealing with linear ordinary differential equations in the setting of such an algebra in the spirit of the classical theory is done in [4]: an enlargement of the reservoir of mathematical objects that could be solutions is always really needed.

It is well known that ultradistributions are generalizations of distributions, they are useful for concrete problems. However, they are also less adapted to nonlinear operations. Algebras of generalized ultradistributions containing ultradistributions are nowadays an important subject of research, see [1], [2], [8]. The almost periodicity of Beurling ultradistributions is studied in the paper [6]. So the aim of this paper is to study systems of linear ordinary differential equations in the framework of an algebra of almost periodic generalized ultradistributions containing the classical almost periodic ultradistributions of [6]. Therefore in the case, in which there are no classical almost periodic generalized solutions, we have the new concept of almost periodic generalized solutions in an ultradistributional sense. This work is an extension and a generalization of the works [3] and [4].

The paper is organised as follows. This introduction is followed by the second section where we introduce the algebra of almost periodic generalized ultradistributions $\mathcal{G}_{\mathrm{ap}}^{M}$ and then we show some properties it satisfies. In the third and last section we study systems of linear ordinary differential equations $u'(t) = Au + f(t)$ in the context of the algebra of almost periodic generalized ultradistributions $\mathcal{G}_{\mathrm{ap}}^{M}$. We prove a result of Bohr-Neugebauer type and then we give conditions on the existence of bounded solutions in a generalized sense.

2. Almost periodic generalized ultradistributions

We consider functions and generalized functions defined on the whole space of real numbers $\mathbb{R}$. For the definition and properties of almost periodic functions see [10]. We denote by $\mathcal{C}_{\mathrm{ap}}$ the space of classical Bohr almost periodic functions on $\mathbb{R}$. We recall the following spaces with some of their properties, see [3],

$$\mathcal{B}_{\mathrm{ap}} := \{\varphi \in C^{\infty} \colon \forall j \in \mathbb{Z}_{+}, \varphi^{(j)} \in C_{\mathrm{ap}}\},$$
$$\mathcal{B} := \{\varphi \in C^{\infty} \colon \forall j \in \mathbb{Z}_{+}, \varphi^{(j)} \in L^{\infty}\}.$$

Proposition 2.1.
(1) *$\mathcal{B}_{\mathrm{ap}}$ is a closed differential subalgebra of $\mathcal{B}$ stable under derivation.*
(2) $\mathcal{B}_{\mathrm{ap}} * L^1 \subset \mathcal{B}_{\mathrm{ap}}$.
(3) $\mathcal{B}_{\mathrm{ap}} = \mathcal{B} \cap C_{\mathrm{ap}}$.

We consider weight sequences $M = (M_k)_{k=0}^{\infty}$ of positive numbers satisfying the following conditions (see [12] for the meaning of these conditions):

Logarithmic convexity

$$(\mathrm{M}_1) \qquad M_k^2 \leqslant M_{k-1}M_{k+1} \quad \forall\, k \in \mathbb{Z}_+.$$

Stability under ultraderivation

$$(\mathrm{M}_2) \qquad \exists\, A > 0,\ \exists\, H > 0,\ M_{k+q} \leqslant AH^{k+q}M_kM_q \quad \forall\, k, q \in \mathbb{Z}_+.$$

Non quasi-analyticity

$$(\mathrm{M}_3') \qquad \sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty.$$

Definition 2.1. The associated function of the sequence M is the function defined by

$$M(t) = \sup_k \ln \frac{t^k M_0}{M_k}, \quad t > 0.$$

Example 2.1. If $M_k = (k!)^{\sigma}$, $\sigma > 0$, then $M(t)$ is equivalent to $t^{1/\sigma}$.

A classical important property of the associated function is given by the following result.

