Commentationes Mathematicae Universitatis Carolinae

Sanjaykumar K. Patel; Jayesh B. Vasava

On prime labeling of union of tadpole graphs

Commentationes Mathematicae Universitatis Carolinae, Vol. 63 (2022), No. 1, 33–50

Persistent URL: http://dml.cz/dmlcz/150426

On prime labeling of union of tadpole graphs

Sanjaykumar K. Patel, Jayesh B. Vasava

Abstract. A graph G of order n is said to be a prime graph if its vertices can be labeled with the first n positive integers in such a way that the labels of any two adjacent vertices in G are relatively prime. If such a labeling on G exists then it is called a prime labeling. In this paper we seek prime labeling for union of tadpole graphs. We derive a necessary condition for the existence of prime labelings of graphs that are union of tadpole graphs and further show that the condition is also sufficient in case of union of two or three tadpole graphs.

Keywords: prime labeling; tadpole graph; union of graphs

Classification: 05C78

1. Introduction

We consider only finite, simple and undirected graphs. The vertex set of a graph G is denoted by $V(G)$. The cardinality of an arbitrary set A is denoted by $|A|$. For a positive real number x, by $\lfloor x \rfloor$ we mean the greatest integer less than or equal to x.

Definition 1.1. A bijection $f\colon V(G) \to \{1, 2, \dots, |V(G)|\}$ is said to be a prime labeling of a graph G if $f(u)$ and $f(v)$ are relatively prime (i.e. $\gcd(f(u), f(v)) = 1$) whenever u and v are adjacent vertices of G. A prime graph is a graph that admits a prime labeling.

It is quite easy to observe that every path and every cycle is a prime graph. The following lemma gives a useful condition for claiming the non-existence of prime labeling of certain graphs. The condition is in terms of the independence number of a graph which is usually denoted by $\alpha(G)$.

Lemma 1.2 ([1]). *For a graph G, if $\alpha(G) < \lfloor |V(G)|/2 \rfloor$ then it is not prime.*

The notion of prime labeling was originated by A. P. Entringer and was discussed in a paper by A. Tout et al. in [7] almost forty years ago. Since then, extensive study has been carried out related to prime labeling and some of its

DOI 10.14712/1213-7243.2022.009

interesting variants like Gaussian prime labeling, see [4], and neighborhood-prime labeling, see [5]. A brief summary of results related to prime labeling and its variants is available in [2]. Our recent interest is in the study of prime labeling of union graphs. In [6], we have studied prime labeling of graphs that are obtained as a union of one point union of cycles. In the present paper, we study prime labeling in the context of union of tadpole graphs.

Definition 1.3. The tadpole graph $T_{n,m}$ (also known as dragon or kite) is the graph obtained by linking the cycle C_n and the path P_m with an edge from a vertex of C_n to a pendant vertex of P_m.

Thus the graph $T_{n,m}$ consists of $n+m$ number of vertices and the same number of edges. For instance, the graph of tadpole $T_{5,3}$ is as shown in Figure 1. Tadpole

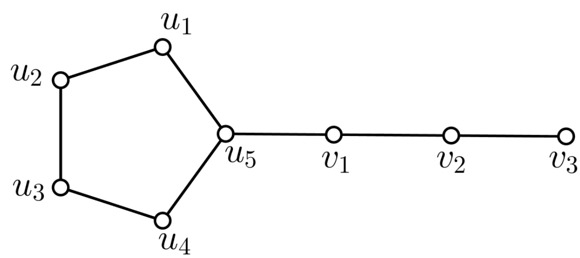

FIGURE 1. Tadpole graph $T_{5,3}$.

graphs have been extensively studied in the context of graceful and super edge magic labelings. M. Truszczynski in [8] has shown that $T_{n,m}$ are graceful for all $n \geq 3$ and $m \geq 1$ whereas S. R. Kim and J. Y. Park in [3] have shown that $T_{n,1}$ is super edge magic if and only if n is odd and that $T_{n,3}$ is super edge magic if and only if n is odd and at least 5. Our results on tadpole graphs in this paper are briefly discussed below.

It is quite easy to show that every tadpole graph $T_{n,m}$ is prime. But we quickly realize that investigating about prime labeling of union of tadpole graphs needs some serious thinking. We begin our investigation by deriving a necessary condition for the existence of a prime labeling for the union of finitely many tadpole graphs. Next we show that this condition is also sufficient if we restrict our attention to graphs that are union of at the most three tadpole graphs. Later, we prove that if n is an even number, then the union of six or fewer copies of $T_{n,n}$ admits a prime labeling. Finally, we show that if $n+m$ is an even number, then the union of $T_{n,m}$ with any other prime graph of order $n+m$, results into a prime graph.

2. Main results

We introduce the concept of parity for the tadpole graph $T_{n,m}$. The idea about parity is quite useful in understanding our main results to follow. We say that $T_{n,m}$ is of r-s parity if the parities of n and m are r and s, respectively. The following table indicates how parity of the tadpole graph defines a relation between its order and its independence number. This relation is useful for getting a necessary condition for the union of tadpole graphs to be prime.

Graph G	**Parity**	$\boldsymbol{\alpha(G)}$	$\boldsymbol{\lvert V(G)\rvert}$	**Relation between $\alpha(G)$ and $\lvert V(G)\rvert$**
$T_{2n,2m}$	even-even	$n+m$	$2n+2m$	$2\alpha(G)=\lvert V(G)\rvert$
$T_{2n+1,2m+1}$	odd-odd	$n+m+1$	$2n+2m+2$	$2\alpha(G)=\lvert V(G)\rvert$
$T_{2n,2m+1}$	even-odd	$n+m+1$	$2n+2m+1$	$2\alpha(G)=\lvert V(G)\rvert+1$
$T_{2n+1,2m}$	odd-even	$n+m$	$2n+2m+1$	$2\alpha(G)=\lvert V(G)\rvert-1$

Table 1. Relation between the independence number and the order of a tadpole graph.

