Carlos Bejines

Aggregation of fuzzy vector spaces

Kybernetika, Vol. 59 (2023), No. 5, 752–767

Persistent URL: http://dml.cz/dmlcz/151986

AGGREGATION OF FUZZY VECTOR SPACES

Carlos Bejines

This paper contributes to the ongoing investigation of aggregating algebraic structures, with a particular focus on the aggregation of fuzzy vector spaces. The article is structured into three distinct parts, each addressing a specific aspect of the aggregation process.

The first part of the paper explores the self-aggregation of fuzzy vector subspaces. It delves into the intricacies of combining and consolidating fuzzy vector subspaces to obtain a coherent and comprehensive outcome.

The second part of the paper centers around the aggregation of similar fuzzy vector subspaces, specifically those belonging to the same equivalence class. This section scrutinizes the challenges and considerations involved in aggregating fuzzy vector subspaces with shared characteristics.

The third part of the paper takes a broad perspective, providing an analysis of the aggregation problem of fuzzy vector subspaces from a general standpoint. It examines the fundamental issues, principles, and implications associated with aggregating fuzzy vector subspaces in a comprehensive manner.

By elucidating these three key aspects, this paper contributes to the advancement of knowledge in the field of aggregation of algebraic structures, shedding light on the specific domain of fuzzy vector spaces.

Keywords: aggregation function, vector spaces, algebraic structures, monotone functions

Classification: 03B52, 94D05

1. INTRODUCTION

Presently, the study of aggregation operators has emerged as a significant and actively pursued research area within the field of Fuzzy Set Theory and its diverse applications. The increasing demand to integrate information, typically represented by numerical values, into a single output for decision-making purposes, has stimulated substantial interest in investigating functions that enable such aggregation. The prominence of aggregation operators is evidenced by their frequent inclusion in various conferences, as well as their dedicated congress, AGOP, which convenes biennially. For a more comprehensive understanding, extensive information on aggregation operators can be found in [6, 7].

A prominent research direction in recent years has been the examination of the preservation of properties within fuzzy algebraic structures under the influence of aggregation

DOI: 10.14736/kyb-2023-5-0752

operators. This framework has garnered significant attention and is actively being developed, as evidenced by the growing number of papers (see [2, 11, 15, 18, 19]). Given an arbitrary set S, an aggregation function $A : [0,1] \times [0,1] \longrightarrow [0,1]$ and two fuzzy subsets $\mu, \eta : S \longrightarrow [0,1]$, the fuzzy set $A(\mu,\eta) : S \longrightarrow [0,1]$ defined pointwise by

$$A(\mu,\eta)(x) := A(\mu(x), \eta(x))$$

is the aggregation of μ and η using A. Our research endeavors are focused on exploring the algebraic structure of a specific set S. In this paper, our attention is directed towards the scenario where S represents a vector space. Consequently, our investigation primarily revolves around the aggregation of fuzzy vector subspaces. This concept was introduced as an extension of the conventional vector space notion to encompass the fuzzy framework in [14]. Similar to other algebraic structures [9], a fuzzy vector subspace can be characterized in terms of its level sets, as indicated in Proposition 2.5.

This article is structured into multiple sections to systematically address the research objectives. In Section 2, we remember two pivotal notions that form the cornerstone of our investigation. Additionally, this section incorporates a preliminary result that serves as a valuable tool for comprehending the underlying framework. Proceeding to Section 3, we embark on an in-depth exploration by examining the aggregation of a fuzzy vector subspace. This analysis provides insights into the intricacies and dynamics of aggregating a single fuzzy vector subspace. Section 4 focuses on the aggregation of fuzzy vector subspaces belonging to the same equivalence class. By investigating this specific scenario, we delve into the challenges, considerations, and implications associated with aggregating fuzzy vector subspaces that share common properties. The central section of our study, Section 5, takes a broader perspective by analyzing the aggregation of fuzzy vector subspaces from a general standpoint. In this section, we present our primary outcome, which serves as a fundamental result characterizing the aggregation of these particular types of algebraic structures.

Through this well-structured organization, our article provides a comprehensive examination of the aggregation process for fuzzy vector subspaces, elucidating its various aspects, and presenting a significant result that contributes to the understanding of these algebraic structures.

2. BASIC NOTIONS AND PRELIMINARIES

Let us commence this paper by revisiting the fundamental concepts that will be examined and analyzed throughout the paper.

Definition 2.1. (Calvo et al. [7]) Let $A : [0,1]^n \longrightarrow [0,1]$ be a function. We say that A is an aggregation function if:

$(A1)$ $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$. (Boundary conditions)

$(A2)$ $A(x_1,\dots,x_n) \le A(y_1,\dots,y_n)$ whenever $x_i \le y_i$ for each $1 \le i \le n$. (Monotonicity)

Two important monographs about aggregation operators are [6, 7]. Aggregation functions are classified into four broad classes:

1. An aggregation function A is called conjunctive if
$$A(x_1, \dots, x_n) \leq \min\{x_1, \dots, x_n\}$$
for all $x_1, \dots, x_n \in [0,1]$. For example: t-norms.
2. An aggregation function A is called averaging if
$$\min\{x_1, \dots, x_n\} \leq A(x_1, \dots, x_n) \leq \max\{x_1, \dots, x_n\}$$
for all $x_1, \dots, x_n \in [0,1]$. For example: OWAs.
3. An aggregation function A is called disjunctive if
$$\max\{x_1, \dots, x_n\} \leq A(x_1, \dots, x_n)$$
for all $x_1, \dots, x_n \in [0,1]$. For example: t-conorms.
4. An aggregation function A is called mixed if A does not belong to the previous classes. For example: uninorms.

Notice that each averaging aggregation function A is idempotent because it satisfies $A(x, \dots, x) = x$ for each $x \in [0,1]$. In the manual [7] can be found an analysis of these operators.

From now, E denotes a vector space on a field $\mathbb{K}$ and e_+ and e_* denote to the neutral elements of the inner operations of the field $\mathbb{K}$.

Definition 2.2. (Katsaras and Liu [14]) Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{K}$ and $\mu : E \longrightarrow [0,1]$ a fuzzy set of E. We say that μ is a fuzzy vector subspace of E if it satisfies the following axioms:

$(E1)$ $\mu(x+y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in E$.