Proposition 2.2. *If the sequence M satisfies the condition* (M_1) *then it satisfies* (M_2) *if and only if exist $A, H > 0$ for all $t > 0$,*

$$2M(t) \leqslant M(Ht) + \ln(AM_0).$$

Let $I :=]0,1]$, if $(f_\varepsilon)_{\varepsilon \in I}$ is a net of functions, the notation

$$\|f_\varepsilon\|_\infty = O(\mathrm{e}^{M(k/\varepsilon)}), \quad \varepsilon \to 0,$$

means that exists $c > 0$, exists $\varepsilon_0 \in I$, for all $\varepsilon \leqslant \varepsilon_0$, $\|f_\varepsilon\|_\infty \leqslant c\mathrm{e}^{M(k/\varepsilon)}$.

Definition 2.2. (1) The space of almost periodic moderate elements is the space defined by

$$\mathcal{M}_{\mathrm{ap}}^{M} := \{(f_\varepsilon)_{\varepsilon \in I} \in (\mathcal{B}_{\mathrm{ap}})^I \colon\ \forall\, j \in \mathbb{Z}_+,\ \exists\, k > 0,\ \|f_\varepsilon^{(j)}\|_\infty = O(\mathrm{e}^{M(k/\varepsilon)}),\ \varepsilon \to 0\}.$$

(2) The space of almost periodic null elements is the space defined by

$$\mathcal{N}_{\mathrm{ap}}^{M} := \{(f_\varepsilon)_{\varepsilon \in I} \in (\mathcal{B}_{\mathrm{ap}})^I \colon\ \forall\, j \in \mathbb{Z}_+,\ \forall\, k > 0,\ \|f_\varepsilon^{(j)}\|_\infty = O(\mathrm{e}^{-M(k/\varepsilon)}),\ \varepsilon \to 0\}.$$

The main properties of $\mathcal{M}_{\mathrm{ap}}^M$ and $\mathcal{N}_{\mathrm{ap}}^M$ are given in the following proposition.

Proposition 2.3.
(1) *An element* $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathrm{ap}}^M$ *is null if and only if for all* $k > 0$,

$$\|f_\varepsilon\|_\infty = O(\mathrm{e}^{-M(k/\varepsilon)}), \quad \varepsilon \to 0. \tag{2.1}$$

(2) *The space* $\mathcal{M}_{\mathrm{ap}}^M$ *is an algebra stable under derivation.*
(3) *The space* $\mathcal{N}_{\mathrm{ap}}^M$ *is an ideal of* $\mathcal{M}_{\mathrm{ap}}^M$.

P r o o f. (1) The proof is based on the Landau-Kolmogorov inequality

$$\|f^{(p)}\|_\infty \leqslant 2\pi \|f\|_\infty^{1-p/m} \|f^{(m)}\|_\infty^{p/m},$$

where $0 < p < m \in \mathbb{Z}_+$ and the function f is of class C^m.

Let $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathrm{ap}}^M$, i.e. for all $i \in \mathbb{Z}_+$, exists $k_i > 0$, exists $c_i > 0$, exists $\varepsilon_i \in I$ for all $\varepsilon \leqslant \varepsilon_i$,

$$\|f_\varepsilon^{(i)}\|_\infty \leqslant c_i \exp\Big(M\Big(\frac{k_i}{\varepsilon}\Big)\Big). \tag{2.2}$$

Suppose that $(f_\varepsilon)_\varepsilon$ satisfies (2.1), i.e. for all $k > 0$, exists $c > 0$, exists $\varepsilon_0 \in I$, for all $\varepsilon \leqslant \varepsilon_0$, $\|f_\varepsilon\|_\infty \leqslant c\mathrm{e}^{-M(k/\varepsilon)}$. Fix $i \in \mathbb{N}$ and $k_0 \in \mathbb{Z}_+$ arbitrary, then the Landau-Kolmogorov inequality for $m = 2i$, $p = i$, (2.1) and (2.2) give for ε small enough,

$$\begin{aligned}\|f_\varepsilon^{(i)}\|_\infty &\leqslant 2\pi \|f_\varepsilon\|_\infty^{1-1/2} \|f_\varepsilon^{(2i)}\|_\infty^{1/2}\\ &\leqslant 2\pi (c\mathrm{e}^{-M(k/\varepsilon)})^{1/2} (c_{2i}\mathrm{e}^{M(k_{2i}/\varepsilon)})^{1/2} \leqslant C_i \mathrm{e}^{-M(k_0/\varepsilon)},\end{aligned}$$

where $k = H\max(k_{2i}, Hk_0)$ and $C_i = 2\pi A M_0 c^{1/2} c_{2i}^{1/2}$, hence the result follows.

