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STRONG ENDOMORPHISM KERNEL PROPERTY FOR FINITE BROUWERIAN SEMILATTICES AND RELATIVE STONE ALGEBRAS

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Abstract. We show that all finite Brouwerian semilattices have strong endomorphism kernel property (SEKP), give a new proof that all finite relative Stone algebras have SEKP and also fully characterize dual generalized Boolean algebras which possess SEKP.

Keywords: (strong) endomorphism kernel property; congruence relation; Brouwerian semilattice; Brouwerian algebra; dual generalized Boolean algebra; direct sum; factorable congruences

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1. Introduction

The notions of endomorphism kernel property was introduced and first studied by Blyth, Fang and Silva in [1]. A slight improvement of one of proofs of [1] was given in [11]. Blyth and Silva in paper [3] defined a strong endomorphism kernel property (SEKP) for a universal algebra (see Definition 2.2). They considered the case of Ockham algebras and in particular of MS-algebras. They proved e.g. that a finite Boolean algebra has SEKP if and only if it is 2 element BA, a finite bounded distributive lattice possesses SEKP if and only if it is a chain and they proved full characterization of MS-algebras having SEKP. Blyth, Fang and Wang proved a full characterization of finite distributive double p-algebras and finite double Stone algebras having SEKP in [2]. SEKP for distributive p-algebras and Stone algebras has been studied and fully characterized by Fang and Fang, see [5]. Fang and Sun fully characterized semilattices with SEKP in [7]. Guričan and Ploščica fully char-

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acterized unbounded distributive lattices which possess SEKP in [13]. The main approach in papers [3], [2] and [5] was done by regarding algebras in question as Ockham algebras and using the duality theory of Priestley. Priestley duality is also used in [13]. Guričan used another approach to prove e.g. that all finite relative Stone algebras have SEKP in [12]. Halušková described monounary algebras with SEKP in [14] and some monounary algebras with a weaker property called endomorphism kernel property (EKP) in [15]. Double MS-algebras with SEKP were described by Fang in [6]. Finite abelian groups with SEKP were described by Fang and Sun in [8], Ghumashyan and Guričan proved that all finite Abelian groups have EKP in [9].

There is one important universal assumption in the original paper [3] of Blyth and Silva, namely, all algebras considered in this paper must contain two nullary operations (denoted by 0 and 1, $0 \neq 1$). This assumption is necessary to prove all important statements in their paper and therefore it seems to be impossible to directly adapt their methods to algebras which do not satisfy this assumption (e.g. lattices with the top element or unbounded lattices). Let us mention three of these results:

Theorem 1.1 ([3], Theorem 1). *If an algebra A has SEKP, then it has at most one maximal congruence.*

Corollary 1.2 ([3], Corollary 1). *A finite algebra that has SEKP is directly indecomposable.*

Theorem 1.3 ([3], Theorem 3). *A semisimple algebra has SEKP if and only if it is simple.*

As it is easy to check, $\{0,1\}^2$ considered as 4 element distributive lattice with a top element ($\{1\}$-lattice) has SEKP and none of these statements is true for this algebra. Also, these statements are not true e.g. for Brouwerian semilattices and for Brouwerian algebras.

In this paper we show that all finite Brouwerian semilattices have strong endomorphism kernel property (SEKP) (see Theorem 3.10), give a new proof that all finite relative Stone algebras have SEKP (see Corollary 3.11), describe an infinite family of infinite Brouwerian semilattices and relative Stone algebras which enjoy SEKP and also fully characterize dual generalized Boolean algebras (one of subvarieties of the variety of relative Stone algebras) which possess SEKP (see Theorem 4.5).

2. Preliminaries

Let A be a universal algebra. We denote $\omega_A = \{(a, a) \colon a \in A\}$ and $\iota_A = A \times A$, trivial and universal congruences on A, respectively, $\mathrm{Con}(A)$ the set of all congruences of A, $\mathrm{End}(A)$ the set of all endomorphisms of A. For an endomorphism $f \in \mathrm{End}(A)$, $\ker(f) = \{(a, b) \in A^2 \colon f(a) = f(b)\}$ denotes the kernel of f and $\mathrm{Im}(f) = \{f(a)\colon a \in A\}$ denotes the image of f. We denote $[a]\theta = \{b \in A \colon (a, b) \in \theta\}$ for $a \in A$, $\theta \in \mathrm{Con}(A)$.