Theorem 2.1. *Let G be a graph obtained by taking the union of a finite number of tadpole graphs. Suppose $G=\bigcup_{i=1}^{k} T_{n_i,m_i}$. If G is prime, then*

$$\begin{aligned} &|\{i\colon\ n_i \text{ is odd and where } 1\le i\le k\}| \\ &\qquad+|\{i\colon\ m_i \text{ is even and where } 1\le i\le k\}|\le k+1. \end{aligned} \tag{1}$$

Proof: Assume that

$$\begin{aligned} &|\{i\colon\ n_i \text{ is odd and where } 1\le i\le k\}| \\ &\qquad+|\{i\colon\ m_i \text{ is even and where } 1\le i\le k\}|\ge k+2. \end{aligned}$$

Then G must contain at least two odd-even tadpoles. We assume (without loss of generality) that T_{n_1,m_1} and T_{n_2,m_2} are of odd-even parities. As a result

$$\begin{aligned} &|\{i\colon\ n_i \text{ is odd and where } 3\le i\le k\}| \\ &\qquad+|\{i\colon\ m_i \text{ is even and where } 3\le i\le k\}|\ge k-2. \end{aligned}$$

Now set $H=\bigcup_{i=3}^{k} T_{n_i,m_i}$ and consider a non negative integer λ such that

$$\begin{aligned} &|\{i\colon\ n_i \text{ is odd and where } 3\le i\le k\}| \\ &\qquad+|\{i\colon\ m_i \text{ is even and where } 3\le i\le k\}|=k-2+\lambda. \end{aligned}$$

We use this identity to show that

$$2\alpha(H) = |V(H)| - \lambda.$$

For this, we set

$$\begin{aligned}
c_{\mathrm{e,o}} &= |\{i\colon\ T_{n_i,m_i} \text{ is even-odd and where } \ 3 \le i \le k\}|,\\
c_{\mathrm{o,e}} &= |\{i\colon\ T_{n_i,m_i} \text{ is odd-even and where } \ 3 \le i \le k\}|,\\
c_{\mathrm{e,e}} &= |\{i\colon\ T_{n_i,m_i} \text{ is even-even and where } \ 3 \le i \le k\}|, \quad \text{and}\\
c_{\mathrm{o,o}} &= |\{i\colon\ T_{n_i,m_i} \text{ is odd-odd and where } \ 3 \le i \le k\}|.
\end{aligned}$$

It is obvious that $c_{\mathrm{e,o}} + c_{\mathrm{o,e}} + c_{\mathrm{e,e}} + c_{\mathrm{o,o}} = k - 2$ and moreover

$$\begin{aligned}
(c_{\mathrm{o,e}} + c_{\mathrm{o,o}}) + (c_{\mathrm{o,e}} + c_{\mathrm{e,e}}) &= (|\{i\colon\ n_i \text{ is odd and where } \ 3 \le i \le k\}|)\\
&\quad + (|\{i\colon\ m_i \text{ is even and where } 3 \le i \le k\}|)\\
&= \ -2 + \lambda\\
&= c_{\mathrm{e,o}} + c_{\mathrm{o,e}} + c_{\mathrm{e,e}} + c_{\mathrm{o,o}} + \lambda.
\end{aligned}$$

Consequently, $c_{\mathrm{o,e}} = c_{\mathrm{e,o}} + \lambda$ and so using the relation between the independence number and the order of a tadpole graph, see Table 1, we deduce that

$$2\alpha(H) = 2\sum_{i=3}^{k} \alpha(T_{n_i,m_i}) = \sum_{i=3}^{k}(n_i + m_i) + (c_{\mathrm{e,o}} - c_{\mathrm{o,e}}) = |V(H)| - \lambda.$$

This means that $2\alpha(H) \le |V(H)|$ and as a result

$$\begin{aligned}
2\alpha(G) &= 2\alpha(H) + 2\alpha\Big(T_{n_1,m_1} \bigcup T_{n_2,m_2}\Big)\\
&\le |V(H)| + \Big|V\Big(T_{n_1,m_1} \bigcup T_{n_2,m_2}\Big)\Big| - 2\\
&= |V(G)| - 2 < |V(G)| - 1.
\end{aligned}$$

Thus $\alpha(G) < \lfloor |V(G)|/2 \rfloor$ and so in view of Lemma 1.2, we conclude that G is not a prime graph. □

Theorem 2.1 gives a necessary condition (1) for the union of k tadpole graphs to be prime. We shall show that this condition is also sufficient when $k = 2$ or 3. For this, we first ask the reader to verify the statements of the following two lemmas (which simply state condition (1) in terms of the parities of the tadpole graphs).

Lemma 2.2. *For the graph $G = T_{n_1,m_1} \bigcup T_{n_2,m_2}$, the condition*

$$|\{i\colon\ n_i \text{ is odd and where } 1 \le i \le 2\}| \\ + |\{i\colon\ m_i \text{ is even and where } 1 \le i \le 2\}| \le 3$$

holds if and only if the parity of at least one of the two tadpoles is different from being odd-even.

Lemma 2.3. *For the graph $G = T_{n_1,m_1} \bigcup T_{n_2,m_2} \bigcup T_{n_3,m_3}$, the condition*

$$|\{i\colon\ n_i \text{ is odd and where } 1 \le i \le 3\}| \\ + |\{i\colon\ m_i \text{ is even and where } 1 \le i \le 3\}| \le 4$$

holds if and only if one of the following four possibilities occur:

(a) *At least one of the three tadpoles is of even-odd parity.*
(b) *There exists a pair of tadpoles with even-even and odd-odd parity.*
(c) *At least two of the three tadpoles are of odd-odd parity.*
(d) *At least two of the three tadpoles are of even-even parity.*

In view of Lemma 2.2, Lemma 2.3 and Theorem 2.1, we conclude that the parity conditions stated in these two lemmas are necessary for claiming the primality of the respective union graphs. Our next few results focus on showing that these parity conditions are also sufficient for the said results.

Proposition 2.4. *The graph $G = T_{n_1,m_1} \bigcup T_{n_2,m_2}$ is prime if the parity of at least one of the two tadpoles is different from being odd-even.*

PROOF: For $j = 1, 2$, let C_{n_j} and P_{m_j} denote the cycle and the path involved in the tadpole T_{n_j,m_j}. Further, we denote the vertices of C_{n_j} by $u^j_1, u^j_2, u^j_3, \ldots, u^j_{n_j}$ and that of P_{m_j} by $v^j_1, v^j_2, v^j_3, \ldots, v^j_{m_j}$ such that $u^j_{n_j}$ is adjacent to v^j_1.