(2) The stability with respect to the derivation is obvious. Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathrm{ap}}^M$, then they satisfy (2.2) and we have for all $n \in \mathbb{Z}_+$,

$$\|\partial^n (f_\varepsilon g_\varepsilon)\|_\infty \leqslant \sum_{i+j=n} \frac{n!}{i!\,j!} |f_\varepsilon^{(j)}(x)||g_\varepsilon^{(i)}(x)| \leqslant \sum_{i+j=n} \frac{n!}{i!\,j!} c_j \mathrm{e}^{M(k_j/\varepsilon)} c_i' \mathrm{e}^{M(k_i'/\varepsilon)}.$$

The function M being increasing and taking $t_1 = k_j/\varepsilon$, $t_2 = k_i'/\varepsilon$, $k = H \max\limits_{i+j=n}(k_j, k_i')$ and $\varepsilon \leqslant \min\limits_{i+j=n}(\varepsilon_j, \varepsilon_i')$, we obtain by Proposition 2.2

$$M\Big(\frac{k_j}{\varepsilon}\Big) + M\Big(\frac{k_i'}{\varepsilon}\Big) \leqslant M\Big(\frac{k}{\varepsilon}\Big) + \ln(AM_0),$$

and consequently

$$\|\partial^n(f_\varepsilon g_\varepsilon)\|_\infty \leqslant \Big(AM_0 \sum_{i+j=n} \frac{n!}{i!\,j!} c_j c'_i\Big) \mathrm{e}^{M(k/\varepsilon)},$$

which gives $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{M}^M_{\mathrm{ap}}$.

(3) Let $(f_\varepsilon)_\varepsilon \in \mathcal{M}^M_{\mathrm{ap}}$ and $(g_\varepsilon)_\varepsilon \in \mathcal{N}^M_{\mathrm{ap}}$, then for all $i \in \mathbb{Z}_+$, exists $k_i > 0$, exists $c_i > 0$, exists $\varepsilon_i \in I$, for all $\varepsilon \leqslant \varepsilon_i$,

$$\|f^{(i)}_\varepsilon\|_\infty \leqslant c_i \exp\Big(M\Big(\frac{k_i}{\varepsilon}\Big)\Big)$$

and for all $i \in \mathbb{Z}_+$, for all $k_1 > 0$, exists $c'_i > 0$, exists $\varepsilon'_i \in I$, for all $\varepsilon \leqslant \varepsilon'_i$,

$$\|g^{(i)}_\varepsilon\|_\infty \leqslant c'_i \mathrm{e}^{-M(k_1/\varepsilon)}. \tag{2.3}$$

Since $\mathcal{N}^M_{\mathrm{ap}} \subset \mathcal{M}^M_{\mathrm{ap}}$, (2) gives $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{M}^M_{\mathrm{ap}}$. It remains to prove (2.1). Indeed,

$$\|f_\varepsilon g_\varepsilon\|_\infty \leqslant \|f_\varepsilon\|_\infty \|g_\varepsilon\|_\infty \leqslant c_0 c'_0 \mathrm{e}^{M(k_0/\varepsilon)} \mathrm{e}^{-M(k_1/\varepsilon)}.$$

Fix $k \in \mathbb{Z}_+$ arbitrary. We obtain due to Proposition 2.2 with $t_1 = k_0/\varepsilon$, $t_2 = k/\varepsilon$, $k_1 = H\max(k_0, k)$ and $\varepsilon \leqslant \min(\varepsilon_0, \varepsilon'_0)$, that

$$M\Big(\frac{k_0}{\varepsilon}\Big) - M\Big(\frac{k_1}{\varepsilon}\Big) \leqslant -M\Big(\frac{k}{\varepsilon}\Big) + \ln(AM_0),$$

then

$$\|f_\varepsilon g_\varepsilon\|_\infty \leqslant c_0 c'_0 A M_0 \mathrm{e}^{-M(k/\varepsilon)},$$

and according to (1), we have $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{N}^M_{\mathrm{ap}}$. □

