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EXISTENCE RESULTS FOR SOME NONLINEAR PARABOLIC
EQUATIONS WITH DEGENERATE COERCIVITY
AND SINGULAR LOWER-ORDER TERMS

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Abstract. In this paper, we study the existence results for some parabolic equations with degenerate coercivity, singular lower order term depending on the gradient, and positive initial data in L^1 .

Keywords: singular equation; nonlinear parabolic equation; degenerate coercivity

MSC 2020: 35K55, 35K65, 35K67

1. INTRODUCTION

We study the existence and regularity results for the following parabolic problem:

$$(P) \quad \begin{cases} \partial_t u + Lu + B \frac{|\nabla u|^p}{u^\theta} = u^r & \text{in } Q_T = (0, T) \times \Omega, \\ u(0, x) = u_0(x) \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_T = (0, T) \times \partial\Omega, \end{cases}$$

where $T > 0$, $B > 0$ are real numbers, $u_0 \in L^1(\Omega)$, Ω is a bounded open subset of \mathbb{R}^N ($N > 2$) with boundary denoted by $\partial\Omega$, p is a real number such that $p \geq 2$ and L is the operator given by

$$Lu = -\operatorname{div}(A(t, x, u)|\nabla u|^{p-2}\nabla u).$$

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Here, we suppose that $A: (0, T) \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function and for almost every $(t, x) \in (0, T) \times \Omega$, for all $s \in \mathbb{R}$ satisfies

$$(1.1) \quad \frac{\beta}{(a(t, x) + |s|)^\varrho} \leq A(t, x, s) \leq \alpha,$$

where α, β are strictly positive real numbers and $\varrho \geq 0$, $a: (0, T) \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a measurable nonnegative function verifying

$$(1.2) \quad a(t, x) \leq \delta,$$

where δ is a strictly positive real number. We furthermore suppose that

$$(1.3) \quad 0 < \theta < 1, \quad 0 < r < p - \theta.$$

If (1.1) holds true, the differential operator L is not coercive when u is large. Moreover, the lower order term is singular as u tends to zero. We overcome these two difficulties by approximation of (P) by a sequence of nondegenerate and nonsingular problems (in the case $u_0 \in L^\infty(\Omega)$), and passing to the limit in the approximate problems we prove that (P) admits at least one solution $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(\Omega)$. In the case $u_0 \in L^1(\Omega)$ we approach u_0 by $u_{0n} \in L^\infty(\Omega)$ and we use the results of the first case to achieve the passage to the limit in the approximate problems by proving the existence of the solution $u \in L^q(0, T; W_0^{1,q}(\Omega))$, $q = p - \theta N / (N + 1)$.

For the case $p = 2$, $\varrho = 0$ and for positive initial data, the existence of solutions to problem (P) is proved in [3] under the operator $v \rightarrow -\operatorname{div}(M(t, x)\nabla v)$, where $M: Q_T \rightarrow \mathbb{R}^{N^2}$ is a measurable bounded and uniformly elliptic matrix.

If the nonlinear right-hand term is not present, i.e., in the evolutive case, problems as

$$(P1) \quad \begin{cases} \partial_t u - \Delta_p u + B \frac{|\nabla u|^p}{u^\theta} = f & \text{in } Q_T = (0, T) \times \Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_T = (0, T) \times \partial\Omega, \end{cases}$$

under various assumptions on the summability of the source f , have been considered in the case $p = 2$ and $\theta < 1$ in [8]. If $\theta = 1$, the existence solution has been considered in [12] and [13] for smooth strictly positive data, while degenerate problems were studied in [15] in the one dimensional case and $p > 2$.

Let us also mention that in [4] the authors proved the existence and nonexistence of solutions for a general class of singular homogeneous (i.e., $f \equiv 0$) parabolic problems as (P1) with $p \geq 2$.

In [11], the author showed the existence of positive solutions of elliptic equations with degenerate coercivity and singular quadratic lower-order terms

$$-\operatorname{div}(M(x, u)\nabla u) + b(x)\frac{|\nabla u|^2}{u^\theta} = u^r + f, \quad f \in L^1(\Omega).$$

The aim of this paper is to extend the results in [3] to the case of degenerate parabolic equations with $p \geq 2$ and establish the existence of weak solutions of problem (P) for nonnegative initial data $u_0 \in L^1(\Omega)$.

This paper is organized as follows. In Section 2, we define the weak solution and prove the existence of weak solutions u for the first case $u_0 \in L^\infty(\Omega)$. Section 3 is devoted to the study of (P) with an initial datum $u_0 \in L^1(\Omega)$. We give a better regularity result compared to [9] because if $\theta \in (0, 1)$, we have

$$1 - \frac{\theta N}{N + 1} > 1 - \frac{N}{N + 1},$$

so Theorem 3.1 improves (see Theorem 1, [9]).

2. BOUNDED INITIAL DATA ($u_0 \in L^\infty(\Omega)$)

In this section, we prove that there exists a weak solution of problem (P) for u_0 bounded. For this, we use the result in [1] and then an L^∞ -estimate procedure introduced by [5]. Given a real positive number k , we define the functions

$$T_k(r) = \begin{cases} k & \text{if } r \geq k, \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leq -k, \end{cases} \quad r \in \mathbb{R}.$$

Its primitive $\Theta_k: \mathbb{R} \rightarrow \mathbb{R}^+$ is defined by

$$\Theta_k(r) = \int_0^r T_k(t) dt = \begin{cases} \frac{r^2}{2} & \text{if } |r| \leq k, \\ k|r| - \frac{k^2}{2} & \text{if } |r| > k. \end{cases}$$

We then use the following results:

$$(2.1) \quad \int_0^T \langle \partial_t v, T_k(v) \rangle dt = \int_\Omega \Theta_k(v(T)) dx - \int_\Omega \Theta_k(v(0)) dx$$

and

$$(2.2) \quad k|r| - \frac{k^2}{2} \leq \Theta_k(r) \leq k|r| \quad \forall r \in \mathbb{R}.$$

Definition 2.1. A function u is a weak solution of problem (P) if $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ such that $\partial_t u \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q_T)$, $u > 0$, $|\nabla u|^p/u^\theta$ belongs to $L^1(Q_T)$, and

$$\begin{aligned} \int_{\Omega} u(T)\varphi(T) \, dx - \int_{\Omega} u(0)\varphi(0) \, dx + \int_{Q_T} u \partial_t \varphi \, dx \, dt \\ + \int_{Q_T} A(t, x, u) |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \, dt \\ + B \int_{Q_T} \frac{|\nabla u|^p}{u^\theta} \varphi \, dx \, dt = \int_{Q_T} u^r \varphi \, dx \, dt \end{aligned}$$

for every $\varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ such that $\partial_t \varphi \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q_T)$.

Remark 2.2. Notice that because of the fact that $u \in C([0, T]; L^2(\Omega))$ (see [7]), the functions $\varphi(T)$ and $\varphi(0)$ in the above definition have sense and the meaning of the initial condition $u(0) = u_0$ is clear.