$(E2)$ $\mu(ax) \geq \mu(x)$ for all $x \in E$ and $a \in \mathbb{K}$.

It is worth pointing out that Das in [8] published a significant study of the properties of fuzzy vector spaces using an arbitrary t-norm in the axiom $(E1)$ instead of the minimum operator.

Remark 2.3. It is important to note that axiom $(E2)$ within this context cannot be expressed as the equality $\mu(ax) = \mu(x)$ as observed in other fuzzy algebraic structures. This distinction is particularly evident when considering fuzzy subgroups, where the degree of each element is equivalent to the degree of its inverse. As a consequence, the second axiom is often substituted with an equivalent formulation for the sake of consistency and coherency [16].

As a consequence of Definition 2.2, $\mu(e_+) \geq \mu(x)$ for each $x \in E$, where e_+ denotes to the neutral element of the abelian group $(E, +)$. Also, $\mu(x) = \mu(-x)$ for all $x \in E$. A significant characterization of a fuzzy vector subpace is shown in Proposition 2.5 using level sets.

Definition 2.4. Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{K}$ and $\mu : E \longrightarrow [0,1]$ a fuzzy set of E. For each $t \in [0,1]$, the level set μ_t and strict level set μ^t are defined as follows:

$$\mu_t = \{x \in E \mid \mu(x) \geq t\} \qquad \text{and} \qquad \mu^t = \{x \in E \mid \mu(x) > t\}.$$

The support of μ is defined by supp $\mu := \mu^0$.

Proposition 2.5. Let $\mu : E \longrightarrow [0,1]$ be a fuzzy set of a vector space E on a field $\mathbb{K}$. The following assertions are equivalent:

1. μ is a fuzzy vector subspace of E.
2. $\mu(ax + by) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in E$ and $a, b \in \mathbb{K}$.
3. Each non-empty level set of μ is a vector subspace of E.

Proof. We provide the proof for the sake of completeness.

(1) $\Rightarrow$ (3). Suppose that μ_t is a non-empty level set of μ, where $t \in [0,1]$. First, take $x, y \in \mu_t$. Since μ satisfies $(E1)$, $\mu(x + y) \geq \mu(x) \wedge \mu(y) \geq t$, hence $x + y \in \mu_t$. Second, take $x \in E$ and $a \in \mathbb{K}$. Since μ satisfies $(E2)$, $\mu(ax) \geq \mu(x) \geq t$, hence $ax \in \mu_t$. Therefore, μ_t is a vector subspace of E.

(3) $\Rightarrow$ (2). Let us consider $t = \mu(x) \wedge \mu(y)$. Clearly, the level set μ_t is not empty because $x, y \in \mu_t$, therefore μ_t is a vector subspace of E. Since $x, y \in \mu_t$, $ax + by \in \mu_t$, equivalently, $\mu(ax + by) \geq t = \mu(x) \wedge \mu(y)$.

(2) $\Rightarrow$ (1). Considering $a = b = 1$, we have $(E1)$ and taking $b = 0$ and $x = y$, we have $(E2)$, that is, $\mu(ax) = \mu(ax + by) \geq \mu(x) \wedge \mu(y) = \mu(x)$. □

In preparation for our analysis of the aggregation procedure for fuzzy vector subspaces, we have introduced the two fundamental notions outlined in this paper.

3. SELF-AGGREGATION OF FUZZY VECTOR SPACES

Given a vector space E on a field $\mathbb{K}$, in this section we present the results concerning to the self-aggregation $A(\mu, \mu)$, where μ is a fuzzy vector subspace of E and A is an aggregation function.

Proposition 3.1. Let $A : [0,1]^2 \longrightarrow [0,1]$ be an aggregation function and $\mu : E \longrightarrow [0,1]$ a fuzzy vector subspace of a vector space E. Then $A(\mu, \mu)$ is a fuzzy vector subspace of E.

Proof. In order to prove $(E1)$, without loss of generality, let us consider $x, y \in E$ satisfying $\mu(x) \leq \mu(y)$. Since $\mu(x + y) \geq \mu(x)$ and A satisfies $(A2)$,

$$A(\mu, \mu)(x + y) = A(\mu(x + y), \mu(x + y)) \geq A(\mu(x), \mu(x)) = A(\mu, \mu)(x).$$

Moreover, $A(\mu(x),\mu(x)) \leq A(\mu(y),\mu(y))$. Therefore,

$$A(\mu,\mu)(x+y) \geq A(\mu,\mu)(x) \wedge A(\mu,\mu)(y).$$

Finally, the axiom $(E2)$ for $A(\mu,\mu)$ is a straightforward consequence of $(A2)$ and $\mu(ax) \geq \mu(x)$. □

Corollary 3.2. If $A : [0,1]^n \longrightarrow [0,1]$ is an aggregation function and $\mu : E \longrightarrow [0,1]$ is a fuzzy vector subspace of E, then $A(\mu,\ldots,\mu)$ is a fuzzy vector subspace of E.

In [3], a comparison was presented between μ and $A(\mu,\mu)$ in the context of a group as the universal set. Likewise, we have derived the same result for fuzzy vector subspaces.

Proposition 3.3. Let $\mu : E \longrightarrow [0,1]$ be a fuzzy vector subspace of a vector space E and $A : [0,1]^2 \longrightarrow [0,1]$ an aggregation function.

1. If A is conjunctive, then $A(\mu,\mu) \leq \mu$.
2. If A is averaging, then $A(\mu,\mu) = \mu$.
3. If A is disjunctive, then $A(\mu,\mu) \geq \mu$.

Proof. Let us take $x \in E$. If A is conjunctive, then

$$A(\mu(x),\mu(x)) \leq \mu(x) \wedge \mu(x) = \mu(x).$$

If A is averaging, then A is idempotent and $A(\mu,\mu)(x) = \mu(x)$. Finally, if A is disjunctive, then

$$A(\mu(x),\mu(x)) \geq \mu(x) \vee \mu(x) = \mu(x).$$

□

However, if A is mixed, each case can happen, even $A(\mu,\mu)$ could be incomparable to μ.