Definition 2.3. The algebra of almost periodic generalized ultradistributions, denoted by $\mathcal{G}^M_{\mathrm{ap}}$, is the quotient algebra

$$\mathcal{G}^M_{\mathrm{ap}} := \frac{\mathcal{M}^M_{\mathrm{ap}}}{\mathcal{N}^M_{\mathrm{ap}}}.$$

In order to obtain the next results we give definitions of some algebras of generalized functions. The algebra of bounded generalized ultradistributions is defined by

$$\mathcal{G}^M_{L^\infty} := \frac{\mathcal{M}^M_{L^\infty}}{\mathcal{N}^M_{L^\infty}},$$

where

$$\mathcal{M}^M_{L^\infty} := \{(f_\varepsilon)_{\varepsilon\in I} \in \mathcal{B}^I\colon\ \forall\, j \in \mathbb{Z}_+,\, \exists\, k > 0,\, \|f^{(j)}_\varepsilon\|_{L^\infty} = O(\mathrm{e}^{M(k/\varepsilon)}),\, \varepsilon \to 0\}$$

and

$$\mathcal{N}^M_{L^\infty} := \{(f_\varepsilon)_{\varepsilon I} \in \mathcal{B}^I\colon\ \forall\, j \in \mathbb{Z}_+,\, \forall k > 0,\, \|f^{(j)}_\varepsilon\|_{L^\infty} = O(\mathrm{e}^{-M(k/\varepsilon)}),\, \varepsilon \to 0\}.$$

The algebra of $\mathcal{C}_{\mathrm{ap}}$-generalized functions is defined by

$$\mathcal{G}^M_{C_{\mathrm{ap}}} := \frac{\mathcal{M}^M_{C_{\mathrm{ap}}}}{\mathcal{N}^M_{C_{\mathrm{ap}}}},$$

where

$$\mathcal{M}^M_{C_{\mathrm{ap}}} := \{(f_\varepsilon)_{\varepsilon\in I} \in (\mathcal{C}_{\mathrm{ap}})^I\colon\ \exists\, k > 0,\, \|f_\varepsilon\|_\infty = O(\mathrm{e}^{M(k/\varepsilon)}),\, \varepsilon \to 0\}$$

and

$$\mathcal{N}^M_{C_{\mathrm{ap}}} := \{(f_\varepsilon)_{\varepsilon\in I} \in (\mathcal{C}_{\mathrm{ap}})^I\colon\ \forall\, k > 0,\, \|f_\varepsilon\|_\infty = O(\mathrm{e}^{-M(k/\varepsilon)}),\, \varepsilon \to 0\}.$$

Proposition 2.4. *We have*

(1) $\mathcal{G}^M_{\mathrm{ap}}$ *is embedded into* $\mathcal{G}^M_{L^\infty}$ *and* $\mathcal{G}^M_{C_{\mathrm{ap}}}$.

(2) $\mathcal{G}^M_{\mathrm{ap}}$ *is stable under derivation and translation.*

P r o o f. (1) We have $\mathcal{G}^M_{\mathrm{ap}} \subset \mathcal{G}^M_{L^\infty}$. Indeed, let $\tilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{\mathrm{ap}}$, i.e. $(u_\varepsilon)_\varepsilon$ satisfies (2.2), as $u_\varepsilon \in \mathcal{B}_{\mathrm{ap}} = C_{\mathrm{ap}} \cap \mathcal{B} \subset \mathcal{B}$ for all $\varepsilon > 0$, then $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^\infty}$. In the same way, if $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{\mathrm{ap}}$, then $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{L^\infty}$. As obviously $\mathcal{N}^M_{L^\infty} \cap \mathcal{M}^M_{\mathrm{ap}} \subset \mathcal{N}^M_{\mathrm{ap}}$, then the embedding is clear. In the same way, considering (2.2) with $i = 0$ we obtain that $\mathcal{G}^M_{\mathrm{ap}} \subset \mathcal{G}^M_{C_{\mathrm{ap}}}$. The inclusion $\mathcal{N}^M_{C_{\mathrm{ap}}} \cap \mathcal{M}^M_{\mathrm{ap}} \subset \mathcal{N}^M_{\mathrm{ap}}$, giving the embedding, is obtained from the null characterisation of $\mathcal{N}^M_{\mathrm{ap}}$, i.e. Propositions 2.3 (1).