Theorem 2.3. Let $p \geq 2$, $u_0 \in L^\infty(\Omega)$, suppose that (1.3) holds true. Then problem (P) has at least one weak solution $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \cap C([0, T]; L^1(\Omega))$.

2.1. Proof of Theorem 2.3. We approximate problem (P) by following nonsingular problem:

$$(P_n^*) \quad \begin{cases} \partial_t u_n - \operatorname{div}(A(t, x, T_n(u_n)) |\nabla u_n|^{p-2} \nabla u_n) \\ \quad + B \frac{u_n |\nabla u_n|^p}{(|u_n| + 1/n)^{\theta+1}} = T_n(|u_n|^r) & \text{in } Q_T, \\ u_n(0, x) = u_0(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_T. \end{cases}$$

Note that by (1.1) we have

$$A(t, x, T_n(u_n)) \geq \frac{\beta}{(a(t, x) + |T_n(u_n)|)^q} \geq \frac{\beta}{(\delta + n)^q},$$

so the operator $B: v \mapsto \operatorname{div}(A(t, x, T_n(v)) |\nabla v|^{p-2} \nabla v)$ is coercive. Thus, the existence of the approximate solution is proved as in [7]. We begin by proving that $u_n \geq 0$, using $u_n^- = \min(0, u_n)$ as a test function in (P_n^*) and by (1.1), we have

$$(2.3) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^T (u_n^-)^2 \, dt + \beta \int_{Q_T} \frac{|\nabla u_n^-|^p}{(a(t, x) + |T_n(u_n)|)^q} \, dx \, dt \\ + B \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(|u_n| + 1/n)^{\theta+1}} u_n^- \, dx \, dt \leq \int_{Q_T} |u_n|^r u_n^- \, dx \, dt. \end{aligned}$$

Since $u_0 \geq 0$, we have

$$\frac{1}{2} \frac{d}{dt} \int_0^T (u_n^-)^2 dt = \frac{1}{2} \int_{\Omega} (u_n^-(T, x))^2 dx - \frac{1}{2} \int_{\Omega} (u_n^-(0, x))^2 dx \geq 0.$$

The lower-order term has the same sign of the solution and dropping nonnegative terms, we get

$$\beta \int_{Q_T} \frac{|\nabla u_n^-|^p}{(a(t, x) + |T_n(u_n)|)^{\theta}} dx dt \leq \int_{Q_T} |u_n|^r u_n^- dx dt \leq 0.$$

Thus, $u_n^- = 0$ and so $u_n \geq 0$. Therefore, u_n solves

$$(P_n) \quad \begin{cases} \partial_t u_n - \operatorname{div}(A(t, x, T_n(u_n)) |\nabla u_n|^{p-2} \nabla u_n) \\ \quad + B \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} = T_n(u_n^r) & \text{in } Q_T, \\ u_n(0, x) = u_0(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_T. \end{cases}$$

Lemma 2.4. *Let $p \geq 2$ and u_n be the solutions to problems (P_n) . Then we have for all $k > 0$*

$$(2.4) \quad \int_{Q_T} |\nabla T_k(u_n)|^p dx dt \leq \frac{k(\delta + k)^{\theta}}{\beta} \left(\int_{Q_T} u_n^r dx dt + \|u_0\|_{L^1(\Omega)} \right).$$

Proof. Choosing $T_k(u_n)$ as test function in (P_n) and the fact that $T_n(u_n^r) \leq u_n^r$, we obtain

$$(2.5) \quad \begin{aligned} \int_{\Omega} \Theta_k(u_n)(T) dx + \int_{Q_T} A(t, x, T_n(u_n)) |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n) dx dt \\ + B \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} T_k(u_n) dx dt \\ \leq \int_{Q_T} u_n^r T_k(u_n) dx dt + \int_{\Omega} \Theta_k(u_n)(0) dx. \end{aligned}$$

The first term is positive since we have $\Theta_k \geq 0$, so after dropping nonnegative terms and using (2.2), we obtain

$$(2.6) \quad \int_{Q_T} A(t, x, T_n(u_n)) |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n) dx dt \leq \int_{Q_T} u_n^r |T_k(u_n)| dx dt + k \|u_0\|_{L^1(\Omega)}.$$

According to conditions (1.1) and for $n > k > 0$, we get

$$(2.7) \quad \frac{\beta}{(\delta + k)^{\theta}} \int_{Q_T} |\nabla T_k(u_n)|^p dx dt \leq k \int_{Q_T} u_n^r dx dt + k \|u_0\|_{L^1(\Omega)}.$$

Therefore (2.4) is established. □

Lemma 2.5. *Let u_n be the solutions to problems (P_n) . Then*

$$(2.8) \quad B \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt \leq \int_{Q_T} u_n^r dx dt + \|u_0\|_{L^1(\Omega)}.$$

Proof. Choosing $T_h(u_n)/h$ as a test function in (P_n) , dropping the nonnegative terms, we obtain

$$B \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \frac{T_h(u_n)}{h} dx dt \leq \int_{Q_T} u_n^r \left| \frac{T_h(u_n)}{h} \right| dx dt + \frac{1}{h} \int_{\Omega} |\Theta_h(u_n)(0)| dx.$$

Using the fact that $|T_h(u_n)/h| \leq 1$ and (2.2), we have

$$B \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \frac{T_h(u_n)}{h} dx dt \leq \int_{Q_T} u_n^r dx dt + \|u_0\|_{L^1(\Omega)}.$$

Letting h tend to 0 and by Fatou's Lemma, we deduce (2.8). \square

We shall denote by C or C_j various constants depending only on the structure of A , p , θ , r , T , u_0 , $|\Omega|$ for $j \in \mathbb{N}$.

Lemma 2.6. *Let u_n be the solutions to problems (P_n) . Then there exists a positive constant C such that*

$$\|u_n\|_{L^\infty(Q_T)} \leq C, \quad \|u_n\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C \quad \forall n \in \mathbb{N}.$$

Proof. Choosing $\varphi = (u_n + \delta)^\nu - \delta^\nu$ as a test function in (P_n) , where $\nu > 0$, using (1.1), we obtain

$$\begin{aligned} & \int_0^T \langle \partial_t u_n, (u_n + \delta)^\nu - \delta^\nu \rangle dt + \beta \nu \int_{Q_T} \frac{|\nabla u_n|^p}{(\delta + u_n)^\varrho} (u_n + \delta)^{\nu-1} dx dt \\ & \quad + B \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} (u_n + \delta)^\nu dx dt \\ & \leq B \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \delta^\nu dx dt \\ & \quad + \int_{Q_T} u_n^r (u_n + \delta)^\nu - \int_{Q_T} u_n^r \delta^\nu dx dt, \end{aligned}$$

dropping the nonpositive term on the right-hand side and putting $\nu = 1$, we get

$$\begin{aligned} & \int_0^T \langle \partial_t u_n, u_n \rangle dt + \int_{Q_T} |\nabla u_n|^p (u_n + \delta)^{1-\theta} \left(\frac{\beta}{(\delta + u_n)^{\varrho-\theta+1}} + \frac{B u_n (u_n + \delta)^\theta}{(u_n + 1/n)^{\theta+1}} \right) dx dt \\ & \leq B \delta \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt + \int_{Q_T} u_n^r (u_n + \delta) dx dt. \end{aligned}$$