Example 3.4. Consider the real vector space $\mathbb{R}^2$ and $\mu : \mathbb{R}^2 \longrightarrow [0,1]$ defined by

$$\mu(x,y) = \begin{cases} 1 & \text{if} \quad x = y = 0, \\ a & \text{if} \quad x = 0, y \neq 0, \\ b & \qquad \text{otherwise}, \end{cases}$$

where $a,b \in \mathbb{R}$ satisfying $a \geq b$. It can be readily verified that the level sets of μ constitute vector subspaces of $\mathbb{R}^2$. By applying Proposition 2.5, we can conclude that μ is indeed a fuzzy vector subspace.

Now, let us consider the mixed aggregation function $A : [0,1]^2 \longrightarrow [0,1]$ given by

$$A(x,y) = \begin{cases} 1 & \text{if} \quad x > 0.5, y > 0.5, \\ 0 & \text{if} \quad x < 0.5, y < 0.5, \\ 0.5 & \qquad \text{otherwise}. \end{cases}$$

We have the following facts.

1. If $a, b \in (0.5, 1)$, we have that $A(\mu, \mu) : \mathbb{R}^2 \longrightarrow [0, 1]$ is given by

$$A(\mu, \mu)(x, y) = 1 \qquad \text{for all } x, y \in \mathbb{R}.$$

Therefore, $A(\mu, \mu) \geq \mu$.

2. If $a, b \in (0, 0.5)$, we have that $A(\mu, \mu) : \mathbb{R}^2 \longrightarrow [0, 1]$ is given by

$$A(\mu, \mu)(x, y) = 0 \qquad \text{for all } x, y \in \mathbb{R}.$$

Therefore, $A(\mu, \mu) \leq \mu$.

3. If $b \in (0, 0.5)$ and $a \in (0.5, 1)$, we have that $A(\mu, \mu) : \mathbb{R}^2 \longrightarrow [0, 1]$ is given by

$$A(\mu, \mu)(x, y) = \begin{cases} 1 & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x \neq 0. \end{cases}$$

Since $A(\mu, \mu)(0, 1) = 1 > a = \mu(0, 1)$ and $A(\mu, \mu)(1, 1) = 0 < b = \mu(1, 1)$, $A(\mu, \mu)$ is not comparable to μ.

4. If $a = b = 0.5$, we have that $A(\mu, \mu) : \mathbb{R}^2 \longrightarrow [0, 1]$ is given by

$$A(\mu, \mu)(x, y) = \begin{cases} 1 & \text{if} \quad x = y = 0, \\ 0.5 & \quad \text{otherwise.} \end{cases}$$

Therefore, $A(\mu, \mu) = \mu$.

Remark 3.5. Notice that Proposition 3.1 can be easily extended to more components too.

Proposition 3.1 provides a compelling assurance that the self-aggregation of any given fuzzy vector subspace results in another fuzzy vector subspace, without any limitations or constraints. This inherent property of aggregating the same fuzzy algebraic structure holds significant strength. However, as we will further explore in subsequent sections, it becomes evident that certain conditions on fuzzy vector spaces or aggregation functions are necessary for a comprehensive analysis and meaningful outcomes to be achieved.

4. AGGREGATION OF FUZZY VECTOR SPACES FROM THE SAME EQUIVALENT CLASS

We already know that the aggregation of $A(\mu, \eta)$ is again a fuzzy vector space assuming $\mu = \eta$. Dedicated to broadening the scope of this observation, this section aims at an extension of this fact to a larger class. Our particular focus lies in the examination of the aggregation process pertaining to similar fuzzy vector subspaces.

Definition 4.1. (Das [9]) Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{K}$ and $\mu, \eta : E \longrightarrow [0, 1]$ two fuzzy vector subspaces of E. We say that μ is similar to η, denoted by $\mu \sim \eta$, if they have the same level sets, i. e.,

$$\mu \sim \eta \qquad \iff \qquad \{\mu_t\}_{t \in \operatorname{Im} \mu} = \{\eta_s\}_{s \in \operatorname{Im} \eta} \tag{1}$$

Proposition 4.2. The relation $\sim$ is an equivalence relation.

Proof. Straightforward. $\square$

Remark 4.3. The equivalence relation $\sim$ has been widely studied in the framework of fuzzy subgroups (see [1, 5, 10, 12, 17]).

If $A : [0,1] \times [0,1] \longrightarrow [0,1]$ is an aggregation operator and $\mu, \eta : E \longrightarrow [0,1]$ two fuzzy vector subspaces of E, then, in general, we cannot assure that the aggregation $A(\mu, \eta)$ is again a fuzzy vector subspace as the following example shows.

Example 4.4. Let us consider the real vector space $\mathbb{R}^3$ and the following fuzzy vector subspaces $\mu, \eta : \mathbb{R}^3 \longrightarrow [0,1]$ defined by

$$\mu(x,y,z) = \begin{cases} 1 & \text{if} \quad (x,y,z) = (0,0,0), \\ 0.7 & \text{if} \quad x+y+z = 0 \text{ and } (x,y,z) \neq (0,0,0), \\ 0.4 & \qquad \text{otherwise.} \end{cases}$$

and

$$\eta(x,y,z) = \begin{cases} 1 & \text{if} \quad (x,y,z) = (0,0,0), \\ 0.5 & \text{if} \quad x-y = 0 \text{ and } (x,y,z) \neq (0,0,0), \\ 0.3 & \qquad \text{otherwise.} \end{cases}$$

Choose any aggregation function A satisfying $A(0.4, 0.3) < A(0.4, 0.5)$. For instance: the maximum operator, product t-norm or arithmetic mean can be chosen. Then,

$$A(\mu,\eta)(1,1,-2) = A(\mu(1,1,-2), \eta(1,1,-2)) = A(0.7, 0.5),$$

$$A(\mu,\eta)(0,0,3) = A(\mu(0,0,3), \eta(0,0,3)) = A(0.4, 0.5),$$

and

$$A(\mu,\eta)(1,1,1) = A(\mu(1,1,1), \eta(1,1,1)) = A(0.4, 0.3).$$

Therefore,

$$A(\mu,\eta)(1,1,1) = A(0.4,0.3) < A(0.4,0.5) = A(\mu,\eta)(1,1,-2) \wedge A(\mu,\eta)(0,0,3)$$

which means that $A(\mu, \eta)$ does not satisfy $(E1)$, the first axiom of a fuzzy vector subspace.