(2) The stability with respect to derivation and translation is obvious. $\square$

Now, we introduce the algebra of generalized numbers with asymptotic M.

Definition 2.4. The ring of generalized numbers of type M is defined by

$$\widetilde{\mathbb{K}}^M := \frac{\mathcal{M}^M[\mathbb{K}]}{\mathcal{N}^M[\mathbb{K}]},$$

where

$$\mathcal{M}^M[\mathbb{K}] := \{(z_\varepsilon)_\varepsilon \in \mathbb{K}^I,\, \exists\, k \in \mathbb{Z}_+,\, |z_\varepsilon| = O(\mathrm{e}^{M(k/\varepsilon)}),\, \varepsilon \to 0\}$$

and

$$\mathcal{N}^M[\mathbb{K}] := \{(z_\varepsilon)_\varepsilon \in \mathbb{K}^I,\, \forall\, k \in \mathbb{Z}_+,\, |z_\varepsilon| = O(\mathrm{e}^{-M(k/\varepsilon)}),\, \varepsilon \to 0\}.$$

Here $\mathbb{K}$ is the field $\mathbb{C}$ or $\mathbb{R}$.

Proposition 2.5. *The set* $\widetilde{\mathbb{K}}^M$ *is an algebra.*

Proof. The result follows easily from the condition (M_2) which gives that $\mathcal{M}^M[\mathbb{K}]$ is an algebra and $\mathcal{N}^M[\mathbb{K}]$ is an ideal of it. $\square$

Example 2.2. The number $[(\mathrm{e}^{-M(k/\varepsilon)})_\varepsilon] \in \widetilde{\mathbb{K}}^M$, $k > 0$.

A generalized trigonometric polynomial $\widetilde{P}$ is defined as

$$\widetilde{P}(x) := \sum_{k=1}^{m} \tilde{c}_k \mathrm{e}_\varepsilon^{\mathrm{i}\tilde{\lambda}_k x}, \quad x \in \mathbb{R},$$

where $\tilde{c}_k \in \widetilde{\mathbb{C}}^M$ and $\tilde{\lambda}_k \in \widetilde{\mathbb{R}}^M$.

Proposition 2.6. *Every generalized trigonometric polynomial is an almost periodic generalized ultradistribution.*

Proof. It suffices to prove that if $\tilde{\lambda} \in \widetilde{\mathbb{K}}^M$, then $(\tilde{\lambda})^j \in \widetilde{\mathbb{K}}^M$ for all $j \in \mathbb{N}$. This is a consequence of the fact that $\widetilde{\mathbb{K}}^M$ is an algebra by Proposition 2.5. $\square$

Remark 2.1. The last result gives examples of almost periodic generalized ultradistributions. Actually, we have other important examples. Let $B'_{\mathrm{ap},(M)}$ be the space of almost periodic Beurling ultradistributions of [6], under the conditions (M_1), (M_2) and (M'_3) on the weight sequence M we have that the map

$$\begin{aligned} J\colon\ B'_{\mathrm{ap},(M)} &\to \mathcal{G}^M_{\mathrm{ap}}, \\ T &\mapsto T = [(T * \varphi_\varepsilon)_\varepsilon] \end{aligned}$$

is a linear embedding, see [5]. The mollifier φ is a function taken in a suitable function space and $\varphi_\varepsilon(\cdot) := \varepsilon^{-1}\varphi(\cdot/\varepsilon)$ for $\varepsilon > 0$.