Since $\varrho - \theta + 1 > 0$, there exists a positive constant C_0 such that

$$\frac{\beta}{(\delta + t)^{\varrho - \theta + 1}} + \frac{Bt(t + \delta)^\theta}{(t + 1/n)^{\theta + 1}} \geq C_0 > 0 \quad \forall t \geq 0.$$

So, after dropping nonnegative terms, we obtain

$$C_0 \int_{Q_T} |\nabla u_n|^p (u_n + \delta)^{1 - \theta} \, dx \, dt \leq B\delta \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta + 1}} \, dx \, dt + \int_{Q_T} u_n^r (u_n + \delta) \, dx \, dt + \frac{1}{2} \int_{\Omega} u_0^2 \, dx.$$

Using (2.8) and the fact that $u_n^r \leq (u_n + \delta)^r$, $u_0 \in L^\infty(\Omega)$, we get

$$\begin{aligned} C_0 \int_{Q_T} |\nabla u_n|^p (u_n + \delta)^{1 - \theta} \, dx \, dt &\leq B\delta \int_{Q_T} (u_n + \delta)^r \, dx \, dt + \int_{Q_T} (u_n + \delta)^{r + 1} \, dx \, dt + C \\ &\leq C \int_{Q_T} (u_n + \delta)^{r + 1} \, dx \, dt + C, \end{aligned}$$

which implies

$$\int_{Q_T} |\nabla (u_n + \delta)^{(p + 1 - \theta)/p}|^p \, dx \, dt \leq C \int_{Q_T} (u_n + \delta)^{r + 1} \, dx \, dt + C.$$

Using Poincaré inequality, we have

$$\int_{Q_T} (u_n + \delta)^{p + 1 - \theta} \, dx \, dt \leq C \int_{Q_T} (u_n + \delta)^{r + 1} \, dx \, dt + C,$$

since $r + 1 < p + 1 - \theta$, Young inequality yields

$$\int_{Q_T} (u_n + \delta)^{p + 1 - \theta} \, dx \, dt \leq \frac{1}{2} \int_{Q_T} (u_n + \delta)^{p + 1 - \theta} \, dx \, dt + C,$$

which implies that $(u_n + \delta)_n$ is bounded in $L^{p + 1 - \theta}(Q_T)$, so $(u_n)_n$ is bounded in $L^{p + 1 - \theta}(Q_T)$. Now, we prove that the sequence $(u_n^r)_n$ is bounded in $L^m(Q_T)$ for some $m > \frac{1}{2}N + 1$. We choose u_n^η as a test function in (P_n) , where $\eta > 1$, we find

$$\int_{Q_T} \frac{\beta\eta}{(u_n + \delta)^\varrho} |\nabla u_n|^p u_n^{\eta - 1} \, dx \, dt + B \int_{Q_T} \frac{u_n^{\eta + 1} |\nabla u_n|^p}{(u_n + 1/n)^{\theta + 1}} \, dx \, dt \leq \int_{Q_T} u_n^{r + \eta} \, dx \, dt + C.$$

With the same previous calculations, we find

$$\begin{aligned} \int_{Q_T} |\nabla u_n|^p u_n^{\eta - \theta} \left(\frac{\beta\eta}{u_n^{1 - \theta} (\delta + u_n)^\varrho} + \frac{B u_n^{1 + \theta}}{(u_n + 1/n)^{\theta + 1}} \right) \, dx \, dt \\ \leq \int_{Q_T} u_n^{r + \eta} \, dx \, dt + \frac{1}{\eta + 1} \int_{\Omega} u_0^{\eta + 1} \, dx. \end{aligned}$$

Since $1 - \theta > 0$, there exists a positive constant C_1 such that

$$\frac{\beta\eta}{t^{1-\theta}(\delta+t)^e} + \frac{Bt^{1+\theta}}{(t+1/n)^{\theta+1}} \geq C_1 > 0 \quad \forall t > 0.$$

So,

$$(2.9) \quad \int_{Q_T} u_n^{p+\eta-\theta} dx dt \leq C \int_{Q_T} u_n^{r+\eta} dx dt + C.$$

We now choose $\eta+r = p+1-\theta$ (observe that $\eta > 1$), so $p+\eta-\theta = 2(p+1-\theta)-(r+1)$. Then by (1.3) and (2.9), we obtain that u_n is bounded in $L^{2(p+1-\theta)-(r+1)}(Q_T)$. Consequently, an iterating procedure gives us that (u_n) is bounded in $L^\mu(Q_T)$ for all $\mu < \infty$. Indeed, if we consider $\eta_1 > 1$ such that $r + \eta_1 = 2(p+1-\theta) - (r+1)$, (2.9) and the fact that (u_n) is bounded in $L^{2(p+1-\theta)-(r+1)}(Q_T)$, then it is bounded in $L^{3(p+1-\theta)-2(r+1)}$. Now consider $\eta_2 > 1$ such that $r + \eta_2 = 3(p+1-\theta) - 2(r+1)$ and deduce that (u_n) is bounded in $L^{4(p+1-\theta)-3(r+1)}(Q_T)$. Hence, we can obtain that (u_n) is bounded in $L^{(q+1)(p+1-\theta)-q(r+1)}(Q_T)$ for all $q \in \mathbb{N}$. Since

$$(q+1)(p+1-\theta) - q(r+1) = q(p-r-\theta) + p+1-\theta \rightarrow \infty \quad \text{as } q \rightarrow \infty,$$

we deduce that (u_n) is bounded in $L^\mu(Q_T)$ for all $\mu < \infty$. Because there is $n' > 0$ such that $(n'(p-r-\theta) + p+1-\theta)/r > (N/p) + 1$,

$$(u_n^r) \text{ is bounded in } L^m(Q_T) \quad \text{for some } m > \frac{N}{p} + 1.$$

Standard parabolic estimates, performed using only the principal part of the operator (see for example [5]), and taking advantage of the nonnegativity of the lower order gradient term, then imply that $(u_n)_n$ is bounded in $L^\infty(Q_T)$. Therefore by (2.7), we have

$$\|u_n\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C.$$

Taking n large enough, we get $T_n(u_n^r) = u_n^r$ and $T_n(u_n) = u_n$, so we conclude that u_n is a weak solution of

$$(2.10) \quad \begin{cases} \partial_t u_n - \operatorname{div}(A(t, x, u_n)|\nabla u_n|^{p-2}\nabla u_n) + B \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} = u_n^r & \text{in } Q_T, \\ u_n(0, x) = u_0(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma_T. \end{cases}$$

□

Now, we are going to prove the strict positivity of the sequence of approximated solutions u_n .