However, under the premise of Eq. (1), we can guarantee that the aggregation $A(\mu, \eta)$ is again a fuzzy vector subspace. Drawing inspiration from Theorem 3.14 in [13], which deals with fuzzy subgroups, we present the following lemma that satisfies a similar criterion. For the sake of completeness, we provide the proof below.

Lemma 4.5. Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{K}$ and $\mu, \eta : E \longrightarrow [0,1]$ two fuzzy vector subspaces of E. The following assertions are equivalent

1. $\mu(x) < \mu(y) \iff \eta(x) < \eta(y)$.
2. $\mu \sim \eta$.

P r o o f. Let us assume that $\mu \sim \eta$. We consider $x, y \in E$ such that $\mu(x) > \mu(y)$ and we put $t_1 = \mu(x)$ and $t_2 = \mu(y)$. Then,

$$x \in \mu_{t_1}, \quad y \in \mu_{t_2} \quad \text{and} \quad y \notin \mu_{t_1}.$$

Since $\mu \sim \eta$, we have $\mu_{t_1} = \eta_{r_1}$ for some $r_1 \in \text{Im } \eta \subseteq [0,1]$. Taking into account the fact of $x \in \mu_{t_1} = \eta_{r_1}$ and $y \notin \mu_{t_1} = \eta_{r_1}$, we conclude that

$$\eta(x) \geq r_1 \quad \text{and} \quad \eta(y) < r_1.$$

Hence, $\eta(x) > \eta(y)$. Analogously, we can prove if $\eta(x) > \eta(y)$.

Conversely, let us assume that $\mu(x) < \mu(y) \iff \eta(x) < \eta(y)$. In order to prove that $\mu \sim \eta$, without loss of generality let us choose $\mu_a \in \{\mu_t\}_{t \in \text{Im } \mu}$. On the one hand, if $\mu_a = E$, clearly

$$a = \bigwedge \{t \in [0,1] \mid t \in \text{Im } \mu\},$$

i. e., μ attains its infimum and $\mu(z) = a$ for some $z \in E$. We are going to prove that η attains its minimum. By reductio ad absurdum, let us suppose that η does not attain its infimum. Put $b = \bigwedge\{t \in [0,1] \mid t \in \text{Im } \eta\}$. Therefore, $\eta(x) > b$ for each $x \in E$. In particular, $\eta(z) > b$, and consequently, there exists some $z' \in E$ such that $\eta(z) > \eta(z') > b$. Under our premise, we obtain that $\mu(z) > \mu(z')$, but this is a contradiction because $\mu(z) = a$. On the other hand, we study the case $\mu_a \neq E$. We know that there is $z \in E$ such that $\mu(z) = a$ and put $b = \eta(z)$. We have that

$$y \in E \setminus \{\mu_a\} \iff \mu(y) < a = \mu(z) \iff \eta(y) < \eta(z) = b \iff y \in E \setminus \{\eta_b\}.$$

Therefore, the complementary sets of μ_a and η_b are equal, i. e., $\mu_a = \eta_b$. We conclude that

$$\{\mu_t\}_{t \in \text{Im } \mu} \subseteq \{\eta_s\}_{s \in \text{Im } \eta}.$$

Using the same idea, it can be to prove

$$\{\eta_s\}_{s \in \text{Im } \eta} \subseteq \{\mu_t\}_{t \in \text{Im } \mu}.$$

Thus, $\mu \sim \eta$. □

Theorem 4.6. Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{K}$, $A : [0,1]^2 \longrightarrow [0,1]$ an aggregation function and $\mu, \eta : E \longrightarrow [0,1]$ two fuzzy vector subspaces of E. If μ is similar to η, then $A(\mu, \eta)$ is a fuzzy vector space of E.

P r o o f. Firstly, let us suppose that $x, y \in E$. By monotonicity,

$$A(\mu, \eta)(x + y) = A(\mu(x + y), \eta(x + y)) \geq A(\mu(x) \wedge \mu(y), \eta(x) \wedge \eta(y))$$

Now, taking into account Lemma 4.5, we conclude that

$$A(\mu, \eta)(x + y) \geq A(\mu, \eta)(x) \qquad \text{or} \qquad A(\mu, \eta)(x + y) \geq A(\mu, \eta)(y).$$

Hence, $A(\mu,\eta)(x+y) \geq A(\mu,\eta)(x) \wedge A(\mu,\eta)(y)$. Secondly, let us suppose $x \in E$ and $a \in \mathbb{K}$. From axiom $(A2)$,

$$A(\mu,\eta)(ax) = A(\mu(ax),\eta(ax)) \geq A(\mu(x),\eta(x)) = A(\mu,\eta)(x).$$

Hence, $A(\mu,\eta)$ is a fuzzy vector space of E. □

Corollary 4.7. Let $(E,+,\cdot)$ be a vector space on a field $\mathbb{K}$, $A : [0,1]^n \longrightarrow [0,1]$ an aggregation function and $\mu_1,\ldots,\mu_n : E \longrightarrow [0,1]$ fuzzy vector subspaces of E. If μ_i is similar to μ_j for all $1i, j \in \{1,2,\ldots,n\}$, then $A(\mu_1,\ldots,\mu_n)$ is a fuzzy vector space of E.

We raise the question of whether the aggregation of two similar fuzzy vector subspaces necessarily belongs to the same equivalence class. However, it is important to note that in general, this assumption does not hold true (as demonstrated in Example 4.8). Consequently, we introduce a condition on the aggregation function that characterizes this particular property (as stated in Proposition 4.10).