3. Systems of differential equations

Consider the system of linear ordinary differential equations

$$\dot{\tilde{u}} = A\tilde{u} + \tilde{f}, \tag{E}$$

where $\tilde{f} = ([(f_{1,\varepsilon})_\varepsilon], \ldots, [(f_{n,\varepsilon})_\varepsilon]) \in (\mathcal{G}^M_{\mathrm{ap}})^n$ and $A = (a_{ij})_{0 \leqslant i,j \leqslant n}$ is a square matrix of order n of elements of $\mathbb{C}$. The unknown generalized ultradistribution is $\tilde{u} = ([(u_{1,\varepsilon})_\varepsilon], \ldots, [(u_{n,\varepsilon})_\varepsilon])$.

R e m a r k 3.1. We say that $\tilde{u} = ([(u_{1,\varepsilon})_\varepsilon], \ldots, [(u_{n,\varepsilon})_\varepsilon])$ is bounded (or almost periodic) if each component $[(u_{i,\varepsilon})_\varepsilon]$, $0 \leqslant i \leqslant n$, is bounded (or almost periodic, respectively).

The following result is a generalized version of the Bohr-Neugebauer theorem.

Proposition 3.1. *Let a bounded generalized ultradistribution $\tilde{u} \in (\mathcal{G}^M_{L^\infty})^n$ satisfy*

$$(\mathrm{H}') \qquad (\dot{u}_\varepsilon)_\varepsilon - A(u_\varepsilon)_\varepsilon - (f_\varepsilon)_\varepsilon \in (\mathcal{N}^M_{\mathrm{ap}})^n,$$

where $(u_\varepsilon)_\varepsilon$ and $(f_\varepsilon)_\varepsilon$ are representatives of $\tilde{u}$ and $\tilde{f}$, respectively. Then $\tilde{u}$ is an almost periodic generalized ultradistribution.

P r o o f. There exists, see [10], an invertible matrix $P = (P_{ij})_{0 \leqslant i,j \leqslant n}$ such that $A = PTP^{-1}$ and

$$T = \begin{pmatrix} \lambda_1 & b_{12} & \ldots & b_{1n} \\ 0 & \lambda_2 & \ldots & b_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the matrix A. Let $\tilde{v} = ([(v_{1,\varepsilon})_\varepsilon], \ldots, [(v_{n,\varepsilon})_\varepsilon]) = P^{-1}\tilde{u}$ and $\tilde{g} = ([(g_{1,\varepsilon})_\varepsilon], \ldots, [(g_{n,\varepsilon})_\varepsilon]) = P^{-1}\tilde{f}$, then (H′) is equivalent to the system

$$(\mathrm{E}') \qquad \begin{cases} \dot{v}_{1,\varepsilon}(t) - (\lambda_1 v_{1,\varepsilon}(t) + b_{12}v_{2,\varepsilon}(t) + \ldots + b_{1n}v_{n,\varepsilon}(t) + g_{1,\varepsilon}(t)) = h_{1,\varepsilon}(t), \\ \dot{v}_{2,\varepsilon}(t) - (\lambda_2 v_{2,\varepsilon}(t) + b_{23}v_{2,\varepsilon}(t) + \ldots + b_{2n}v_{n,\varepsilon}(t) + g_{2,\varepsilon}(t)) = h_{2,\varepsilon}(t), \\ \quad\vdots \\ \dot{v}_{n,\varepsilon}(t) - (\lambda_n v_{n,\varepsilon}(t) + g_{n,\varepsilon}(t)) = h_{n,\varepsilon}(t), \end{cases}$$

where $((h_{1,\varepsilon})_\varepsilon, \ldots, (h_{n,\varepsilon})_\varepsilon) \in (\mathcal{N}^M_{\mathrm{ap}})^n$. The result is then reduced to proving that if $\tilde{v} = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{L^\infty}$ satisfies

$$(\mathrm{H}'') \qquad (\dot{v}_\varepsilon)_\varepsilon - \lambda(v_\varepsilon)_\varepsilon - (g_\varepsilon)_\varepsilon \in \mathcal{N}^M_{\mathrm{ap}},$$