Proposition 2.7. *Let ω be a compactly contained open subset of Ω . Then there exists a positive constant $C_{\omega,T}$ such that $u_n \geq C_{\omega,T}$ in $(0,T) \times \omega$.*

Proof. Following the ideas in [11], we define for $s \geq 0$,

$$H_n(s) = \int_0^s \frac{(\delta + \tau)^\theta}{(\tau + 1/n)^\theta} d\tau, \quad \Phi_n(s) = e^{-BH_n(s)/\beta},$$

where $0 < \theta < 1$ and $B > 0$. Taking $\Phi_n(u_n)v$, with $v \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$, $v \geq 0$, as test function in (2.10) and using (1.1)–(1.2) and that

$$\Phi'_n(s) = \frac{-B}{\beta} \frac{(\delta + s)^\theta}{(s + 1/n)^\theta} \Phi_n(s),$$

we obtain

$$\begin{aligned} & \int_0^T \langle \partial_t u_n, \Phi_n(u_n)v \rangle dt + \int_{Q_T} A(t,x,u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla v \Phi_n(u_n) dx dt \\ & \geq \int_{Q_T} \frac{B}{(u_n + 1/n)^\theta} |\nabla u_n|^p \Phi_n(u_n)v dx dt \\ & \quad - B \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \Phi_n(u_n)v dx dt + \int_{Q_T} u_n^r \Phi_n(u_n)v dx dt \\ & \geq 0. \end{aligned}$$

After dropping the nonnegative term, we derive

$$(2.11) \quad \begin{aligned} & \int_0^t \langle \partial_t u_n, \Phi_n(u_n)v \rangle dt + \int_{Q_T} A(t,x,u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla v \Phi_n(u_n) dx dt \\ & \geq \int_{Q_T} u_n^r \Phi_n(u_n)v dx dt. \end{aligned}$$

Now, we consider the nonincreasing function ψ :

$$\psi(s) = \int_s^1 \Phi_n(t) dt = \int_s^1 e^{-BH_n(t)/\beta} dt.$$

Then, inequality (2.11) implies that

$$(2.12) \quad \begin{aligned} & - \int_0^T \langle \partial_t(\psi(u_n)), v \rangle dt - \int_{Q_T} A(t,x,u_n) |\nabla u_n|^{p-2} \nabla \psi(u_n) \nabla v dx dt \\ & \geq \int_{\{0 \leq u_n \leq 1\}} \Phi_n(u_n) u_n^r v dx dt \geq \int_{\{0 \leq u_n \leq 1\}} (\Phi_n(u_n) - 1) u_n^r v dx dt. \end{aligned}$$

We call

$$\tilde{A}(t,x,s) = A(t,x,\psi^{-1}(s)) |\nabla \psi^{-1}(s)|^{p-2},$$

and

$$H(s) = (1 - \Phi_n(\psi^{-1}(s))) u_n^r \chi_{\{0 \leq u_n \leq 1\}}.$$

Thus, see [2] for instance, we deduce that $\psi(u_n)$ is a sub-solution of

$$\partial_t z - \operatorname{div}(\tilde{A}(t, x, z)\nabla z) = H(z) \quad \text{in } Q_T.$$

Since H is a nonnegative term and $u_0 > 0$ in Ω , we can apply Lemma 3.12 in [6] to the previous equation to obtain the existence of $c_{\omega, T} > 0$ such that

$$\psi(u_n) \leq c_{\omega, T} \quad \forall (t, x) \in (0, T) \times \omega \text{ and } \forall n > 1.$$

By the definition of ψ , there exists $C_{\omega, T} > 0$ (independent of n) such that

$$u_n \geq \psi^{-1}(c_{\omega, T}) = C_{\omega, T} \quad \text{in } (0, T) \times \omega.$$

□

2.1.1. Passage to the limit.

Lemma 2.8. *Let A be a function satisfying (1.1) and let $u_n \in L^p(0, T; W_0^{1,p}(\Omega))$ be a sequence of weak solutions to (2.10). Then there exists a subsequence of u_n (still denoted by u_n) converging to a measurable function u a.e. in Q_T , and*

$$(2.13) \quad \nabla u_n \rightarrow \nabla u \text{ a.e. in } Q_T.$$

Proof. Going back again to (2.10), the sequence $(\partial_t u_n)$ remains in a bounded set of the space

$$L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q_T), \quad p' = \frac{p}{p-1}.$$

Therefore, $(\partial_t u_n)$ is bounded in $L^1(0, T; W^{-1,s}(\Omega))$, for all $s < N/(N-1)$. So, we can use Corollary 4 of [10] to see that

$$u_n \text{ is relatively compact in } L^1(Q_T).$$

Summing up, there exists a function $u \in L^p(0, T; W_0^{1,p}(\Omega))$ and a subsequence, still denoted by (u_n) , such that

$$(2.14) \quad u_n \rightharpoonup u \text{ weakly in } L^p(0, T; W_0^{1,p}(\Omega)),$$

$$(2.15) \quad u_n \rightarrow u \text{ strongly in } L^p(Q_T) \text{ and a.e. in } Q_T.$$

Now, we prove that

$$(2.16) \quad \nabla T_k(u_n) \rightarrow \nabla T_k(u) \text{ strongly in } (L^p(Q_T))^N \quad \forall k \in \mathbb{N}.$$

We also introduce another time-regularization of truncations, we will use the sequence $(T_k(u))_\nu$ as approximation of $T_k(u)$. For $\nu > 0$, we define the regularization in time of the function $T_k(u)$ given by

$$(2.17) \quad (T_k(u))_\nu(t, x) := \nu \int_{-\infty}^t e^{\nu(s-t)} T_k(u(s, x)) \, ds + e^{-\nu t} T_k(u_0),$$

where $T_k(u(s, x))$ is the zero extension of u for $s < 0$ (see [14]). Applying this regularization to the truncatures $T_k(u_m)$, we have the following properties:

- ▷ $((T_k(u_m))_\nu)_t = \nu(T_k(u_m) - (T_k(u_m))_\nu)$,
- ▷ $((T_k(u_m))_\nu)(0, x) = T_k(u_0)$,
- ▷ $|((T_k(u_m))_\nu)| \leq k$,
- ▷ $(T_k(u_m))_\nu \rightarrow T_k(u_m)$ strongly in $L^p(0, T; W_0^{1,p}(\Omega))$ as $\nu \rightarrow \infty$.