Next definition defines the notion of EKP (see [1]).

Definition 2.1. An algebra A has the *endomorphism kernel property* (EKP for short) if every congruence relation on A different from the universal congruence ι_A is the kernel of an endomorphism on A. In addition, we say that $f \in \mathrm{End}(A)$ is *associated* to $\theta \in \mathrm{Con}(A)$ if $\theta = \ker(f)$.

Next important notion is strong endomorphism property defined in [3]. Let A be a universal algebra, $f \colon A \to A$ be an endomorphism, $\theta \in \mathrm{Con}(A)$ be a congruence on A. Endomorphism f is *compatible* with θ if and only if $(a, b) \in \theta \Rightarrow (f(a), f(b)) \in \theta$ and is *strong* (on A) if it is compatible with every congruence $\theta \in \mathrm{Con}(A)$.

Definition 2.2. An algebra A has the *strong endomorphism kernel property* (SEKP for short) if and only if every congruence relation on A different from the universal congruence ι_A is the kernel of a strong endomorphism on A.

3. Finite Brouwerian semilattices and finite relative Stone algebras

We are going to show that all finite Brouwerian semilattices and all finite relative Stone algebras have SEKP.

Definition 3.1. *Brouwerian semilattice* is an algebra $(L, \wedge, *, 1)$ of type $(2, 2, 0)$ such that the reduct $(L, \wedge, 1)$ is a meet-semilattice with greatest element 1, and $a * b$ is the relative pseudocomplement of a with respect to b; it means that $a * b$ is the greatest element of the set $\{x \in L;\ a \wedge x \leqslant b\}$.

The list of most important properties which hold in a Brouwerian semilattice L is as follows (see e.g. [23], [22], we assume that $*$ binds stronger than $\wedge$).

For all $a, b, c \in L$:

$$1 * a = a, \tag{3.1}$$

$$a * a = 1, \tag{3.2}$$

$$a * b \geqslant b, \tag{3.3}$$

(3.4) $$a \wedge a * b = a \wedge b,$$

(3.5) $$a * (b * c) = (a \wedge b) * c,$$

(3.6) $$a * (b * c) = (a * b) * (a * c),$$

(3.7) $$a * (b \wedge c) = a * b \wedge a * c,$$

(3.8) $$a \leqslant b \Leftrightarrow a * b = 1,$$

(3.9) $$a \leqslant b \Rightarrow c * a \leqslant c * b.$$

Definition 3.2. Meet-semilattice L is called *distributive* if for $t, x, y \in L$, $t \geqslant x \wedge y$, there are $x_1, y_1 \in L$ such that $x_1 \geqslant x$, $y_1 \geqslant y$ and $t = x_1 \wedge y_1$.

It is known that Brouwerian semilattices are distributive in this sense (see [18], Section 2.7). Let L be a Brouwerian semilattice, F be a filter of L. Denote θ_F to be the relation on L given by

$$(x, y) \in \theta_F \text{ if and only if } x \wedge d = y \wedge d \quad \text{for some } d \in F.$$

Also denote $[a) = \{x \in L;\ a \leqslant x\}$ for $a \in L$.

The characterization of congruences of a Brouwerian semilattice was given in [23] as:

Lemma 3.3. *Let L be a Brouwerian semilattice. If θ is a congruence of L, then $[1]\theta$ is a filter of L and*

$$(x, y) \in \theta \text{ if and only if } x \wedge d = y \wedge d \quad \text{for some } d \in [1]\theta$$

and conversely, if F is a filter of L, then the relation θ_F is a congruence of L with $[1]\theta_F = F$.

Definition 3.4. *Brouwerian algebra* is an algebra $(L, \vee, \wedge, *, 1)$ of type $(2, 2, 2, 0)$ such that the reduct $(L, \vee, \wedge)$ is (necessarily distributive) lattice with greatest element 1, and $a * b$ is a relative pseudocomplement of a with respect to b (in the reduct $(L, \wedge, *, 1)$).