We assume (without loss of generality) that the parity of T_{n_1,m_1} is different from odd-even. Thus either n_1 is even or both n_1 and m_1 are odd. We show that G is prime in either case.

Case-I: Assume that n_1 is even. Define $f\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$\begin{aligned} f(u^1_i) &= i + 1 && \text{for all } i,\\ f(v^1_i) &= n_1 + i + 1 && \text{for all } i,\\ f(u^2_i) &= n_1 + m_1 + i + 1, && i \ne n_2,\\ f(u^2_{n_2}) &= 1, &&\\ f(v^2_i) &= n_1 + m_1 + n_2 + i && \text{for all } i. \end{aligned}$$

The definition of f is illustrated in Figure 2 and it suggests that except for the pair $(u_1^1, u_{n_1}^1)$, all the other pairs of adjacent vertices in G have either consecutive labels or one of the two labels is equal to 1. Moreover, $f(u_1^1) = 2$ whereas $f(u_{n_1}^1) = n_1 + 1$ is an odd number and so we conclude that any two adjacent vertices in G have relatively prime labels under f.

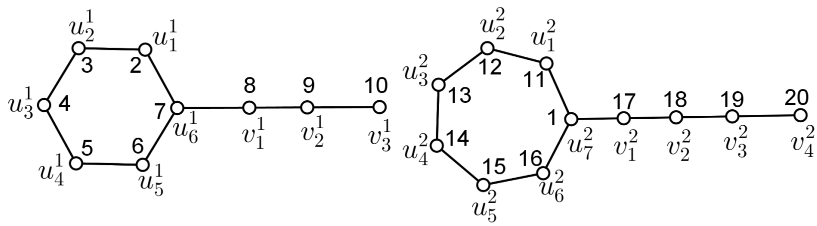

FIGURE 2. Prime labeling of $T_{6,3} \bigcup T_{7,4}$.

Case-II: Assume that n_1 and m_1 both are odd.
Subcase-1: Let $n_1 \not\equiv 1 \pmod 3$. Define $g\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$\begin{aligned} g(u_i^1) &= i + 2 && \text{for all } i, \\ g(v_i^1) &= n_1 + i + 2 && \text{for all } i, \\ g(u_1^2) &= 2, \end{aligned}$$

$$\begin{aligned} g(u_i^2) &= n_1 + m_1 + i + 1, && i \neq 1, n_2, \\ g(u_{n_2}^2) &= 1, \\ g(v_i^2) &= n_1 + m_1 + n_2 + i && \text{for all } i. \end{aligned}$$

The definition of g is illustrated in Figure 3. Note that except for the pairs $(u_1^1, u_{n_1}^1)$ and (u_1^2, u_2^2), every other pair of adjacent vertices have either consecutive labels or one of them is labelled as 1. Also $g(u_1^1) = 3$ is relatively prime to $g(u_{n_1}^1) = n_1 + 2$ since $n_1 \not\equiv 1 \pmod 3$, whereas $g(u_1^2) = 2$ and $g(u_2^2) = n_1 + m_1 + 3$ are relatively prime since $n_1 + m_1 + 3$ is an odd number.
Subcase-2: Let $n_1 \equiv 1 \pmod 3$. If $n_1 \equiv 1 \pmod 3$ then the labeling g is no longer prime and so we modify it as per our requirement. Define $h\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$\begin{aligned} h(x) &= g(x), && x \neq u_i^1, \text{ where } i = 1, 2, \ldots, n_1 - 1, \\ h(u_1^1) &= g(u_{n_1-1}^1), \\ h(u_i^1) &= g(u_{i-1}^1), && i = 2, 3, \ldots, n_1 - 1. \end{aligned}$$

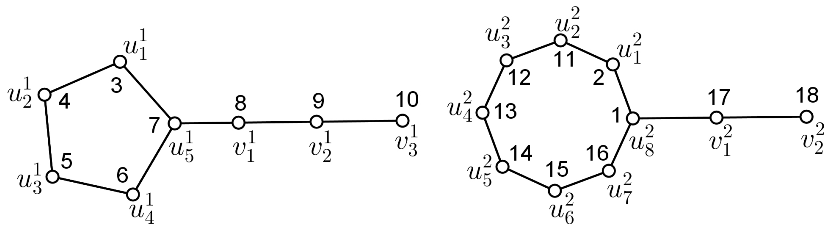

FIGURE 3. Prime labeling of $T_{5,3} \bigcup T_{8,2}$.

We ask the reader to verify that h defines a prime labeling on G and thus concludes the proof of the proposition. □

Our next goal is to establish that the graph obtained by taking the union of three tadpoles is a prime graph under any of the four parity conditions stated in Lemma 2.3. We achieve this by proving the next four propositions in which the vertex set of the graph $T_{n_1,m_1} \bigcup T_{n_2,m_2} \bigcup T_{n_3,m_3}$ is fixed as follows. For $j = 1, 2, 3$; let C_{n_j} and P_{m_j} denote the cycle and the path involved in the tadpole T_{n_j,m_j}. Further, we denote the vertices of C_{n_j} by $u_1^j, u_2^j, u_3^j, \ldots, u_{n_j}^j$ and that of P_{m_j} by $v_1^j, v_2^j, v_3^j, \ldots, v_{m_j}^j$ such that $u_{n_j}^j$ is adjacent to v_1^j.

Proposition 2.5. *The graph $G = T_{n_1,m_1} \bigcup T_{n_2,m_2} \bigcup T_{n_3,m_3}$ is prime if at least one of the three tadpole graphs is of even-odd parity.*

PROOF: We assume (without loss of generality) that T_{n_1,m_1} is of even-odd parity and prove the result by considering two cases.