where $\tilde{g} = [(g_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{\mathrm{ap}}$ and $\lambda \in \mathbb{C}$, then $\tilde{v} \in \mathcal{G}^M_{\mathrm{ap}}$. The general solution of (H″) is given by the representative

$$(v_\varepsilon(t))_\varepsilon = \left(\mathrm{e}^{\lambda t}\left(C_\varepsilon + \int_0^t \mathrm{e}^{-\lambda s}(g_\varepsilon(s) + h_\varepsilon(s))\,\mathrm{d}s\right)\right)_\varepsilon,$$

where $(C_\varepsilon)_\varepsilon \in \mathcal{M}^M[\mathbb{C}]$ and $(h_\varepsilon)_\varepsilon \in \mathcal{N}^M_{\mathrm{ap}}$. Since $\tilde{v} \in \mathcal{G}^M_{L^\infty}$, we have three cases:

(1) $v_\varepsilon(t) = -\int_t^\infty \mathrm{e}^{\lambda(t-s)}(g_\varepsilon(s) + h_\varepsilon(s))\,\mathrm{d}s$, if $\Re\lambda > 0$.

(2) $v_\varepsilon(t) = \int_{-\infty}^t \mathrm{e}^{\lambda(t-s)}(g_\varepsilon(s) + h_\varepsilon(s))\,\mathrm{d}s$, if $\Re\lambda < 0$.
(3) $v_\varepsilon(t) = \mathrm{e}^{\mathrm{i}\theta t}(C_\varepsilon + \int_0^t \mathrm{e}^{-\mathrm{i}\theta s}(g_\varepsilon(s) + h_\varepsilon(s))\,\mathrm{d}s)$, if $\Re\lambda = 0$.
In the case $\Re\lambda > 0$, we have for all $\varepsilon > 0$,

$$\sup_{t\in\mathbb{R}} |v_\varepsilon(t+\tau) - v_\varepsilon(t)| = \sup_{t\in\mathbb{R}} \left| \int_t^\infty \mathrm{e}^{\lambda(t-s)}(g_\varepsilon(s+\tau) + h_\varepsilon(s+\tau) - (g_\varepsilon(s) + h_\varepsilon(s)))\,\mathrm{d}s \right|$$
$$\leqslant \frac{1}{|\Re\lambda|} \sup_{t\in\mathbb{R}} |g_\varepsilon(s+\tau) + h_\varepsilon(s+\tau) - (g_\varepsilon(s) + h_\varepsilon(s))|,$$

which gives that v_ε is almost periodic because $g_\varepsilon + h_\varepsilon$ is almost periodic. Hence $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^\infty} \cap \mathcal{M}^M_{C_{\mathrm{ap}}} = \mathcal{M}^M_{\mathrm{ap}}$, where the last equality follows from Propositions 2.1–2.3. $\square$

R e m a r k 3.2. If we say that a bounded generalized ultradistribution $\tilde{u} \in (\mathcal{G}^M_{L^\infty})^n$ is a solution of the system (E) if it satisfies

$$(\dot{u}_\varepsilon)_\varepsilon - A(u_\varepsilon)_\varepsilon - (f_\varepsilon)_\varepsilon \in (\mathcal{N}^M_{\mathrm{ap}})^n.$$

As $\tilde{u} \in (\mathcal{G}^M_{\mathrm{ap}})^n \Rightarrow \tilde{u} \in (\mathcal{G}^M_{L^\infty})^n$ is obvious since $\mathcal{G}^M_{\mathrm{ap}} \subset \mathcal{G}^M_{L^\infty}$ due to Proposition 2.4 (1), then we have proved that a solution $\tilde{u}$ of the system (E) is an almost periodic generalized ultradistribution if and only if it is a bounded generalized ultradistribution.

A primitive of $\tilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{\mathrm{ap}}$ is defined by the representative $(U_\varepsilon)_\varepsilon$ as a Colombeau generalized function, where

$$U_\varepsilon(x) = \left(\int_{x_0}^x u_\varepsilon(y)\,\mathrm{d}y \right)_\varepsilon, \quad x_0 \in \mathbb{R}.$$

As a consequence, we have a generalized version of the Bohl-Bohr theorem.