Brouwerian algebra is called a *relative Stone algebra* if each of its (closed) intervals is a Stone algebra, or equivalently, if it satisfies the identity

$$(x * y) \vee (y * x) = 1$$

(see [18]).

As Brouwerian algebras are distributive lattices, the characterization and relationship described in Lemma 3.3 hold also between filters and congruences in a Brouwerian algebra (the operation $\vee$ do not change congruences in the case of Brouwerian algebras, but it changes subalgebras and endomorphisms).

We shall start with the characterization of strong endomorphisms in Brouwerian semilattices (algebras).

Theorem 3.5. *Let L be either a Brouwerian semilattice or a Brouwerian algebra, $\varphi\colon L \to L$ be an endomorphism of the corresponding algebra. The following conditions are equivalent:*

(i) *φ is a strong endomorphism,*
(ii) *$\varphi(F) \subseteq F$ for every filter F of L,*
(iii) *$a \leqslant \varphi(a)$ for all $a \in L$.*

Proof. (i) $\Rightarrow$ (ii): Let F be a filter of L, θ be a congruence of L with $F = [1]\theta$, which means $\theta = \theta_F$. As φ is a strong endomorphism, for $f \in F$ we have $(f, 1) \in \theta$ and therefore $(\varphi(f), \varphi(1)) = (\varphi(f), 1) \in \theta$. It means that $\varphi(f) \in F$, so that $\varphi(F) \subseteq F$.

(ii) $\Rightarrow$ (i): Let θ be a congruence on L, $F = [1]\theta$. Let $(a, b) \in \theta$. Then there is $f \in F$ with $a \wedge f = b \wedge f$. As φ is an endomorphism,

$$\varphi(a) \wedge \varphi(f) = \varphi(a \wedge f) = \varphi(b \wedge f) = \varphi(b) \wedge \varphi(f).$$

By (ii), $\varphi(f) \in F$ and therefore $(\varphi(a), \varphi(b)) \in \theta$.

(ii) $\Rightarrow$ (iii): Let $a \in L$. Then $[a)$ is a filter in L and by (ii), $\varphi([a)) \subseteq [a)$, so that $a \leqslant \varphi(a)$.

(iii) $\Rightarrow$ (ii): Let F be a filter of L and $f \in F$. By (iii), $f \leqslant \varphi(f)$. As F is a filter, this means that $\varphi(f) \in F$, so that $\varphi(F) \subseteq F$. □

We remind a result from [21]. Even if it is not used in our proofs, it provides us a suggestion where to search for relevant strong endomorphisms for Brouwerian semilattices and Brouwerian algebras. First we utilize one new notion and new operation between filters.

Definition 3.6. Let L be a distributive meet-semilattice with 1. A filter F of L is called *comonomial* if every congruence class of the congruence θ_F has a greatest element. Let F_1, F_2 be filters of L. Denote

$$F_1 \veebar F_2 = \{t \in L;\ (\exists\, x \in F_1)(\exists\, y \in F_2)\ t = x \wedge y\}.$$

The set $F_1 \veebar F_2$ is a filter of L and using the distributivity of L it is the smallest filter containing both F_1 and F_2. Both these notions are relevant for Brouwerian semilattices and Brouwerian algebras, as they have a distributive meet-semilattice as a reduct.

Theorem 3.7 ([21], 2.1). *Let L be a Brouwerian semilattice or a Brouwerian algebra, F_1, F_2 be filters of L such that there exists $d \in L$ with $F_1 \cap F_2 = [d)$ (it means that this intersection is a principal filter generated by d) and $F_1 \veebar F_2 = L$. Then both F_1 and F_2 are comonomial and $d * t$ for $t \in F_2$ is the greatest element in the block $[t]\theta_{F_1}$.*

Corollary 3.8. *Let L be a Brouwerian semilattice or a Brouwerian algebra, F be a filter of L such that $F = [d)$ for some $d \in F$. Then F is comonomial and for every $t \in L$, $d * t$ is the greatest element in the block $[t]\theta_F$.*

P r o o f. It is enough to take $F_1 = F$ and $F_2 = L$. The assumptions of Theorem 3.7 are satisfied, therefore F is comonomial and $d * t$ for $t \in F_2 = L$ is the greatest element in the block $[t]\theta_{F_1}$. □

As a consequence, the function $\varphi_d\colon L \to L$ which assigns $t \mapsto d * t$ is a function, which is constant on blocks of the congruence $[t]\theta_{[d)}$ (the kernel of φ_d is $\theta_{[d)}$) and to each element of $[t]\theta_{[d)}$ it assignes the greatest element of this block. Thus, for all $a \in L$ we have $a \leqslant \varphi_d(a)$.