Example 4.8. Let us consider the real vector space $\mathbb{R}^2$ and the following fuzzy vector subspaces $\mu,\eta : \mathbb{R}^2 \longrightarrow [0,1]$ defined by

$$\mu(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0), \\ 0.5 & \text{if } x+y=0 \text{ and } (x,y) \neq (0,0), \\ 0.25 & \text{otherwise.} \end{cases}$$

and

$$\eta(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0), \\ 0.4 & \text{if } x+y=0 \text{ and } (x,y) \neq (0,0), \\ 0.1 & \text{otherwise.} \end{cases}$$

Clearly, their level sets are the same, hence $\mu \sim \eta$. Choose any aggregation function A satisfying $A(0.5, 0.4) = 0$, and consequently, $A(x,y) = 0$ for all $x \leq 0.5$ and $y \leq 0.4$. For instance: Łukasiewicz t-norm or the drastic conjunctor. Then,

$$A(\mu,\eta)(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $A(\mu,\eta) \not\sim \mu$.

Jointly strictly monotone property serves as a necessary and sufficient condition that can be imposed on the aggregation operator to ensure the preservation of the inner structure of an equivalence class when aggregating fuzzy vector subspaces.

Definition 4.9. (Calvo et al. [7]) Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. We say that A is jointly strictly monotone if assuming $x_i < y_i$ for all $i \in \{1,2,\ldots,n\}$, then $A(x_1,\ldots,x_n) < A(y_1,\ldots,y_n)$.

Proposition 4.10. Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{K}$ and $A : [0,1]^2 \longrightarrow [0,1]$ an aggregation function. The following assertions are equivalent.

1. A is jointly strictly monotone.
2. Given any two fuzzy vector subspaces μ and η of E satisfying $\mu \sim \eta$, we have that $A(\mu, \eta) \sim \mu$.

Proof. (1) $\Rightarrow$ (2). Let us suppose that $\mu \sim \eta$. Theorem 4.6 provides $A(\mu, \eta)$ is a fuzzy vector subspace. Assume $x, y \in E$ such that $\mu(x) < \mu(y)$. Since $\mu \sim \eta$, $\eta(x) < \eta(y)$. Under the premise that A is jointly strictly monotone,

$$A(\mu, \eta)(x) = A(\mu(x), \eta(x)) < A(\mu(y), \eta(y)) = A(\mu, \eta)(y),$$

hence $\mu \sim A(\mu, \eta)$.

(2) $\Rightarrow$ (1). For this converse, we are going to prove that if A is not a jointly strictly monotone aggregation function, then there are two similar fuzzy vector subspaces μ and η such that $\mu \sim \eta \not\sim A(\mu, \eta)$. Since A is not jointly strictly monotone, there exist $x_1, x_2, y_1, y_2 \in [0,1]$ satisfying $x_1 < x_2$ and $y_1 < y_2$ such that $A(x_1, y_1) \geq A(x_2, y_2)$. By monotonicity, $A(x_1, y_1) = A(x_2, y_2))$. Now, let us choose the following fuzzy vector subspaces $\mu, \eta : E \longrightarrow [0,1]$ given by

$$\mu(x) = \begin{cases} a_2 & \text{if} \quad x = e, \\ a_1 & \text{if} \quad x \neq e. \end{cases} \qquad \eta(x) = \begin{cases} b_2 & \text{if} \quad x = e, \\ b_1 & \text{if} \quad x \neq e. \end{cases}$$

Trivially, μ and η are fuzzy vector subspaces of E because their level sets are vector subspaces of any vector space. Clearly, $\mu \sim \eta$. Eventually, we prove that $A(\mu, \eta) \not\sim \mu$. Choose $z \neq e$, we have

$$A(\mu, \eta)(e) = A(\mu(e), \eta(e)) = A(a_2, b_2) = A(a_1, b_1) = A(\mu(z), \eta(z)) = A(\mu, \eta)(z).$$

Since $\mu(e) = a_2 > a_1 = \mu(z)$, taking into account Lemma 4.5, we conclude that $A(\mu, \eta) \not\sim \mu$. □

5. AGGREGATION OF FUZZY VECTOR SPACES

In this concluding section, our attention shifts towards arbitrary fuzzy vector subspaces. It is important to note that, in general, the aggregation of such subspaces does not result in a fuzzy vector subspace, as demonstrated in Example 4.4. However, as we have observed in previous sections, certain conditions can be established to ensure that their aggregation does form a new fuzzy vector subspace. Our primary objective is to characterize this phenomenon.

To facilitate our analysis, we introduce a notion that plays a crucial role in the aggregation process. It is noteworthy that this notion holds broader significance compared to Definition 3.12 in [4].

Definition 5.1. Let S be a set, and consider two fuzzy sets $\mu, \eta : S \longrightarrow [0,1]$. We denote by $\mathcal{L}(\mu, \eta)$ the subset of the Cartesian product $S \times S$ defined by:

$$\mathcal{L}(\mu, \eta) := \{(x, y) \in S \times S \mid \mu(x) < \mu(y) \text{ and } \eta(x) > \eta(y)\}.$$

We are interested in the algebraic structure of the set S. For instance, if S is a group, Definition 5.1 is equal to the given in [4]. If $S = E$ is a vector space, then we can create a binary relation on the set of fuzzy vector subspaces of a vector space E:

$$\mu \simeq \eta \qquad \Longleftrightarrow \qquad \mathcal{L}(\mu, \eta) = \emptyset.$$

It is evident that the relation $\simeq$ is reflexive and symmetric. However, it does not possess the transitive property in general. To illustrate this, consider the fuzzy vector subspace ν defined by $\nu(x) = 1$ for all $x \in E$. It can be observed that every fuzzy vector subspace μ trivially satisfies $\mu \simeq \nu$. If $\simeq$ were a transitive relation, then for any pair (μ, η) of fuzzy vector subspaces, we would have $\mu \simeq \eta$, which is clearly not the case (as shown in Example 4.4).

In order to establish a characterization of the aggregation of fuzzy vector subspaces, we present a preliminary lemma.