Considering the function $\varphi_\lambda(s)$ defined by

$$\varphi_\lambda(s) = se^{\lambda s^2}, \quad \lambda > 0,$$

in what follows we use that for every $a, b > 0$ we have

$$(2.18) \quad a\varphi'_\lambda(s) - b|\varphi_\lambda(s)| \geq \frac{a}{2} \quad \text{if } \lambda > \frac{b^2}{4a^2}.$$

We also denote by $\tau(m, n, \nu)$ any quantity I such that

$$\lim_{\nu \rightarrow \infty} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} I = 0,$$

likewise $\tau(n, \nu)$ denotes a quantity I such that $\lim_{\nu \rightarrow \infty} \lim_{n \rightarrow \infty} I = 0$. Let ϕ be a function in $C_c^\infty(\Omega)$ such that $\phi \geq 0$. By the same technique as in [1] we have that

$$(2.19) \quad \int_0^T \langle \partial_t u_n, \varphi_\lambda(T_k(u_n) - (T_k(u_m))_\nu) \phi \rangle \, dt \geq \tau(m, n, \nu).$$

Using (2.19) and taking $\psi_\lambda = \varphi_\lambda(T_k(u_n) - (T_k(u_m))_\nu) \phi$ as a test function in (2.10), we obtain

$$(2.20) \quad \begin{aligned} \tau(m, n, \nu) &+ \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \\ &\quad \times \nabla (T_k(u_n) - (T_k(u_m))_\nu) \varphi'_\lambda(T_k(u_n) - (T_k(u_m))_\nu) \phi \, dx \, dt \\ &+ B \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \psi_\lambda \, dx \, dt \\ &\leq \int_{Q_T} u_n^r \psi_\lambda \, dx \, dt \\ &\quad - \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \varphi_\lambda(T_k(u_n) - (T_k(u_m))_\nu) \, dx \, dt. \end{aligned}$$

By (2.15), $(T_k(u_m))_\nu \rightarrow (T_k(u))_\nu$ a.e. in Q_T and we have

$$|u_n^r \psi_\lambda| \leq \|u_n^r\|_{L^\infty(Q_T)} \varphi_\lambda(2k) \in L^1(Q_T),$$

by the Lebesgue dominated convergence theorem,

$$\lim_{\nu \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \left(\lim_{m \rightarrow \infty} \int_{Q_T} u_n^r \psi_\lambda \right) \right) = 0.$$

By writing $Q_T = \{u_n \leq k\} \cup \{u_n > k\}$ and adopting the technique used in [1], we have

$$\lim_{\nu \rightarrow \infty} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \varphi_\lambda (T_k(u_n) - (T_k(u_m))_\nu) = 0.$$

Therefore

$$(2.21) \quad \int_{Q_T} u_n^r \psi_\lambda - \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \varphi_\lambda (T_k(u_n) - (T_k(u_m))_\nu) = \tau(m, n, \nu).$$

We next turn to consider the last term on the left-hand side of (2.20). Choosing $\omega \subset \subset \Omega$ with $\text{supp } \phi \subset \omega$, by the nonnegativity of $\varphi_\lambda(k - (T_k(u))_\nu)$, we have that

$$(2.22) \quad \lim_{m \rightarrow \infty} \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \psi_\lambda \, dx \, dt \geq \int_{\{C_{\omega, T} \leq u_n \leq k\}} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \psi_\lambda \, dx \, dt \\ \geq -C_{k, T}(\omega) \int_{Q_T} |\nabla T_k(u_n)|^p |\psi_\lambda| \, dx \, dt,$$

where $C_{k, T}(\omega)$ is a positive constant such that

$$(2.23) \quad \frac{u_n}{(u_n + 1/n)^{\theta+1}} \leq \max_{u_n \in [C_{\omega, T}, k]} \frac{1}{u_n^\theta} = C_{k, T}(\omega) \quad \forall n \gg 1.$$

From the convergence

$$\nabla(T_k(u_m))_\nu \rightharpoonup \nabla(T_k(u))_\nu \text{ weakly in } (L^p(Q_T))^N \text{ as } m \rightarrow \infty$$

we get, by using (2.20)–(2.21) and (2.22), that

$$(2.24) \quad \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla (T_k(u_n) - (T_k(u))_\nu) \varphi'_\lambda (T_k(u_n) - (T_k(u))_\nu) \phi \, dx \, dt \\ - BC_{k, T}(\omega) \int_{Q_T} |\nabla T_k(u_n)|^p |\psi_\lambda| \, dx \, dt \leq \tau(\nu, n).$$

Note that

$$\begin{aligned}
 & \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \\
 & \quad \times \nabla (T_k(u_n) - (T_k(u))_\nu) \varphi'_\lambda (T_k(u_n) - (T_k(u))_\nu) \phi \chi_{\{u_n \geq k\}} \, dx \, dt \\
 & = - \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla (T_k(u))_\nu \varphi'_\lambda (k - (T_k(u))_\nu) \phi \chi_{\{u_n \geq k\}} \, dx \, dt \\
 & = \tau(\nu, n),
 \end{aligned}$$

so adding

$$\begin{aligned}
 & - \int_{Q_T} A(t, x, u_n) |\nabla (T_k(u))_\nu|^{p-2} \nabla (T_k(u))_\nu \\
 & \quad \times \nabla (T_k(u_n) - (T_k(u))_\nu) \varphi'_\lambda (T_k(u_n) - (T_k(u))_\nu) \phi \, dx \, dt = \tau(\nu, n).
 \end{aligned}$$

On both sides of (2.24) and since

$$\begin{aligned}
 (2.25) \quad & \int_{Q_T} |\nabla T_k(u_n)|^p |\psi_\lambda| \, dx \, dt \leq 2^{p-1} \int_{Q_T} |\nabla (T_k(u_n) - (T_k(u))_\nu)|^p |\psi_\lambda| \, dx \, dt \\
 & \quad + 2^{p-1} \int_{Q_T} |\nabla (T_k(u))_\nu|^p |\psi_\lambda| \, dx \, dt \\
 & = 2^{p-1} \int_{Q_T} |\nabla (T_k(u_n) - (T_k(u))_\nu)|^p |\psi_\lambda| \, dx \, dt + \tau(\nu, n),
 \end{aligned}$$

and using the following well-known inequalities that hold for any two real vectors ξ, η and a real $p \geq 2$,

$$(2.26) \quad (|\xi|^{p-2} \xi - |\eta|^{p-2} \eta)(\xi - \eta) \geq 2^{2-p} |\xi - \eta|^p,$$

we find, by using also (1.1) and (2.25), for all $n > k > 0$,

$$\begin{aligned}
 & 2^{2-p} \frac{\beta}{(\delta + k)^e} \int_{Q_T} |\nabla (T_k(u_n) - (T_k(u))_\nu)|^p \varphi'_\lambda (T_k(u_n) - (T_k(u))_\nu) \phi \, dx \, dt \\
 & \quad - 2^{p-1} BC_{k,T}(\omega) \int_{Q_T} |\nabla (T_k(u_n) - (T_k(u))_\nu)|^p |\varphi_\lambda (T_k(u_n) - (T_k(u))_\nu)| \phi \, dx \, dt \\
 & \leq \tau(\nu, n).
 \end{aligned}$$