Lemma 3.9. *Let L be a Brouwerian semilattice, $d \in L$. Then φ_d is an endomorphism and consequently,* $\mathrm{Im}(\varphi_d)$ *is a subalgebra of L.*

P r o o f. Let us remind that $\varphi_d\colon L \to L$ is defined by $\varphi_d(a) = d * a$. Let $a, b \in L$. Then

$$\varphi_d(a * b) = d * (a * b) = (d * a) * (d * b) = \varphi_d(a) * \varphi_d(b)$$

by (3.6). Also

$$\varphi_d(a \wedge b) = d * (a \wedge b) = (d * a) \wedge (d * b) = \varphi_d(a) \wedge \varphi_d(b)$$

by (3.7). Thus, φ_d is an endomorphism of L. □

We are ready to prove the main result of this section

Theorem 3.10. *Let L be a finite Brouwerian semilattice. Then L has SEKP.*

P r o o f. Let θ be a congruence of L, $F = [1]\theta$. As L is finite, there is $d \in L$ such that $F = [d)$. We know that $\varphi_d \in \mathrm{End}(L)$ by Lemma 3.9. Also, $a \leqslant d * a = \varphi_d(a)$ holds for every $a \in L$ by (3.3), thus φ_d satisfies condition (iii) of Theorem 3.5 and it is a strong endomorphism of L.

We know that $d * f = 1$ is equivalent to $d \leqslant f$, which means that $F = [d) = \varphi_d^{-1}(1)$ and $\ker(\varphi_d) = \theta$. Therefore φ_d is a strong endomorphism of the Brouwerian semilattice L such that $\ker(\varphi_d) = \theta$. □

The signature of the universal algebra changes the notion of strong endomorphisms. Let us give a simple example. Let $L = \{0, a, b, 1\}$ be such that $0 < a, b < 1$ and a, b are incomparable. We can consider it as the meet-semilattice $L_{\mathrm{ms}} = (L, \wedge)$ and as the Brouwerian semilattice $L_{\mathrm{bs}} = (L, \wedge, *, 1)$.

The meet-semilattice L_{ms} has 7 congruences—trivial, universal and congruences (written as partitions) $c_1 = \{\{0, a\}, \{b, 1\}\}$, $c_2 = \{\{0, b\}, \{a, 1\}\}$, $c_3 = \{\{0, a\}, \{b\}, \{1\}\}$, $c_4 = \{\{0, b\}, \{a\}, \{1\}\}$, $c_5 = \{\{0, a, b\}, \{1\}\}$. Let $\varphi\colon L \to L$ be a map which maps $0, a \mapsto 0$; $b, 1 \mapsto b$. The map φ is an endomorphism of L_{ms} and by a routine check, it is a strong endomorphism of L_{ms}. That means that as a map, φ preserves also all congruences of the Brouwerian semilattice L_{bs} (trivial, universal, c_1, c_2), but it is not an endomorphism of L_{bs}.

Let $(L, \vee, \wedge, *, 1)$ be a Brouwerian algebra. Then the join operation $\vee$ is fully determined by $\wedge$, but in general there is no term in the language of $\wedge$, $*$ defining $\vee$. But Katriňák in [20] proved that if $(L, \vee, \wedge, *, 1)$ is a relative Stone algebra, than the join operation $\vee$ is defined by the formula

$$x \vee y = (x * y) * y \wedge (y * x) * x. \tag{3.10}$$

This fact allows us to formulate:

Corollary 3.11. *Let L be a nontrivial finite relative Stone algebra. Then it has SEKP.*

Proof. Using results and denotation from Lemma 3.9 we know that φ_d is an endomorphism preserving operations $*$ and $\wedge$ and from formula (3.10) we see that

$$(\forall\, a, b \in L) \quad \varphi_d(a \vee b) = \varphi_d(a) \vee \varphi_d(b),$$

so that φ_d is the endomorphism of the relative Stone algebra L which is strong and has $[d)$ as its kernel. □

Corollary 3.11 was proved also in [12], but using a complicated induction via the chain of equational classes of relative Stone algebras, and the proof did not give the description of needed strong endomorphisms.