Case-I: Let $n_1 + m_1 + n_2 \not\equiv 2 \pmod 3$. Define $f\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$
\begin{aligned}
f(u_i^1) &= i + 3 && \text{for all } i,\\
f(v_i^1) &= i + n_1 + 3 && \text{for all } i,\\
f(u_i^2) &= 4 - i, && i = 1, 2,\\
f(u_i^2) &= i + n_1 + m_1 + 1, && i \neq 1, 2,\\
f(v_i^2) &= i + n_1 + m_1 + n_2 + 1 && \text{for all } i,\\
f(u_i^3) &= i + n_1 + m_1 + n_2 + m_2 + 1, && i \neq n_3,\\
f(u_{n_3}^3) &= 1, &&\\
f(v_i^3) &= i + n_1 + m_1 + n_2 + m_2 + n_3 && \text{for all } i.
\end{aligned}
$$

The definition of f is illustrated in Figure 4. The definition of f suggests that except for the pairs of vertices $(u_1^1, u_{n_1}^1)$, (u_2^2, u_3^2) and $(u_1^2, u_{n_2}^2)$, every other pair of adjacent vertices has either consecutive labels or one of them has label as 1.

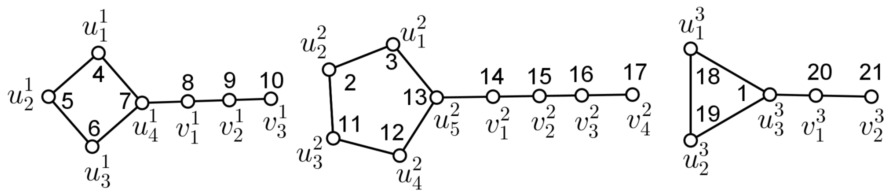

FIGURE 4. Prime labeling of $T_{4,3} \bigcup T_{5,4} \bigcup T_{3,2}$.

Further, since $f(u_{n_1}^1) = n_1 + 3$ and $f(u_3^2) = n_1 + m_1 + 4$ are odd integers and labels of u_1^1 and u_2^2 are 4 and 2, respectively, we have $\gcd(f(u_1^1), f(u_{n_1}^1)) = 1$ and $\gcd(f(u_2^2), f(u_3^2)) = 1$. Finally $f(u_1^2) = 3$ and $f(u_{n_2}^2) = n_1 + m_1 + n_2 + 1$ are relatively prime as $n_1 + m_1 + n_2 \not\equiv 2 \pmod 3$ and thus f is a prime labeling on G.

Case-II: Let $n_1 + m_1 + n_2 \equiv 2 \pmod 3$. If $n_1 + m_1 + n_2 \equiv 2 \pmod 3$ then the function f defined above is not a prime labeling of G since

$$\gcd(f(u_1^2), f(u_{n_2}^2)) = \gcd(3, n_1 + m_1 + n_2 + 1) = 3.$$

So to overcome this problem we modify f as per the following two subcases.

Subcase-1: Assume that n_2 is odd. Define $g\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$\begin{aligned} g(x) &= f(x), && x \neq u_i^2, \quad \text{where } i = 1, 2, \ldots, n_2 - 1, \\ g(u_1^2) &= f(u_{n_2-1}^2), && \\ g(u_i^2) &= f(u_{i-1}^2), && i = 2, 3, \ldots, n_2 - 1. \end{aligned}$$

Here $g(u_1^2) = n_1 + m_1 + n_2$ and $g(u_2^2) = 3$ are relatively prime because $n_1 + m_1 + n_2 \equiv 2 \pmod 3$. Further, note that the vertices adjacent to $u_{n_2}^2$ are u_1^2 and $u_{n_2-1}^2$ whose labels are $n_1 + m_1 + n_2$ and $n_1 + m_1 + n_2 - 1$, respectively. But $g(u_{n_2}^2) = n_1 + m_1 + n_2 + 1$ is an odd number now and so it is relatively prime to $g(u_1^2)$ and $g(u_{n_2-1}^2)$. Concluding that g is a required labeling is now easy.

Subcase-2: Assume that n_2 is even. Define $h\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$\begin{aligned} h(x) &= f(x), && x \neq v_1^2, \ x \neq u_i^2, \\ h(u_i^2) &= f(u_{i+2}^2), && i = 1, 2, 3, \ldots, n_2 - 2, \\ h(u_{n_2-1}^2) &= f(v_1^2), && \\ h(u_{n_2}^2) &= f(u_2^2), && \\ h(v_1^2) &= f(u_1^2). && \end{aligned}$$

Observe that $h(v_1^2) = 3$ and $h(v_2^2) = n_1+m_1+n_2+3$ are relatively prime because $n_1 + m_1 + n_2 \equiv 2 \pmod 3$. Further, the neighbors of $u_{n_2}^2$ are u_1^2, $u_{n_2-1}^2$ and v_1^2 whose labels are $n_1 + m_1 + 4$, $n_1 + m_1 + n_2 + 2$ and 3, respectively, which are all odd integers, whereas $h(u_{n_2}^2) = 2$. Therefore verifying that h is a prime labeling on G is not difficult now. □

Proposition 2.6. *The graph $G = T_{n_1,m_1} \bigcup T_{n_2,m_2} \bigcup T_{n_3,m_3}$ is prime if there exists a pair of tadpoles with even-even and odd-odd parity.*

PROOF: Assuming that T_{n_1,m_1} is of even-even parity whereas T_{n_2,m_2} is of odd-odd parity, define $f\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$\begin{aligned}
f(u_1^1) &= 2, && \\
f(u_i^1) &= i + n_2 + m_2 + 1, && i \neq 1,\\
f(v_i^1) &= i + n_1 + n_2 + m_2 + 1 && \text{for all } i,\\
f(u_i^2) &= 5 - i, && i = 1, 2,\\
f(u_i^2) &= i + 2, && i = 3, 4, \ldots, n_2,\\
f(v_i^2) &= i + n_2 + 2 && \text{for all } i,\\
f(u_i^3) &= i + n_1 + m_1 + n_2 + m_2 + 1, && i \neq n_3,\\
f(u_{n_3}^3) &= 1, && \\
f(v_i^3) &= i + n_1 + m_1 + n_2 + m_2 + n_3 && \text{for all } i.
\end{aligned}$$

The definition of f is illustrated in Figure 5. Note that if (u, v) is a pair of adja-