Lemma 5.2. Let $\mu : E \longrightarrow [0, 1]$ be a fuzzy vector subspace of a vector space E on a field $\mathbb{K}$ and $x, y \in E$. If $\mu(x) \neq \mu(y)$, then

$$\mu(x + y) = \mu(x) \wedge \mu(y).$$

Proof. From $(E1)$, $\mu(x + y) \geq \mu(x) \wedge \mu(y)$. By reductio ad absurdum, we assume $\mu(x + y) > \mu(x) \wedge \mu(y)$. Without loss of generality, we can consider $\mu(x) < \mu(y)$. Consequently, $\mu(x+y) > \mu(x)$ and taking into account that μ is a fuzzy vector subspace, we obtain

$$\mu(x) \geq \mu(x + y) \wedge \mu(-y) = \mu(x + y) \wedge \mu(y).$$

This is a contradiction because $\mu(x) < \mu(y)$ and $\mu(x) < \mu(x + y)$. □

Theorem 5.3. Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{K}$ and and $\mu, \eta : E \longrightarrow [0, 1]$ two fuzzy vector subspaces. Then, $\mathcal{L}(\mu, \eta) = \emptyset$ if and only if for any aggregation function $A : [0, 1]^2 \longrightarrow [0, 1]$, we have that $A(\mu, \eta)$ is a fuzzy vector subspace.

Proof. First, let us assume $\mathcal{L}(\mu, \eta) = \emptyset$.

$(E1)$ Choose $x, y \in E$, we have three cases to study:

1. $\mu(x) < \mu(y)$. Since $\mathcal{L}(\mu, \eta) = \emptyset$, necessary $\eta(x) \leq \eta(y)$. By monotonicity of A,

$$A(\mu(x + y), \eta(x + y)) \geq A(\mu(x) \wedge \mu(y), \eta(x) \wedge \eta(y)) = A(\mu(x), \eta(x)).$$

   Hence,

$$A(\mu, \eta)(x + y) \geq A(\mu, \eta)(x) \geq A(\mu, \eta)(x) \wedge A(\mu, \eta)(y)).$$

2. $\mu(x) > \mu(y)$. By a similar argument to the previous case.

3. $\mu(x) = \mu(y)$. We have $\eta(x) \leq \eta(y)$ or $\eta(x) \geq \eta(y)$. Without loss of generality, let us assume $\eta(x) \leq \eta(y)$. By monotonicity,

$$A(\mu(x+y), \eta(x+y)) \geq A(\mu(x) \wedge \mu(y), \eta(x) \wedge \eta(y)) = A(\mu(x), \eta(x)).$$

Hence,

$$A(\mu,\eta)(x+y) \geq A(\mu,\eta)(x) \geq A(\mu,\eta)(x) \wedge A(\mu,\eta)(y).$$

$(E2)$ Choose $x \in E$ and $a \in \mathbb{K}$, then

$$A(\mu,\eta)(ax) = A(\mu(ax), \eta(ax)) \geq A(\mu(x), \eta(x)) = A(\mu,\eta)(x).$$

Conversely, let us assume $\mathcal{L}(\mu,\eta) \neq \emptyset$. We are going to prove that there is an aggregation function A satisfying $A(\mu,\eta)$ is not a fuzzy vector subspace. By hyphotesis, since $\mathcal{L}(\mu,\eta) \neq \emptyset$, there are $x, y \in E$ such that $\mu(x) < \mu(y)$ and $\eta(x) > \eta(y)$. Let us define the following aggregation function $A : [0,1]^2 \longrightarrow [0,1]$,

$$A(r,s) = \begin{cases} 1 & \text{if} \quad r \geq \mu(x) \text{ and } s \geq \eta(x), \\ 1 & \text{if} \quad r \geq \mu(y) \text{ and } s \geq \eta(y), \\ 0 & \quad\quad \text{otherwise.} \end{cases}$$

Clearly, A is well-defined and satisfies the two axioms of an aggregation function. By Lemma 5.2,

$$A(\mu,\eta)(x+y) = A(\mu,\eta(x) \wedge A(\mu,\eta)(y).$$

Now, let us notice that $A(\mu(x),\eta(x)) = 1 = A(\mu(y),\eta(y))$ and $A(\mu(x),\eta(y)) = 0$. Thus,

$$A(\mu,\eta)(x+y) = A(\mu(x+y),\eta(x+y)) = A(\mu(x),\eta(y)) = 0 < 1 = A(\mu,\eta)(x) \wedge A(\mu,\eta)(y),$$

which is a contradiction. □

Building upon Theorem 5.3, we derive three consequential corollaries. The first corollary demonstrates that if there exists a strictly monotone aggregation function B that preserves the algebraic structure of fuzzy vector subspaces, then any aggregation function that satisfies this condition will also preserve the algebraic structure. Notably, Ordered Weighted Averaging (OWA) functions serve as noteworthy examples of strictly monotone aggregation functions in practical applications (see [7]).

Corollary 5.4. Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{K}$ and and $\mu, \eta : E \longrightarrow [0,1]$ two fuzzy vector subspaces. If B is a strictly monotone aggregation function, then the following assertions are equivalent:

1) $\mathcal{L}(\mu,\eta) = \emptyset$.

2) $B(\mu,\eta)$ is a fuzzy vector subspace.

3) $A(\mu,\eta)$ is a fuzzy vector subspace for each aggregation function A.

Proof. Since trivially 3) $\Rightarrow$ 2), taking into account Theorem 5.3, it is only needed to prove 2) $\Rightarrow$ 1). Let us assume that $B(\mu,\eta)$ is a fuzzy vector subspace of E. By reductio ad absurdum, $\mathcal{L}(\mu,\eta) \neq \emptyset$, i. e., there are $x, y \in E$ satisfying $\mu(x) < \mu(y)$ and $\eta(y) < \eta(x)$. By Lemma 5.2, we have that

$$\mu(x+y) = \mu(x) < \mu(y) \quad \text{and} \quad \eta(x+y) = \eta(y) < \eta(x).$$

Choose $t = B(\mu,\eta)(x) \wedge B(\mu,\eta)(y)$ and consider the following level set

$$A(\mu,\eta)_t = \{z \in E \mid B(\mu,\eta)(z) \geq t\}.$$

Trivially $x, y \in B(\mu,\eta)_t$ and taking into account that $B(\mu,\eta)$ is fuzzy vector subspace, using Proposition 2.5, $B(\mu,\eta)_t$ is a vector subspace of E (in the crisp sense). Hence, $x+y \in B(\mu,\eta)_t$. Oppositely, since B is strictly monotone,

$$B(\mu,\eta)(x+y) = B(\mu(x+y),\eta(x+y)) = B(\mu(x),\eta(y)) < B(\mu(y),\eta(y)) = B(\mu,\eta)(y)$$

and

$$B(\mu,\eta)(x+y) = B(\mu(x+y),\eta(x+y)) = B(\mu(x),\eta(y)) < B(\mu(x),\eta(x)) = B(\mu,\eta)(x)$$

concluding $B(\mu,\eta)(x+y) < t$, that is, a contradiction. Therefore, $\mathcal{L}(\mu,\eta) = \emptyset$. □

In various scenarios, the strictly monotone property is only partially required within a subset of the unit interval, such as in the case of t-norms or t-conorms. Subject to certain conditions on the fuzzy vector subspaces, the following two corollaries provide a characterization of the aggregation process for fuzzy vector subspaces, taking into consideration these types of fuzzy operators.