The ternary term $\mu(x, y, z) = (x * y) * z \wedge (z * y) * x \wedge (z * x) * x$ satisfies $\mu(x, z, z) = \mu(x, y, x) = \mu(z, z, x) = x$ (see [22]) and therefore the variety of Brouwerian semilattices is arithmetical, and therefore for a Brouwerian semilattice, $\mathrm{Con}(L)$ is a distributive lattice. Also every Brouwerian algebra L (having a lattice as a reduct) has $\mathrm{Con}(L)$ distributive. This means that Brouwerian semilattices and also Brouwerian algebras have factorable congruences. By [12], Corollary 3.2 the direct sum of a family of finite Brouwerian semilattices (or of a family of finite relative

Stone algebras) has SEKP, so that we have nontrivial infinite examples of Brouwerian semilattices and Brouwerian algebras which have SEKP (see [12] for details).

It is known that a subdirectly irreducible Brouwerian algebra L which is not a relative Stone is a Brouwerian algebra, which is not a chain and it has one coatom (which is the largest element of $L \setminus \{1\}$). It means that finite subdirectly irreducible Brouwerian not relative Stone algebra L has a coatom and the largest join reducible element; we shall call this element *critical* element of L. It is proved in [12] that a finite subdirectly irreducible Brouwerian algebra which is not relative Stone does not have EKP (which means it does not have SEKP).

Lemma 3.12. *Let L be a finite Brouwerian algebra which is not relative Stone. Then*

(1) *there exist $\overline{d} \in L$ such that*
  (a) $\varphi_{\overline{d}} \notin \mathrm{End}(L)$,
  (b) $\mathrm{Im}(\varphi_{\overline{d}})$ *is not a subalgebra of L,*

(2) *the operation $\vee$ of L is not a term function in functional symbols $\wedge$, $*$, 1.*

P r o o f. (1a): Let L be a subdirect product of subdirectly irreducible algebras $L_1, \ldots, L_n$. At least one of $L_1, \ldots, L_n$ is not relative Stone; we shall assume that L_1 is not relative Stone and that L is a subalgebra of $L_1 \times \ldots \times L_n$. Let d be a critical element of L_1. There are elements $a, b \in L_1$, $a \neq b$ such that d covers a and b. We have $a \vee b = d$ and using formula (3.3) we see that $d * a = a$ and $d * b = b$. L is a subdirect product and therefore there are elements $a_2, b_2 \in L_2, \ldots, a_n, b_n \in L_n$ such that $\overline{a} = (a, a_2, \ldots, a_n)$ and $\overline{b} = (b, b_2, \ldots, b_n)$ are elements of L.

Denote $\overline{d} = \overline{a} \vee \overline{b}$. Then $\overline{d} \in L$. We have $\varphi_{\overline{d}}(\overline{d}) = \mathbf{1} = (1, \ldots, 1)$ ($\mathbf{1}$ denotes the top element of L here). Let $\pi_1 \colon L \to L_1$ be the projection homomorphism onto the first coordinate. We have

$$\pi_1(\varphi_{\overline{d}}(\overline{a} \vee \overline{b})) = \pi_1(\varphi_{\overline{d}}(\overline{d})) = 1.$$

But clearly,

$$\pi_1(\varphi_{\overline{d}}(\overline{a}) \vee \varphi_{\overline{d}}(\overline{b})) = \pi_1(\varphi_{\overline{d}}(\overline{a})) \vee \pi_1(\varphi_{\overline{d}}(\overline{b})) = d * a \vee d * b = a \vee b = d, \tag{3.11}$$

which means that $\varphi_{\overline{d}}$ does not preserve joins and it is not an endomorphism of L.