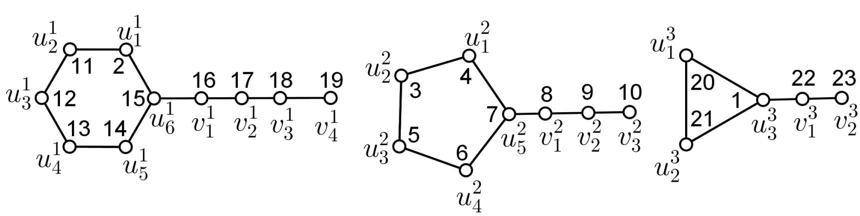

FIGURE 5. Prime labeling of $T_{6,4} \bigcup T_{5,3} \bigcup T_{3,2}$.

cent vertices which is different from (u_1^1, u_2^1), $(u_1^1, u_{n_1}^1)$, $(u_1^2, u_{n_2}^2)$ and (u_2^2, u_3^2) then $f(u)$ and $f(v)$ are relatively prime because either $f(u)$ and $f(v)$ are consecutive integers or one of them is equal to 1. Moreover if $(u, v) = (u_2^2, u_3^2)$ then $f(u) = 3$ whereas $f(v) = 5$ and so they are relatively prime. Finally, if (u, v) is one of the three pairs (u_1^1, u_2^1), $(u_1^1, u_{n_1}^1)$ or $(u_1^2, u_{n_2}^2)$ then the same holds since $f(u)$ is either 2 or 4 whereas $f(v)$ is an odd number for these three pairs. Thus f is a prime labeling on G. □

Proposition 2.7. *The graph $G = T_{n_1,m_1} \bigcup T_{n_2,m_2} \bigcup T_{n_3,m_3}$ is prime if at least two of the three tadpole graphs are of odd-odd parity.*

PROOF: Assume that the tadpoles T_{n_1,m_1} and T_{n_2,m_2} are of odd-odd parity. We consider four cases over here.

Case-I: Let $n_1 \not\equiv 1 \pmod 5$ and $n_1 + m_1 + n_2 \not\equiv 1 \pmod 3$. Define $f\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$
\begin{aligned}
f(u_i^1) &= i + 4 && \text{for all } i,\\
f(v_i^1) &= i + n_1 + 4 && \text{for all } i,\\
f(u_i^2) &= i + 2, && i = 1, 2,\\
f(u_i^2) &= i + n_1 + m_1 + 2, && i \neq 1, 2,\\
f(v_i^2) &= i + n_1 + m_1 + n_2 + 2 && \text{for all } i,\\
f(u_1^3) &= 2, &&\\
f(u_i^3) &= i + n_1 + m_1 + n_2 + m_2 + 1, && i \neq 1, n_3,\\
f(u_{n_3}^3) &= 1, &&\\
f(v_i^3) &= i + n_1 + m_1 + n_2 + m_2 + n_3 && \text{for all } i.
\end{aligned}
$$

The definition of f is illustrated in Figure 6. We see that if (u, v) is a pair of

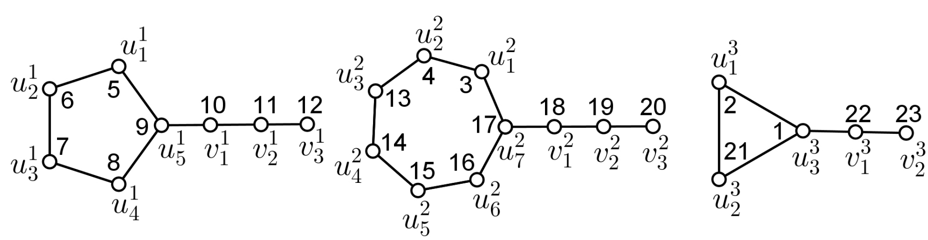

FIGURE 6. Prime labeling of $T_{5,3} \bigcup T_{7,3} \bigcup T_{3,2}$.

adjacent vertices which is different from $(u_1^1, u_{n_1}^1)$, $(u_1^2, u_{n_2}^2)$, (u_2^2, u_3^2) and (u_1^3, u_2^3) then $\gcd(f(u), f(v)) = 1$ because either $f(u)$ and $f(v)$ are consecutive integers or one of them is equal to 1. Further, if (u, v) is one of these four pairs then the same holds due to one of the following three reasons:

- $f(u) = 5$, $f(v) = n_1 + 4$ and $n_1 \not\equiv 1 \pmod 5$;
- $f(u) = 3$, $f(v) = n_1 + m_1 + n_2 + 2$ and $n_1 + m_1 + n_2 \not\equiv 1 \pmod 3$;
- $f(u)$ is a power of 2 and $f(v)$ is an odd number.

Thus f is a prime labeling on G whenever $n_1 \not\equiv 1 \pmod 5$ and $n_1 + m_1 + n_2 \not\equiv 1 \pmod 3$. Note that if one of these assumptions fails then f is not a prime labeling but it may be modified to give a prime labeling in each of the respective cases. We give these modified labelings and ask the reader to verify the details.

Case-II: Let $n_1 \not\equiv 1 \pmod 5$ and $n_1 + m_1 + n_2 \equiv 1 \pmod 3$.
Define $g\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$\begin{aligned} g(x) &= f(x), && x \neq u_i^2, \quad \text{where } i = 1, 2, 3, \ldots, n_2 - 1, \\ g(u_1^2) &= f(u_{n_2-1}^2), && \\ g(u_i^2) &= f(u_{i-1}^2), && i = 2, 3, \ldots, n_2 - 1. \end{aligned}$$

Case-III: Let $n_1 \equiv 1 \pmod 5$ and $n_1 + m_1 + n_2 \not\equiv 1 \pmod 3$. Define $h\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$\begin{aligned} h(x) &= f(x), && x \neq u_i^1, \quad \text{where } \ i = 1, 2, \ldots, n_1 - 1, \\ h(u_1^1) &= f(u_{n_1-1}^1), && \\ h(u_i^1) &= f(u_{i-1}^1), && i = 2, 3, \ldots, n_1 - 1. \end{aligned}$$

The definition of h is illustrated in Figure 7.