Corollary 5.5. Let $(E,+,\cdot)$ be a vector space on a field $\mathbb{K}$ and and $\mu,\eta : E \longrightarrow [0,1]$ two fuzzy vector subspaces satisfying that $\mu(x) > 0$ and $\eta(x) > 0$ for each $x \in E$. If B_0 is an aggregation function which is strictly monotone on the Cartesian product $(0,1] \times (0,1]$, then the following assertions are equivalent:

1) $\mathcal{L}(\mu,\eta) = \emptyset$.

2) $B_0(\mu,\eta)$ is a fuzzy vector subspace.

3) $A(\mu,\eta)$ is a fuzzy vector subspace for any aggregation function A.

Proof. Since trivially 3) $\Rightarrow$ 2), taking into account Theorem 5.3, it is only needed to prove 2) $\Rightarrow$ 1). Suppose that $B_0(\mu,\eta)$ is a fuzzy vector subspace. By reductio ad absurdum, there are $x, y \in E$ satisfying $0 < \mu(x) < \mu(y)$ and $0 < \eta(y) < \eta(x)$. By Lemma 5.2, we have that $0 < \mu(x+y) = \mu(x) < \mu(y)$ and $0 < \eta(x+y) = \eta(y) < \eta(x)$. Choose $t = B_0(\mu,\eta)(x) \wedge B_0(\mu,\eta)(y)$ and let us consider the level set $B_0(\mu,\eta)_t = \{z \in E \mid B_0(\mu,\eta)(z) \geq t\}$. Clearly, $x, y \in B_0(\mu,\eta)_t$. Since $B_0(\mu,\eta)$ is a fuzzy vector subspace by hypothesis, $B_0(\mu,\eta)_t$ is a vector subspace of E (in the crisp sense). Then, $x + y \in B_0(\mu, \eta)_t$. Since B_0 is strictly monotone on $(0, 1] \times (0, 1]$ and $\mu(x) \neq 0 \neq \eta(y)$,

$$B_0(\mu,\eta)(x+y) = B_0(\mu(x+y), \eta(x+y)) = B_0(\mu(x), \eta(y)) < B_0(\mu(y), \eta(y)) = B_0(\mu,\eta)(y)$$

and

$$B_0(\mu,\eta)(x + y) = B_0(\mu(xy), \eta(x + y)) = B_0(\mu(x), \eta(y)) < B_0(\mu(x), \eta(x)) = B_0(\mu,\eta)(x)$$

then $B_0(\mu,\eta)(x+y) < t$. Hence $x+y \notin B_0(\mu,\eta)_t$, which is a contradiction. We conclude that $\mathcal{L}(\mu,\eta) = \emptyset$. □

Corollary 5.6. Let $(E, +, \cdot)$ be a vector space on a field $\mathbb{K}$ and and $\mu, \eta : E \longrightarrow [0, 1]$ two fuzzy vector subspaces satisfying $\mu(x) = 1$ if and only if $x = e_+$ and $\eta(x) = 1$ if and only if $x = e_+$. If B_1 is an aggregation function which is strictly monotone on $[0, 1) \times [0, 1)$, then the following assertions are equivalent:

1) $\mathcal{L}(\mu,\eta) = \emptyset$.

2) $B_1(\mu,\eta)$ is a fuzzy vector subspace.

3) $A(\mu,\eta)$ is a fuzzy vector subspace for any aggregation function A.

Proof. Since trivially 3) $\Rightarrow$ 2), taking into account Theorem 5.3, it is only needed to prove 2) $\Rightarrow$ 1). Let us assume that $B_1(\mu,\eta)$ is a fuzzy vector space. By reductio ad absurdum, there are $x, y \in E$ sastisfying $\mu(x) < \mu(y)$ and $\eta(y) < \eta(x)$. Trivially, $x \neq e \neq y$. By hyphotesis, $\mu(y) < 1$ and $\eta(x) < 1$, and Lemma 5.2, we have that $\mu(x + y) = \mu(x) < \mu(y) < 1$ and $\eta(x + y) = \eta(y) < \eta(x) < 1$. Let us choose $t = B_1(\mu,\eta)(x) \wedge B_1(\mu,\eta)(y)$ and consider the level set $B_1(\mu,\eta)_t = \{z \in E \mid B_1(\mu,\eta)(z) \geq t\}$. Clearly, $x, y \in B_1(\mu,\eta)_t$. Since $B_1(\mu,\eta)$ is a fuzzy vector subspace, $B_1(\mu,\eta)_t$ is a vector subspace of E (in the crisp sense). Thus, $x, y \in B_1(\mu,\eta)_t$. Since B_1 is strictly monotone on $[0, 1) \times [0, 1)$ and $\mu(x) \neq 1 \neq \eta(y)$, we have

$$B_1(\mu,\eta)(x+y) = B_1(\mu(x+y), \eta(x+y)) = B_1(\mu(x), \eta(y)) < B_1(\mu(y), \eta(y)) = B_1(\mu,\eta)(y)$$

and

$$B_1(\mu,\eta)(x+y) = B_1(\mu(x+y), \eta(x+y)) = B_1(\mu(x), \eta(y)) < B_1(\mu(x), \eta(x)) = B_1(\mu,\eta)(x)$$

then $B_1(\mu,\eta)(x+y) < t$. Hence $x+y \notin B_1(\mu,\eta)_t$, which is a contradiction. We conclude that $\mathcal{L}(\mu,\eta) = \emptyset$. □

6. CONCLUDING REMARKS

This paper contributes to the ongoing research on the aggregation of algebraic structures, focusing specifically on the domain of fuzzy vector spaces. The study is divided into several key components, each addressing different aspects of the aggregation process.