(1b): Let d be a critical element of L_1, and $\overline{d}$ be as defined within the proof of (1a). Let $x \in L_1$, $x \geqslant d$. Then $d * x = 1 \neq d$. Let $x \in L_1$, $x < d$. Then there is an element $a \in L_1$ which is covered by d and $x \leqslant a$. We have $d * a = a$, thus $d * x \leqslant d * a = a \neq d$ by (3.9). It means $d \notin \mathrm{Im}(\varphi_d)$ and for any $c_2, \ldots, c_n$, $(d, c_2, \ldots, c_n) \notin \mathrm{Im}(\varphi_{\overline{d}})$. Together with (3.11) this means that $\mathrm{Im}(\varphi_{\overline{d}})$ is not a subalgebra of L.

(2): The map $\varphi_{\overline{d}}$ preserves operations $\wedge$, $*$ and also 1 due to Lemma 3.9. As $\varphi_{\overline{d}}$ does not preserve the operation $\vee$, there does not exist a term function $t(x, y)$ in functional symbols $\wedge$, $*$, 1 such that $x \vee y = t(x, y)$ for all $x, y \in L$. □

So, the set of greatest elements of blocks of the congruence $\theta_{[\overline{d})}$ is not the subalgebra of L. We shall finish this section with two examples.

Let L be a Brouwerian (not relative Stone) algebra as given in Figure 1.

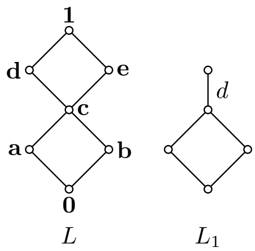

Figure 1.

Algebra L is a subdirect product of L_1^2 with $\mathbf{c} = (d, d)$, $\mathbf{d} = (1, d)$, $\mathbf{e} = (d, 1)$. Maps $\varphi_{\mathbf{x}}$ are (strong) endomorphisms of a Brouwerian algebra L for $\mathbf{x} \in L \setminus \{\mathbf{c}, \mathbf{d}, \mathbf{e}\}$. As we know by Lemma 3.12 (or by an easy check), $\varphi_{\mathbf{c}}$ is not an endomorphism, but $\varphi_{\mathbf{d}}$ and $\varphi_{\mathbf{e}}$ are not endomorphisms of L as well, for example, $\varphi_{\mathbf{e}}$ maps $\mathbf{a} \mapsto \mathbf{a}$; $\mathbf{b} \mapsto \mathbf{b}$ and $\mathbf{c} \mapsto \mathbf{d}$, hence $\varphi_{\mathbf{e}}$ does not preserve joins. It is easy to check that the map φ which maps $\mathbf{0} \mapsto \mathbf{c}$; $\mathbf{a} \mapsto \mathbf{d}$; $\mathbf{b} \mapsto \mathbf{e}$; $\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{1} \mapsto \mathbf{1}$ is a strong endomorphism of L with $\ker(\varphi) = \theta_{[\mathbf{c})}$. But for example $L/\theta_{[\mathbf{e})}$ is in fact isomorphic to a subalgebra $\{\mathbf{0}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{1}\}$ (and that is the only subalgebra of L to which it is isomorphic) and there is only one endomorphism φ (which maps $\mathbf{0} \mapsto \mathbf{0}$; $\mathbf{a} \mapsto \mathbf{a}$; $\mathbf{b} \mapsto \mathbf{b}$; $\mathbf{c}, \mathbf{d} \mapsto \mathbf{c}$; $\mathbf{e}, \mathbf{1} \mapsto \mathbf{1}$) whith $\theta_{[\mathbf{e})} = \ker(\varphi)$, but this endomorphism is not strong, as $\varphi(\mathbf{d}) = \mathbf{c}$, therefore φ does not satisfy condition (iii) of Theorem 3.5. It means that L does not have SEKP, but it has EKP.

Let L be a Brouwerian (not relative Stone) algebra as given in Figure 2.

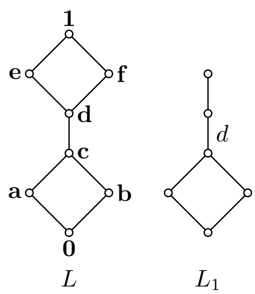

Figure 2.