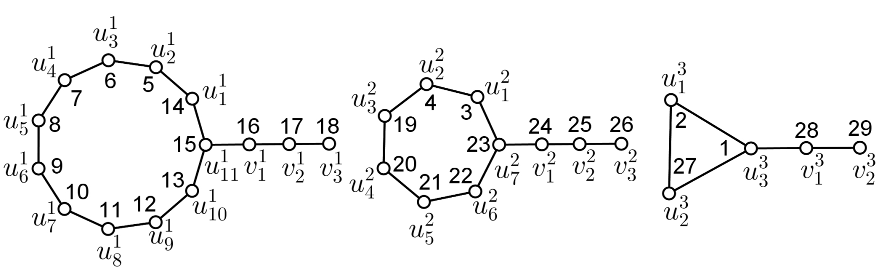

FIGURE 7. Prime labeling of $T_{11,3} \bigcup T_{7,3} \bigcup T_{3,2}$.

Case-IV: Let $n_1 \equiv 1 \pmod 5$ and $n_1 + m_1 + n_2 \equiv 1 \pmod 3$. Define $s\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$\begin{aligned} s(x) &= h(x), && x \neq u_i^2, \quad \text{where } i = 1, 2, \ldots, n_2 - 1, \\ s(x) &= g(x), && x = u_i^2, \quad \text{where } i = 1, 2, \ldots, n_2 - 1. \end{aligned}$$

$\square$

Proposition 2.8. *The graph $G = T_{n_1,m_1} \bigcup T_{n_2,m_2} \bigcup T_{n_3,m_3}$ is prime if two or more tadpoles are of even-even parity.*

Proof: Assume that T_{n_1,m_1} and T_{n_2,m_2} are of even-even parity. We define the required labeling as per the following two cases on the number n_1+m_1. The verification part is left to the reader.

Case-I: Let $n_1+m_1 \not\equiv 2 \pmod 3$. Define $f\colon V(G) \to \{1,2,\dots,|V(G)|\}$ as

$$
\begin{aligned}
f(u_i^1) &= i+3 && \text{for all } i,\\
f(v_i^1) &= i+n_1+3 && \text{for all } i,\\
f(u_i^2) &= i+1, && i=1,2,\\
f(u_i^2) &= i+n_1+m_1+1, && i\neq 1,2,\\
f(v_i^2) &= i+n_1+m_1+n_2+1 && \text{for all } i,\\
f(u_i^3) &= i+n_1+m_1+n_2+m_2+1, && i\neq n_3,\\
f(u_{n_3}^3) &= 1, && \\
f(v_i^3) &= i+n_1+m_1+n_2+m_2+n_3 && \text{for all } i.
\end{aligned}
$$

The definition of f is illustrated in Figure 8.

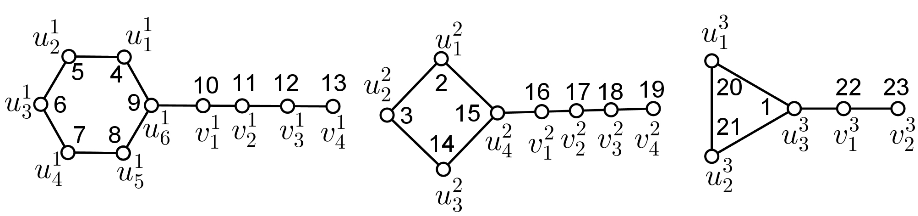

Figure 8. Prime labeling of $T_{6,4} \bigcup T_{4,4} \bigcup T_{3,2}$.

Case-II: Let $n_1+m_1 \equiv 2 \pmod 3$. Define $g : V(G) \to \{1,2,\dots,|V(G)|\}$ as

$$
\begin{aligned}
g(x) &= f(x), && x\neq v_{m_1}^1, u_2^2,\\
g(v_{m_1}^1) &= f(u_2^2), &&\\
g(u_2^2) &= f(v_{m_1}^1). &&
\end{aligned}
$$

□

We may summarise the results proven so far as:

Theorem 2.9. *The graphs $T_{n_1,m_1} \bigcup T_{n_2,m_2}$ and $T_{n_1,m_1} \bigcup T_{n_2,m_2} \bigcup T_{n_3,m_3}$ are prime if and only if the condition* (1) *holds with $k=2$ and $k=3$, respectively.*

Our next two theorems focus on prime labeling of graphs that are obtained as a union of more than three tadpole graphs.

Theorem 2.10. *Let $kT_{n,n}$ denote union of k copies of $T_{n,n}$. If n is even, then the graph $G = kT_{n,n}$ is prime for $1 \le k \le 6$.*

Proof: For $j = 1, 2, \ldots, k$, let C_n^j and P_n^j denote the cycle and the path involved in the jth copy of the tadpole $T_{n,n}$. Further, we denote the vertices of C_n^j by $u_1^j, u_2^j, u_3^j, \ldots, u_n^j$ and that of P_n^j by $v_1^j, v_2^j, v_3^j, \ldots, v_n^j$ such that u_n^j is adjacent to v_1^j.

The case $k = 1$ is trivial whereas $k = 2$ and $k = 3$ are included in our previous results. So here we prove the result for $k = 4, 5$ and 6 only. We give appropriate labeling in each of the three cases and show only nontrivial calculations. The reader can easily verify the other details.

Case-I: Let $k = 4$. Define $f\colon V(G) \to \{1, 2, \ldots, 8n\}$ as

$$
\begin{aligned}
f(u_i^j) &= 2n(j-1) + i + 1, && j = 1, 2 \text{ and for all } i,\\
f(v_i^j) &= 2n(j-1) + i + n + 1, && j = 1, 2 \text{ and for all } i,\\
f(u_i^3) &= 4n + i + 3 && \text{for all } i,\\
f(v_i^3) &= 5n + i + 3, && i = 1, 2, \ldots, n-2,\\
f(v_i^3) &= 3n + i + 3, && i = n-1, n,\\
f(u_1^4) &= 1, && \\
f(u_i^4) &= 6n + i, && i = 2, 3, \ldots, n,\\
f(v_i^4) &= 7n + i && \text{for all } i.
\end{aligned}
$$

The definition of f is illustrated in Figure 9. Since n is an even number, we have

- $\gcd(f(u_1^1), f(u_n^1)) = \gcd(2, n+1) = 1$;
- $\gcd(f(u_1^2), f(u_n^2)) = \gcd(2n+2, 3n+1) = \gcd(2n+2, n-1) = \gcd(2(n-1)+4, n-1) = \gcd(4, n-1) = 1$;
- $\gcd(f(u_1^3), f(u_n^3)) = \gcd(4n+4, 5n+3) = \gcd(4n+4, n-1) = \gcd(4(n-1)+8, n-1) = \gcd(8, n-1) = 1$;
- $\gcd(f(v_{n-2}^3), f(v_{n-1}^3)) = \gcd(6n+1, 4n+2) = \gcd(2n-1, 4n+2) = \gcd(2n-1, 2(2n-1)+4) = \gcd(2n-1, 4) = 1$.