Choosing λ such that (2.18) holds with $a = 2^{2-p} \beta / (\delta + k)^e$ and $b = 2^{p-1} BC_{k,T}(\omega)$, we obtain (2.16) by setting $\nu \rightarrow \infty$. From this result we also deduce that (up to subsequences)

$$\nabla u_n \rightarrow \nabla u \quad \text{almost everywhere in } Q_T.$$

□

Lemma 2.9. *We have*

$$(2.27) \quad \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \rightarrow \frac{|\nabla u|^p}{u^\theta} \text{ strongly in } L^1(Q_T).$$

Proof. In view of (2.13) and (2.15), we have

$$\frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \rightarrow \frac{|\nabla u|^p}{u^\theta} \text{ a.e. in } Q_T.$$

Now, we shall obtain local equi-integrability of $u_n |\nabla u_n|^p / (u_n + 1/n)^{\theta+1}$ on Q_T . Observe that

$$\int_0^T \int_{u_n \geq k} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt = \frac{1}{k} \int_0^T \int_{u_n \geq k} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} T_k(u_n) dx dt.$$

We choose $\varphi = T_k(u_n)$ as a test function in problems (2.10), we find

$$\begin{aligned} \int_{\Omega} dx \int_0^{u_n(T,x)} T_k(\sigma) d\sigma + \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n) dx dt \\ + \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} T_k(u_n) dx dt \\ = \int_{Q_T} u_n^r T_k(u_n) dx dt + \int_{\Omega} dx \int_0^{u_n(0,x)} T_k(\sigma) d\sigma. \end{aligned}$$

So, after dropping the nonnegative terms, we derive

$$\int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} T_k(u_n) dx dt \leq \int_{Q_T} u_n^r |T_k(u_n)| dx dt + \int_{\Omega} dx \int_0^{u_n(0,x)} |T_k(\sigma)| d\sigma.$$

Taking into account that for any $M > 0$, $0 \leq |T_k(s)| \leq M + k \mathbf{1}_{s > M}$, $s \in \mathbb{R}^+$, we have

$$\int_{Q_T} u_n^r |T_k(u_n)| dx dt \leq MC \|u_n\|_{L^p(Q_T)}^r + k \int_0^T \int_{u_n > M} u_n^r dx dt,$$

and

$$\int_{\Omega} dx \int_0^{u_n(0,x)} |T_k(\sigma)| d\sigma \leq M \|u_0\|_{L^1(\Omega)} + k \int_0^T \int_{u_n > M} u_{0n} dx dt.$$

Consequently, we have

$$\begin{aligned} \int_0^T \int_{u_n \geq k} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt \\ \leq \frac{1}{k} \left(MC + k \int_0^T \int_{u_n > M} u_n^r dx dt + M \|u_0\|_{L^1(\Omega)} + k \int_0^T \int_{u_n > M} u_{0n} dx dt \right) \\ \leq C \frac{M}{k} + \int_0^T \int_{\Omega} \chi_{\{u_n > M\}} u_n^r dx dt + \int_0^T \int_{\Omega} \chi_{\{u_n > M\}} u_{0n} dx dt. \end{aligned}$$

We take $M = \sqrt{k}$, we obtain

$$(2.28) \quad \int_0^T \int_{u_n \geq k} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt \xrightarrow{k \rightarrow \infty} 0 \text{ uniformly with respect to } n,$$

then, there exists $k_0 > 1$ such that

$$(2.29) \quad \int_0^T \int_{u_n \geq k} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt \leq \frac{\varepsilon}{2} \quad \forall k \geq k_0 \text{ and } \forall n \in \mathbb{N}.$$

Consequently, if $E \subset\subset \omega$, we have

$$(2.30) \quad \begin{aligned} & \int_0^T \int_E \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt \\ &= \int_0^T \int_{E \cap \{u_n \geq k\}} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt + \int_0^T \int_{E \cap \{u_n \leq k\}} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt \\ &\leq \int_0^T \int_{E \cap \{u_n \geq k\}} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt + C_{k,T}(\omega) \int_0^T \int_{E \cap \{u_n \leq k\}} |\nabla T_k(u_n)|^p dx dt. \end{aligned}$$

From (2.14) there exist n_ε and δ_ε such that for every $E \subset\subset \Omega$ with $\text{meas}(E) < \delta_\varepsilon$ we have

$$\int_0^T \int_{E \cap \{u_n \leq k\}} |\nabla T_k(u_n)|^p dx dt < \frac{\varepsilon}{2C_{k,T}(\omega)} \quad \forall n \geq n_\varepsilon.$$

By (2.29), (2.30), and taking $n \geq n_\varepsilon$, $k \geq k_0$, we see that $\text{meas}(E) < \delta_\varepsilon$ implies

$$\int_0^T \int_E \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} dx dt < \varepsilon.$$

We deduce that $u_n |\nabla u_n|^p / (u_n + 1/n)^{\theta+1}$ is equi-integrable in Q_T , then by Vitali's theorem convergence, we have (2.27) and $|\nabla u^p / u^\theta \in L^1(Q_T)$. \square

Lemma 2.10. *The sequence (u_n) is a Cauchy sequence in $C([0, T]; L^1(\Omega))$, hence u_n converges to $u \in C([0, T]; L^1(\Omega))$.*

Proof. To do this, fix $t \in [0, T]$. Taking $T_k(u_n - u_m)$ as a test function in (2.10) for u_n and u_m , subtracting up both identities, we deduce that

$$\begin{aligned} & \int_\Omega \Theta_k(u_n(t) - u_m(t)) dx \\ &+ \int_{Q_t} (A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n - A(t, x, u_m) |\nabla u_m|^{p-2} \nabla u_m) \nabla T_k(u_n - u_m) dx dt \\ &+ B \int_{Q_t} \left(\frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} - \frac{u_m |\nabla u_m|^p}{(u_m + 1/m)^{\theta+1}} \right) T_k(u_n - u_m) dx dt \\ &\leq \int_{Q_t} |u_n^r - u_m^r| |T_k(u_n - u_m)| dx dt + \int_\Omega |\Theta_k(u_n(0) - u_m(0))| dx. \end{aligned}$$

So, by (2.2) we obtain

$$\begin{aligned}
& \int_{\Omega} \Theta_k(u_n(t) - u_m(t)) \, dx \\
& \leq \int_{Q_t} |A(t, x, u_m)| |\nabla u_m|^{p-2} \nabla u_m - A(t, x, u_n)|\nabla u_n|^{p-2} \nabla u_n| |\nabla T_k(u_n - u_m)| \, dx \, dt \\
& \quad + Bk \int_{Q_t} \left| \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} - \frac{u_m |\nabla u_m|^p}{(u_m + 1/m)^{\theta+1}} \right| \, dx \, dt \\
& \quad + k \int_{Q_t} |u_n^r - u_m^r| \, dx \, dt + k \int_{\Omega} |u_n(0) - u_m(0)| \, dx.
\end{aligned}$$