Algebra L is a subdirect product of L_1^2 with $\mathbf{c} = (d, d)$. Maps $\varphi_{\mathbf{x}}$ are (strong) endomorphisms of a Brouwerian algebra L for $\mathbf{x} \in L \setminus \{\mathbf{c}\}$. As we know by Lemma 3.12 (or by an easy check), $\varphi_{\mathbf{c}}$ is not an endomorphism of L, but it is easy to check that the map φ which maps $\mathbf{0} \mapsto \mathbf{d}$; $\mathbf{a} \mapsto \mathbf{e}$; $\mathbf{b} \mapsto \mathbf{f}$; $\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{1} \mapsto \mathbf{1}$ is a strong endomorphism of L with $\ker(\varphi) = \theta_{[\mathbf{c})}$. Thus, L has SEKP.

4. Dual generalized Boolean algebras

We know that all finite relative Stone algebras have SEKP and we also know that there are infinite relative Stone algebras which possess this property, but that is not the full characterization of relative Stone algebras with SEKP. The purpose of this section is to give a full characterization of relative Stone algebras within one subvariety of the variety of all relative Stone algebras.

From the point of view of Brouwerian algebras, DGBA's are relative Stone algebras which form a subvariety S_2 of the variety of relative Stone algebras generated by a Brouwerian algebra $\{0, 1\}$ (DGBA's are relative Stone algebras which satisfy the identity $x * y \vee y * z = 1$), for more details see [19] and [17]. For example, the full duality for the category of DGBA's is described in [4].

As it is shown in [19], a distributive lattice with a top element 1 in which every principal filter $[a)$ is a Boolean lattice (a distributive lattice in which all elements have complements) is a Brouwerian algebra. We shall use the following definition:

Definition 4.1. A Brouwerian algebra L is a *dual generalized Boolean algebra* (DGBA for short) if its reduct $(L, \vee, \wedge)$ is a distributive lattice in which every principal filter $[a)$ is a Boolean lattice.

The great importance of DGBA's lies in the fact that for a DGBA L (more precisely for a Brouwerian algebra $(L, \vee, \wedge, *, 1)$ which is DGBA), congruences of $(L, \vee, \wedge, *, 1)$ and congruences of its lattice reduct $(L, \vee, \wedge)$ are the same, as stated in the following theorem (this is a dual form of the original result by Hashimoto [16], see also [10], Theorems 145, 146). This property of DGBA's is most important also for our purposes.

Theorem 4.2. *Let* $(L, \vee, \wedge, *, 1)$ *be a DGBA. Then* $\mathrm{Con}((L, \vee, \wedge, *, 1)) = \mathrm{Con}((L, \vee, \wedge))$ *and every congruence of* $\mathrm{Con}((L, \vee, \wedge))$ *is of the form* θ_F *for a filter* F *of* L*. Let* $(L', \vee, \wedge, 1)$ *be a lattice such that all its congruences are of the form* θ_F *for some filter* F *of* L'*. Then all principal filters of* L' *are Boolean lattices.*

We have the obvious corollary.

Corollary 4.3. *Let a Brouwerian algebra* $(L, \vee, \wedge, *, 1)$ *be a DGBA. If it has SEKP, then the distributive lattice with top* $(L, \vee, \wedge, 1)$ *has SEKP.*

Unbounded distributive lattices with a top element which have SEKP were characterized in [13] as follows:

Theorem 4.4 ([13], Theorem 2.7)**.** *Let* L *be an unbounded distributive lattice with a top element (which means that the top element must be preserved by homomorphisms/endomorphisms). Then* L *has SEKP if and only if it is isomorphic to the lattice of all cofinite subsets of a set* Z*.*

Let Z be a set. Denote $\mathrm{Cof}(Z) = \{X \subset Z \colon Z \setminus X \text{ is finite}\}$. Clearly, $(\mathrm{Cof}(Z), \cup, \cap, Z)$ is an (unbounded) distributive lattice with top element Z. The proof of Theorem 4.4 does not assure that strong endomorphisms which witness the SEKP for the lattice $(\mathrm{Cof}(Z), \cup, \cap, Z)$ of a set Z are (could be choosen as) Brouwerian algebra endomorphisms (i.e., that also the operation $*$ is preserved).