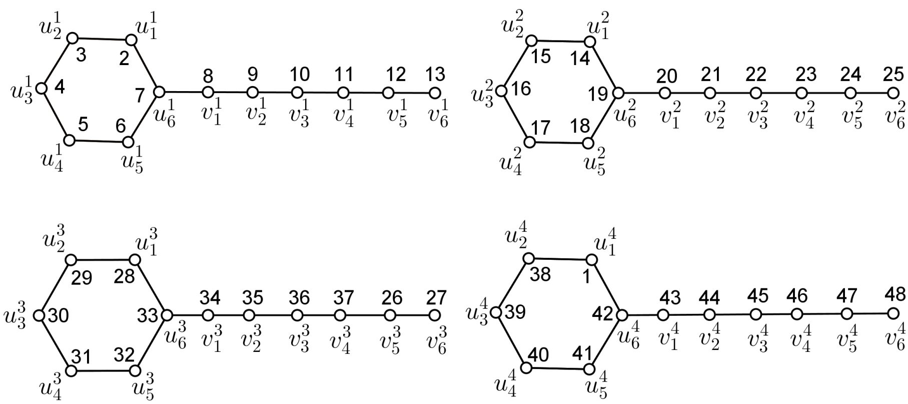

FIGURE 9. Prime labeling of $4T_{6,6}$.

Case-II: Let $k = 5$. Using the labeling f, we define $g\colon V(G) \to \{1, 2, \dots, 10n\}$ as

$$
\begin{aligned}
g(x) &= f(x), && x = u_i^j, v_i^j,\ j = 1,2,3 \text{ and for all } i,\\
g(u_i^4) &= 6n + i + 1 && \text{for all } i,\\
g(v_i^4) &= 7n + i + 1 && \text{for all } i,\\
g(u_1^5) &= 1, &&\\
g(u_i^5) &= 8n + i, && i = 2,3,\dots,n,\\
g(v_i^5) &= 9n + i && \text{for all } i.
\end{aligned}
$$

As n is even, $\gcd(g(u_1^4), g(u_n^4)) = \gcd(6n+2, 7n+1) = \gcd(6n+2, n-1) = \gcd(6(n-1)+8, n-1) = \gcd(8, n-1) = 1$.

Case-III: Let $k = 6$. Using the labeling f, we define $h\colon V(G) \to \{1, 2, \dots, 12n\}$ as

$$
\begin{aligned}
h(x) &= f(x), && x = u_i^j, v_i^j,\ j = 1,2,3 \text{ and for all } i,\\
h(u_i^4) &= 6n + i + 1 && \text{for all } i,\\
h(v_i^4) &= 7n + i + 1, && i = 1,2,\dots,n-2,\\
h(v_i^4) &= 11n - i - 2, && i = n-1, n,\\
h(u_i^5) &= 8n + i - 1 && \text{for all } i,\\
h(v_i^5) &= 9n + i - 1, && i = 1,2,\dots,n-2,\\
h(v_i^5) &= 11n - i, && i = n-1, n,
\end{aligned}
$$

$$h(u_1^6) = 1,$$
$$h(u_i^6) = 10n + i, \qquad i = 2, 3, \ldots, n,$$
$$h(v_i^6) = 11n + i \qquad \text{for all } i.$$

Since n is even,

- $\gcd(h(u_1^4), h(u_n^4)) = \gcd(6n+2, 7n+1) = \gcd(6n+2, n-1) = \gcd(6(n-1)+8, n-1) = \gcd(8, n-1) = 1$;
- $\gcd(h(v_{n-2}^4), h(v_{n-1}^4)) = \gcd(8n-1, 10n-1) = \gcd(8n-1, 2n) = \gcd(1, 2n) = 1$;
- $\gcd(h(u_1^5), h(u_n^5)) = \gcd(8n, 9n-1) = \gcd(8n, n-1) = \gcd(8(n-1)+8, n-1) = \gcd(8, n-1) = 1$;
- $\gcd(h(v_{n-2}^5), h(v_{n-1}^5)) = \gcd(10n-3, 10n+1) = \gcd(10n-3, 4) = 1$.

□

Theorem 2.11. *If $n \geq 3$, then the graph $G = kT_{n,n-2}$ is prime for all positive integers k.*

Proof: The vertices of the cycle and the path involved in the jth, $1 \leq j \leq k$, copy of the tadpole $T_{n,n-2}$ will be denoted by the symbols u_i^j and v_r^j such that u_n^j is adjacent to v_1^j, where $1 \leq i \leq n$ and $1 \leq r \leq n-2$. We define $f\colon V(G) \to \{1, 2, \ldots, k(2n-2)\}$ as

$$f(u_i^j) = 2(j-1)(n-1) + i \qquad \text{for all } j \text{ and all } i,$$
$$f(v_r^j) = 2(j-1)(n-1) + n + r \qquad \text{for all } j \text{ and all } r.$$