Using (2.2) and dividing this inequality by k , we obtain

$$\begin{aligned}
& \sup_{t \in [0, T]} \int_{\Omega} |u_n(t) - u_m(t)| \, dx \\
& \leq \frac{1}{k} \int_{Q_T} |A(t, x, u_m)| |\nabla u_m|^{p-2} \nabla u_m - A(t, x, u_n)|\nabla u_n|^{p-2} \nabla u_n| |\nabla T_k(u_n - u_m)| \, dx \, dt \\
& \quad + B \int_{Q_T} \left| \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} - \frac{u_m |\nabla u_m|^p}{(u_m + 1/m)^{\theta+1}} \right| \, dx \, dt \\
& \quad + \int_{Q_T} |u_n^r - u_m^r| \, dx \, dt + \int_{\Omega} |u_n(0) - u_m(0)| \, dx + \frac{k}{2}.
\end{aligned}$$

By (1.1), (2.13), (2.14) and (2.15), we have

$$(2.31) \quad |A(t, x, u_m)| |\nabla u_m|^{p-2} \nabla u_m - A(t, x, u_n)|\nabla u_n|^{p-2} \nabla u_n| \rightharpoonup 0 \quad \text{in } L^{p'}(Q_T).$$

Taking into account (2.27) and letting $k \rightarrow 0$, we deduce that (u_n) is a Cauchy sequence in $C([0, T]; L^1(\Omega))$. Consequently, $u_n \rightarrow u$ in $C([0, T], L^1(\Omega))$. This ends the proof of Lemma 2.10. \square

2.2. The end of the proof of Theorem 2.3. Let $\varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$. We have

$$\begin{aligned}
(2.32) \quad & - \int_{\Omega} u_n(0) \varphi(0) \, dx + \int_{Q_T} u_n \partial_t \varphi \, dx \, dt + \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \, dx \, dt \\
& \quad + B \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \varphi \, dx \, dt = \int_{Q_T} u_n^r \varphi \, dx \, dt.
\end{aligned}$$

Arguing as in (2.31), we have

$$\lim_{n \rightarrow \infty} \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \, dx \, dt = \int_{Q_T} A(t, x, u) |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \, dt.$$

Therefore by (2.27), we can easily pass to the limit in (2.32). Theorem 2.3 is proved. \square

3. L^1 INITIAL DATA

Theorem 3.1. *Given $u_0 \in L^1(\Omega)$, suppose that (1.3) holds true. Then problem (P) has at least a weak solution u , i.e., a function u belonging to $L^q(0, T; W_0^{1,q}(\Omega)) \cap C([0, T]; L^1(\Omega))$, $u > 0$, $|\nabla u|^p/u^\theta \in L^1(Q_T)$, such that*

$$(3.1) \quad - \int_{\Omega} u(0)\varphi(0) \, dx + \int_{Q_T} u \partial_t \varphi \, dx \, dt + \int_{Q_T} A(t, x, u) |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \, dt \\ + B \int_{Q_T} \frac{|\nabla u|^p}{u^\theta} \varphi \, dx \, dt = \int_{Q_T} u^r \varphi \, dx \, dt, \quad q = p - \frac{\theta N}{N+1}$$

for every $\varphi \in W^{1,\infty}(0, T; L^\infty(\Omega))$ and such that $\varphi(T) = 0$ in Ω .

3.1. Proof of Theorem 3.1. Let (u_{0n}) , $u_{0n} = T_n(u_0) \geq 0$ be a sequence of bounded functions defined in Ω , which converges to u_0 in $L^1(\Omega)$, such that

$$\begin{cases} \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}, \\ u_{0n} \leq n. \end{cases}$$

A nonnegative weak solution $u_n \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ to problem (P) with $u_n(0, x) = u_{0n}(x)$ does exist by Theorem 2.3. Therefore, u_n satisfies

$$(3.2) \quad \int_0^T \langle \partial_t u_n, \varphi \rangle \, dt + \int_{Q_T} A(t, x, u_n) |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \, dx \, dt \\ + B \int_{Q_T} \frac{|\nabla u_n|^p}{u_n^\theta} \varphi \, dx \, dt = \int_{Q_T} u_n^r \varphi \, dx \, dt$$

for all $\varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$. Using the same technique as in Proposition 2.7, there exists a positive constant $C_{\omega, T}$ such that

$$(3.3) \quad u_n \geq C_{\omega, T} \quad \text{in } (0, T) \times \omega,$$

where ω is a compactly contained open subset of Ω .

Lemma 3.2. *Assume that (1.3) hold with $p \geq 2$ and u_n is the solution to problems (3.2). Then there exists a positive constant C such that*

$$(3.4) \quad \int_{Q_T} u_n^r \, dx \, dt \leq C,$$

$$(3.5) \quad \int_{Q_T} |\nabla u_n|^q \, dx \, dt \leq C, \quad q = p - \frac{\theta N}{N+1}.$$

Proof. Take $\varphi = T_1(u_n)$ as a test function in the weak formulation (3.2). By (1.1), we have

$$(3.6) \quad \int_{\Omega} \Theta_1(u_n)(T) \, dx + \int_{Q_T} \frac{\beta}{(\delta + u_n)^2} |\nabla T_1(u_n)|^p \, dx \, dt + B \int_{Q_T} \frac{|\nabla u_n|^p}{u_n^\theta} T_1(u_n) \, dx \, dt \\ \leq \int_{Q_T} u_n^r T_1(u_n) \, dx \, dt + \int_{\Omega} \Theta_1(u_n)(0) \, dx.$$

Since

$$\int_{Q_T} \frac{|\nabla u_n|^p}{u_n^\theta} T_1(u_n) \, dx \, dt \geq \int_{\{u_n > 1\} \cap Q_T} \frac{|\nabla u_n|^p}{u_n^\theta} \, dx \, dt,$$

dropping nonnegative terms in (3.6), it follows that

$$(3.7) \quad B \int_{\{u_n > 1\} \cap Q_T} \frac{|\nabla u_n|^p}{u_n^\theta} \, dx \, dt \\ \leq \int_{Q_T} u_n^r T_1(u_n) \, dx \, dt + \int_{\Omega} \Theta_1(u_n)(0) \, dx \\ \leq \int_{\{u_n \leq 1\} \cap Q_T} u_n^{r+1} \, dx \, dt + \int_{\{u_n > 1\} \cap Q_T} u_n^r \, dx \, dt + \|u_0\|_{L^1(\Omega)} \\ \leq |Q_T| + C_1 + C \int_{\{u_n > 1\} \cap Q_T} (u_n - 1)^r \, dx \, dt.$$