It is easy to see that the lattice of all cofinite subsets of a set Z is locally finite, i.e., all principal filters are finite (Boolean) lattices. Another way how to describe such lattices is using the notion of a direct sum of algebras with distinguished elements—we have already mentioned this construction after Corollary 3.11, but we will be more specific in this case. Clearly, a lattice of all cofinite subsets of a set Z is isomorphic to

$$\sum_{i \in Z} ((\{0,1\}, \vee, \wedge, 1), 1) = \\ (\{f \colon Z \to \{0,1\};\ f(i) = 1 \text{ for all but finitely many indices } i\}, \vee, \wedge, 1),$$

a sublattice of a lattice $(\{0,1\}^Z, \vee, \wedge, 1)$ with top element. $\sum_{i \in Z} ((\{0,1\}, \vee, \wedge, 1), 1)$ is called a direct sum of Z copies of $(\{0,1\}, \vee, \wedge, 1)$ with distinguished elements 1. By considering $\{0,1\}$ as a Brouwerian algebra $(\{0,1\}, \vee, \wedge, *, 1)$, the direct sum $\sum_{i \in Z} ((\{0,1\}, \vee, \wedge, *, 1), 1)$ is a Brouwerian algebra, in fact a DGBA, as well.

As we have already mentioned, any direct sum of finite relative Stone algebras, it means also $\sum_{i \in Z} ((\{0,1\}, \vee, \wedge, *, 1), 1)$, has SEKP by [12], Corollary 3.2.

Combining previous results we get the following theorem and corollary.

Theorem 4.5. *Let a Brouwerian algebra* $(L, \vee, \wedge, *, 1)$ *be a DGBA. The following statements are equivalent:*

(i) *L has SEKP.*
(ii) *L is isomorphic to the lattice* $(\mathrm{Cof}(Z), \cup, \cap, Z)$ *of a set* Z *(considered as a Brouwerian algebra).*
(iii) *L is isomorphic to the direct sum* $\sum_{i \in Z} ((\{0,1\}, \vee, \wedge, *, 1), 1)$ *for a set* Z*.*
(iv) *The sublattice* $([a), \vee, \wedge)$ *is a finite Boolean lattice for all* $a \in L$*.*

Corollary 4.6. *A Boolean algebra considered as a Brouwerian algebra (a DBGA which has bottom element) has SEKP if and only if it is finite.*

We can also prove an interesting feature of congruences of lattices which fulfill one of the conditions of Theorem 4.5.

Corollary 4.7. *Let L be $(\mathrm{Cof}(Z), \cup, \cap, Z)$ for a set Z. Then all congruences of L are comonomial, it means for any $\theta \in \mathrm{Con}(L)$, every block $[x]\theta$ has the largest element.*

P r o o f. Let $\theta \in \mathrm{Con}(L)$. L is also a Brouwerian algebra and it has SEKP as a Brouwerian algebra by Theorem 4.5. Let φ be a strong endomorphism of a Brouwerian algebra L such that $\ker(\varphi) = \theta$. By Theorem 4.2, φ is a strong endomorphism of a distributive lattice L, as well, and thus by [24], Lemma 2.2, φ is idempotent, $\varphi \circ \varphi = \varphi$. As $\varphi(x) = \varphi(\varphi(x))$, we see that $(x, \varphi(x)) \in \ker(\varphi) = \theta$, or $\varphi(x) \in [x]\theta$. We shall prove that $\varphi(x)$ is the largest element of a block $[x]\theta$. By Theorem 3.5 (iii) we know that $x \leqslant \varphi(x)$ for all $x \in L$. Now, let $y \in [x]\theta$. This means that $\varphi(y) = \varphi(x)$ and we also know that $y \leqslant \varphi(y)$, hence $y \leqslant \varphi(x)$. Therefore $\varphi(x)$ is the largest element of $[x]\theta$, as required. □

This means that congruences of unbounded distributive lattices with a top element which have SEKP (all of those are DGBA's) behave also in a general case similarly to a finite case for relative Stone algebras and also that strong endomorphisms are determined uniquely, just as in a finite case for relative Stone algebras.

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