The first part of the paper delves into a comprehensive analysis of the self-aggregation of fuzzy vector spaces. By exploring the inherent properties and characteristics of these algebraic structures, providing a thorough understanding of the aggregation process within this domain.

Moving forward, the paper presents a series of propositions that investigate the aggregation of fuzzy vector subspaces belonging to equivalent classes. We examine the conditions under which these aggregations preserve the equivalence class, shedding light on the factors that influence the outcome of the aggregation process. This analysis provides valuable insights into the behavior of fuzzy vector subspaces during aggregation and enhances our understanding of the underlying mechanisms at play.

Furthermore, the preservation of these equivalence classes is thoroughly examined in a dedicated section. By studying the properties and constraints that ensure the preservation of equivalence classes, we contribute to the theoretical framework surrounding the aggregation of fuzzy vector subspaces. This analysis adds depth to our knowledge of how aggregation impacts the algebraic structure of these subspaces.

The culmination of the paper is the presentation of the main result: a characterization of the aggregation of fuzzy vector subspaces based on an algebraic condition imposed on the subspaces. This significant finding contributes to our understanding of the aggregation process and provides a valuable tool for further research and analysis in the field of fuzzy vector spaces.

In addition to the main result, the paper concludes with three significant corollaries that explore the role of strictly monotone aggregation functions. These corollaries highlight the impact of such functions on the aggregation of fuzzy vector subspaces, providing further insights and implications for practical applications.

Overall, this paper provides a comprehensive and in-depth investigation of the aggregation of fuzzy vector spaces.

ACKNOWLEDGEMENT

Carlos Bejines gratefully acknowledges the funding support of the Spanish Ministerio de Ciencia e Innovación whose codes are PID2021-124063OB-I00 and PID2022-140630NB-I00. Moreover, we thank the team of reviewers for their thorough reading while reviewing the article. For the purpose of enhancing the article's readability, an artificial intelligence known as chatGPT (OpenAI, 2021) was employed in the writing of the manuscript.

(Received June 16, 2023)

REFERENCES

[1] I. K. Appiah and B. Makamba: Counting distinct fuzzy subgroups of some rank-3 abelian groups. Iranian J. Fuzzy Systems *14* (2017), 1, 163–181.

[2] C. Bejines, S. Ardanza-Trevijano, M. Chasco, and J. Elorza: Aggregation of indistinguishability operators. Fuzzy Sets Systems *446* (2022), 53–67. DOI:10.1016/j.fss.2021.04.023

[3] C. Bejines, S. Ardanza-Trevijano, and J. Elorza: On self-aggregations of min-subgroups. Axioms *10* (2021), 3, 201. DOI:10.3390/axioms10030201

[4] C. Bejines, M. J. Chasco, and J. Elorza: Aggregation of fuzzy subgroups. Fuzzy Sets Systems *418* (2021), 170–184. DOI:10.1016/j.fss.2020.05.017

[5] C. Bejines, M. J. Chasco, J. Elorza, and S. Montes: Equivalence relations on fuzzy subgroups. In: Conference of the Spanish Association for Artificial Intelligence, Springer 2018, pp. 143–153.

[6] G. Beliakov, A. Pradera, T. Calvo, et al.: Aggregation Functions: A Quide for Practitioners, Vol. 221. Springer, 2007.

[7] T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar: Aggregation Operators. Chapter Aggregation Operators: Properties, Classes and Construction Methods, Physica-Verlag, Heidelberg 2002, pp. 3–104.

[8] P. Das: Fuzzy vector spaces under triangular norms. Fuzzy Sets Systems *25* (1988), 1, 73–85. DOI:10.1016/0165-0114(88)90101-7

[9] P. S. Das: Fuzzy groups and level subgroups. J. Math. Anal. Appl. *84* (1981), 1, 264–269. DOI:10.1016/0022-247X(81)90164-5

[10] V. Dixit, R. Kumar, and N. Ajmal: Level subgroups and union of fuzzy subgroups. Fuzzy Sets Systems *37* (1990), 3, 359–371. DOI:10.1016/0165-0114(90)90032-2

[11] J. Drewniak and U. Dudziak: Preservation of properties of fuzzy relations during aggregation processes. Kybernetika *43* (2007), 2, 115–132.

[12] A. Iranmanesh and H. Naraghi: The connection between some equivalence relations on fuzzy subgroups. Iranian J. Fuzzy Systems *8* (2011), 5, 69–80.

[13] A. Jain: Fuzzy subgroups and certain equivalence relations. Iranian J. Fuzzy Systems *3* (2006), 2, 75–91.

[14] A. Katsaras and D. Liu: Fuzzy vector spaces and fuzzy vector topological spaces. J. Math. Anal. App. *58* (1977), 135–146.

[15] F. Kouchakinejad and A. Šipošová: A note on the super-additive and sub-additive transformations of aggregation functions: The multi-dimensional case. Kybernetika *53* (2017), 1, 129–136. DOI:10.14736/kyb-2017-1-0129

[16] J. N. Mordeson, K. R. Bhutani, and A. Rosenfeld: Fuzzy Group Theory. Springer, 2005.

[17] V. Murali and B. B. Makamba: On an equivalence of fuzzy subgroups i. Fuzzy Sets Systems *123* (2001), 2, 259–264. DOI:10.1016/S0165-0114(00)00098-1

[18] T. Pedraza, J. Ramos-Canós, and J. Rodríguez-López: Aggregation of weak fuzzy norms. Symmetry *13* (2021), 10, 1908. DOI:10.3390/sym13101908

[19] T. Pedraza, J. Rodríguez-López, and Ó. Valero: Aggregation of fuzzy quasi-metrics. Inform. Sci. *581* (2021), 362–389. DOI:10.1016/j.ins.2020.08.045

Carlos Bejines, Universidad de Málaga, Andalucía Tech. Spain.
e-mail: cbejines@uma.es