Consequently, denoting $G_1(r) = r - T_1(r)$, we get the inequality

$$\int_{\{u_n > 1\} \cap Q_T} |\nabla u_n|^p u_n^{-\theta} \, dx \, dt \leq C_2 + C_2 \int_{\{u_n > 1\} \cap Q_T} (G_1(u_n))^r \, dx \, dt,$$

so

$$\int_{\{u_n > 1\} \cap Q_T} |\nabla u_n|^p (G_1(u_n) + 1)^{-\theta} \, dx \, dt \leq C_2 + C_2 \int_{\{u_n > 1\} \cap Q_T} (G_1(u_n))^r \, dx \, dt,$$

which yields

$$\left(1 - \frac{\theta}{p}\right)^{-p} \int_{\{u_n > 1\} \cap Q_T} |\nabla (G_1(u_n) + 1)^{(1-\theta/p)}|^p \, dx \, dt \\ \leq C_2 + C_2 \int_{\{u_n > 1\} \cap Q_T} (G_1(u_n))^r \, dx \, dt.$$

Now, the Poincaré inequality implies

$$\int_{\{u_n > 1\} \cap Q_T} (G_1(u_n) + 1)^{p-\theta} \, dx \, dt \leq C_3 + C_3 \int_{\{u_n > 1\} \cap Q_T} (G_1(u_n))^r \, dx \, dt.$$

Observe that $r < p - \theta$. By Young's inequality we obtain

$$(3.8) \quad \int_{\{u_n > 1\} \cap Q_T} |G_1(u_n)|^{p-\theta} \, dx \, dt \leq C_4.$$

Therefore

$$\int_{\{u_n > 1\} \cap Q_T} u_n^r \, dx \, dt = \int_{\{u_n > 1\} \cap Q_T} (G_1(u_n) + 1)^r \, dx \, dt \leq C_5.$$

So (3.4) is proved. To prove (3.5) we have by (3.4) and (3.6)

$$(3.9) \quad \int_{Q_T} |\nabla T_1(u_n)|^p \, dx \, dt = \int_{\{u_n \leq 1\} \cap Q_T} \frac{|\nabla T_1(u_n)|^p}{(\delta + u_n)^\theta} (\delta + u_n)^\theta \, dx \, dt \\ \leq (1 + \delta)^\theta \int_{\{u_n \leq 1\} \cap Q_T} \frac{|\nabla T_1(u_n)|^p}{(\delta + u_n)^\theta} \, dx \, dt \leq C.$$

From (3.4), (3.7) and $q = p - \theta N / (N + 1)$, we write

$$(3.10) \quad \int_{\{u_n > 1\} \cap Q_T} |\nabla u_n|^q \, dx \, dt \\ = \int_{\{u_n > 1\} \cap Q_T} \frac{|\nabla u_n|^q}{u_n^{\theta q/p}} u_n^{\theta q/p} \, dx \, dt \\ \leq \left(\int_{\{u_n > 1\} \cap Q_T} \frac{|\nabla u_n|^p}{u_n^\theta} \, dx \, dt \right)^{q/p} \left(\int_{\{u_n > 1\} \cap Q_T} u_n^{\theta q/(p-q)} \, dx \, dt \right)^{1-q/p} \\ \leq C \left(\int_{Q_T} u_n^s \, dx \, dt \right)^{1-q/p}, \quad s = \frac{q(N+1)}{N}.$$

By (2.2) and (3.6), we have

$$(3.11) \quad \sup_{t \in [0, T]} \int_{\Omega} u_n(t, x) \, dx \leq C.$$

Use the following interpolation argument: $\|u_n\|_{L^s(\Omega)} \leq \|u_n\|_{L^1(\Omega)}^\tau \|u_n\|_{L^{q^*}(\Omega)}^{1-\tau}$ with $1 - \tau = ((1 - s)/(1 - q^*)) (q^*/s)$, where $q^* = Nq/(N - q)$ if $q < N$ and $q^* > 1$ satisfying $(1 - \tau)s = q$ otherwise. Using (3.11) and the Sobolev inequality we obtain

$$\int_0^T \|u_n\|_{L^s(\Omega)}^s \, dt \leq C \int_0^T \|\nabla u_n\|_{L^q(\Omega)}^{(1-\tau)s} \, dt.$$

By this last inequality, (3.10), $q < p$, and (3.9) we have

$$(3.12) \quad \int_{\{u_n > 1\} \cap Q_T} |\nabla u_n|^q \, dx \, dt \leq C \left(\int_{Q_T} |\nabla u_n|^q \, dx \, dt \right)^{1-q/p} \\ \leq C + C \left(\int_{\{u_n > 1\} \cap Q_T} |\nabla u_n|^q \, dx \, dt \right)^{1-q/p},$$

which implies that

$$\int_{\{u_n > 1\} \cap Q_T} |\nabla u_n|^q \, dx \, dt \leq C.$$

Furthermore, (3.9) implies estimate (3.5) and Lemma 3.2 is proved. \square

3.2. Passage to the limit and finishing the proof of Theorem 3.1. Arguing as in Lemma 2.8, we obtain a subsequence (u_n) and a measurable function $u \in L^q(0, T, W_0^{1,q}(\Omega))$ such that

$$(3.13) \quad u_n \rightharpoonup u \text{ weakly in } L^q(0, T; W_0^{1,q}(\Omega)),$$

$$(3.14) \quad u_n \rightarrow u \text{ strongly in } L^q(Q_T) \text{ and a.e. in } Q_T,$$

$$(3.15) \quad \nabla u_n \rightarrow \nabla u \text{ a.e. in } Q_T.$$

From (3.14), (3.15), (1.1), and $q/(p-1) > 1$, we obtain

$$(3.16) \quad A(t, x, u_n)|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup A(t, x, u)|\nabla u|^{p-2}\nabla u \text{ in } (L^{q/(p-1)}(Q_T))^N.$$

By the technique used in the proof of Lemma 2.9,

$$(3.17) \quad \frac{|\nabla u_n|^p}{u_n^\theta} \rightarrow \frac{|\nabla u|^p}{u^\theta} \text{ strongly in } L^1(Q_T).$$

We also deduce that

$$(3.18) \quad u_n^r \rightarrow u^r \text{ strongly in } L^1(Q_T).$$

Indeed, thanks to (3.14), we just have to show that the sequence (u_n^r) is equi-integrable, but this is straightforward taking into account (3.4), (3.8), $r < p - \theta$, and Hölder's inequality. Finally, for $\varphi \in W^{1,\infty}(0, T; L^\infty(\Omega))$,

$$(3.19) \quad - \int_{\Omega} u_n(0)\varphi(0) \, dx + \int_{Q_T} u_n \partial_t \varphi \, dx \, dt + \int_{Q_T} A(t, x, u_n)|\nabla u_n|^{p-2}\nabla u_n \nabla \varphi \, dx \, dt \\ + B \int_{Q_T} \frac{|\nabla u_n|^p}{u_n^\theta} \varphi \, dx \, dt = \int_{Q_T} u_n^r \varphi \, dx \, dt.$$

Using (3.16), (3.17) and (3.18), we can easily pass to the limit in (3.19). Taking into account (3.3) and Lemma 2.10, Theorem 3.1 is proved. \square

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