The definition of f is illustrated in Figure 10. Thus

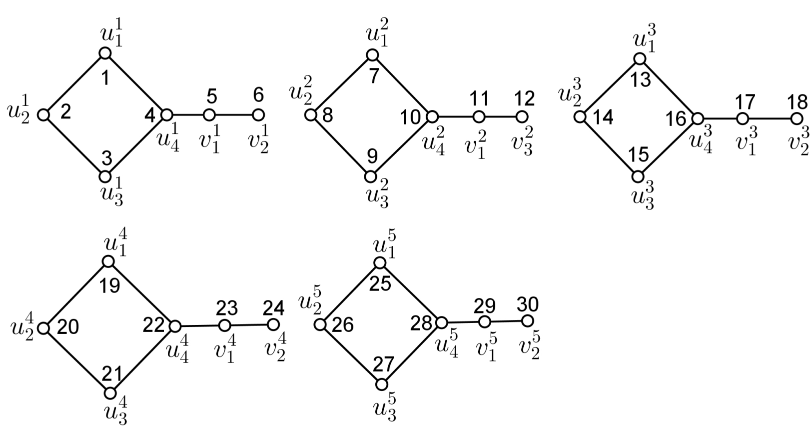

Figure 10. Prime labeling of $5\,T_{4,2}$.

$$\begin{aligned}\gcd(f(u_1^j), f(u_n^j)) &= \gcd(2(j-1)(n-1)+1, 2(j-1)(n-1)+n)\\ &= \gcd(2(j-1)(n-1)+1, n-1)\\ &= \gcd(1, n-1) = 1.\end{aligned}$$

The remaining pairs of adjacent vertices are labelled with consecutive integers and hence f is a prime labeling on G. □

Our final result is about the union of a tadpole graph with a prime graph of the same order.

Theorem 2.12. *Consider a tadpole graph $T_{n,m}$ whose order $n+m$ is an even number. If H is a prime graph of the same order then the graph $G = H \cup T_{n,m}$ is also a prime graph.*

Proof: As usual we shall denote the vertices of the cycle and the path involved in the tadpole $T_{n,m}$ in terms of u and v, respectively. Let f be a prime labeling of H which exists because H is assumed to be a prime graph. Consider an arbitrary prime number p laying strictly between $n+m$ and $2(n+m)$ that exists due to Bertrand's postulate which states that if $N > 1$, then there is a prime number laying strictly between N and $2N$. We prove the theorem by considering the following three cases on the prime number p.

Case-I: Let $p = 2n+m$. Define $g\colon V(G) \to \{1, 2, \dots, 2n+2m\}$ as

$$\begin{aligned} g(x) &= f(x), && x \in V(H),\\ g(u_i) &= m+n+i && \text{for all } i,\\ g(v_i) &= m+2n+i && \text{for all } i.\end{aligned}$$

Since

$$\gcd(g(u_1), g(u_n)) = \gcd(n+m+1, 2n+m) = \gcd(n+m+1, p) = 1,$$

it is obvious that g is a prime labeling.

Case-II: Let $p < 2n+m$. Define $h\colon V(G) \to \{1, 2, \dots, 2n+2m\}$ as

$$\begin{aligned} h(x) &= f(x), && x \in V(H),\\ h(u_i) &= p+m+i, && i = 1, 2, \dots, 2n+m-p,\\ h(u_i) &= p-n+i, && i = 2n+m-p+1, 2n+m-p+2, \dots, n,\\ h(v_i) &= p+i, && \text{for all } i.\end{aligned}$$

Note that the labels of u_1 and u_n are $p+m+1$ and p, respectively, which are relatively prime because p is a prime number strictly greater than $m+1$. Further,

$n + m$ being an even number, we have

$$\begin{aligned}\gcd(h(u_{2n+m-p}), h(u_{2n+m-p+1})) &= \gcd(2n + 2m, n + m + 1)\\ &= \gcd(n + m, n + m + 1) = 1.\end{aligned}$$

The argument that h is a prime labeling is trivial now.

Case-III: Let $p > 2n + m$. Define $s\colon V(G) \to \{1, 2, \ldots, 2n + 2m\}$ as

$$\begin{aligned}
s(x) &= f(x), && x \in V(H),\\
s(u_i) &= p - n + i && \text{for all } i,\\
s(v_i) &= p + i, && i = 1, 2, \ldots, 2n + 2m - p,\\
s(v_i) &= p - n - m + i, && i = 2n + 2m - p + 1, 2n + 2m - p + 2, \ldots, m.
\end{aligned}$$

The verification part is similar and so it is left to the reader. □

3. Conclusion and future scope

We have derived a necessary condition for the union of tadpole graphs to be prime and have shown that the condition is also sufficient if we consider union of at most three tadpole graphs. Investigating the sufficiency part in case of four or more tadpoles could be an interesting and a challenging task. Also, we have proved that $kT_{n,n}$ is prime for all even numbers n and for all $k \leq 6$. Similar investigation for odd n or higher values of k will be also interesting.

References

[1] Fu H.-L, Huang K. C., *On prime labellings*, Discrete Math. **127** (1994), no. 1–3, 181–186.

[2] Gallian J. A., *A dynamic survey of graph labeling*, Electron. J. Combin. **5** (1998), Dynamic Survay 6, 43 pages.

[3] Kim S.-R., Park J. Y., *On super edge-magic graphs*, Ars Combin. **81** (2006), 113–127.

[4] Klee S., Lehmann H., Park A., *Prime labeling of families of trees with Gaussian integers*, AKCE Int. J. Graphs Combin. **13** (2016), no. 2, 165–176.

[5] Patel S. K., Shrimali N. P., *Neighborhood-prime labeling of some product graphs*, Algebra Discrete Math. **25** (2018), no. 1, 118–129.

[6] Patel S. K., Vasava J., *On prime labeling of some union graphs*, Kragujevac J. Math. **42** (2018), no. 3, 441–452.

[7] Tout R., Dabboucy A. N., Howalla K., *Prime labeling of graphs*, Nat. Acad. Sci. Lett. **11** (1982), 365–368.

[8] Truszczyński M., *Graceful unicyclic graphs*, Demonstratio Math. **17** (1984), no. 2, 377–387.

S. K. Patel:
Department of Mathematics, Government Engineering College-Bhuj,
Near Uma Nagar, New Ravalwadi Relocation Site, Mirzapar Road,
Bhuj-370001, India

E-mail: skpatel27@yahoo.com

J. B. Vasava:
Department of Mathematics, Gujarat University, Navrangpura, Ahmedabad,
Gujarat-380009, India

E-mail: jayeshvasava1910@gmail.com

(*Received* October 17, 2020*, revised* May 6, 2021)