Corollary 3.1. *A primitive of an almost periodic generalized ultradistribution is almost periodic if and only if it is a bounded generalized ultradistribution.*

The existence of an almost periodic generalized solution of the system (E) is given by the following result.

Proposition 3.2. *If the matrix A has eigenvalues whose real parts are not zero, then there exists an almost periodic generalized solution $\tilde{u}$ of the system* (E).

P r o o f. If $\Re\lambda \neq 0$, the nth equation of the system (E′) has a solution $v_{n,\varepsilon}$ defined by:
(1) $v_{n,\varepsilon}(t) = -\int_t^\infty \mathrm{e}^{\lambda_n(t-s)}(g_{n,\varepsilon}(t) + h_{n,\varepsilon}(t))\,\mathrm{d}s$, if $\Re\lambda_n > 0$.
(2) $v_{n,\varepsilon}(t) = \int_{-\infty}^t \mathrm{e}^{\lambda_n(t-s)}(g_{n,\varepsilon}(t) + h_{n,\varepsilon}(t))\,\mathrm{d}s$, if $\Re\lambda_n < 0$.

By replacing $v_{n,\varepsilon}$ in the $(n-1)$st equation of the system (E′), we obtain that $v_{n-1,\varepsilon}$ is defined as in cases (1) and (2). The same reasoning is valid for the remaining components which gives that $((v_{1,\varepsilon})_\varepsilon, \dots, (v_{n,\varepsilon})_\varepsilon)$ is a solution of the system (E′), actually this solution is a bounded generalized ultradistribution. Indeed, since $\Re\lambda_i \neq 0$, $1 \leqslant i \leqslant n$, from the formulas (1) and (2), we have

$$\|v_{n,\varepsilon}\|_\infty \leqslant \frac{1}{|\Re\lambda_n|}\|g_{n,\varepsilon}(t) + h_{n,\varepsilon}(t)\|_\infty.$$

The $(n-1)$st equation is

$$\dot{v}_{n-1,\varepsilon}(t) = \lambda_{n-1}v_{n-1,\varepsilon}(t) + (b_{n-1,n}v_{n,\varepsilon}(t) + (g_{n-1,\varepsilon}(t) + h_{n-1,\varepsilon}(t)))$$

so the estimate

$$\|v_{n-1,\varepsilon}\|_\infty \leqslant \Big(\frac{|b_{n-1n}|}{|\Re\lambda_{n-1}||\Re\lambda_n|} + \frac{1}{|\Re\lambda_{n-1}|}\Big) \\ \times \max(\|g_{n,\varepsilon}(t) + h_{n,\varepsilon}(t)\|_\infty, \|g_{n-1,\varepsilon}(t) + h_{n-1,\varepsilon}(t)\|_\infty)$$

holds for all $\varepsilon \in I$. Consequently, we obtain for $i = n, \dots, 1$ that there exists $C > 0$ for all $\varepsilon \in I$ such that

$$\|v_{i,\varepsilon}\|_\infty \leqslant C \max_{1\leqslant i\leqslant n} \|g_{i,\varepsilon}(t) + h_{i,\varepsilon}(t)\|_\infty.$$

Since the second member $((g_{1,\varepsilon} + h_{1,\varepsilon})_\varepsilon, \dots, (g_{n,\varepsilon} + h_{n,\varepsilon})_\varepsilon)$ defines a bounded generalized ultradistribution, then the solution $((v_{1,\varepsilon})_\varepsilon, \dots, (v_{n,\varepsilon})_\varepsilon)$ of the system (E′) is a bounded generalized ultradistribution. From the equality $\tilde{v} = P^{-1}\tilde{u}$, we have $\tilde{u} \in \mathcal{G}^M_{L^\infty} \Leftrightarrow \tilde{u} \in \mathcal{G}^M_{\mathrm{ap}}$ according to Proposition 3.1 and Remark 3.2. □

R e m a r k 3.3. This result generalises the result of [4].

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