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OSCILLATION OF SECOND-ORDER QUASILINEAR RETARDED DIFFERENCE EQUATIONS VIA CANONICAL TRANSFORM

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Abstract. We study the oscillatory behavior of the second-order quasi-linear retarded difference equation

$$\Delta(p(n)(\Delta y(n))^{\alpha}) + \eta(n)y^{\beta}(n-k) = 0$$

under the condition $\sum_{n=n_0}^{\infty} p^{-1/\alpha}(n) < \infty$ (i.e., the noncanonical form). Unlike most existing results, the oscillatory behavior of this equation is attained by transforming it into an equation in the canonical form. Examples are provided to show the importance of our main results.

Keywords: quasi-linear; difference equation; retarded; second-order; oscillation

MSC 2020: 39A10, 39A21

1. Introduction

This paper deals with the oscillatory behavior of solutions of the second-order quasi-linear retarded difference equation of the form

$$\Delta(p(n)(\Delta y(n))^{\alpha}) + \eta(n)y^{\beta}(n-k) = 0, \quad n \geqslant n_0, \tag{1.1}$$

where n_0 is a positive integer, and

(H_1) $\{p(n)\}$ and $\{\eta(n)\}$ are positive real sequences;
(H_2) k is a positive integer;
(H_3) $\alpha \geqslant 1$ and β are ratios of odd positive integers such that $\beta > \alpha - 1$.

The solution of (1.1) is a real sequence $\{y(n)\}$ that is defined for $n \geqslant n_0 - k$ and satisfies (1.1) for all $n \geqslant n_0$. A nontrivial solution of (1.1) is called *oscillatory*

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if it is neither eventually positive nor eventually negative; otherwise we say it is *nonoscillatory.* The equation (1.1) is said to be oscillatory if all its solutions are oscillatory. It follows from [15], that (1.1) is in the canonical form if

$$A(n) = \sum_{n=n_0}^{n-1} p^{-1/\alpha}(n) \to \infty \quad \text{as } n \to \infty \tag{1.2}$$

and it is in the noncanonical form if

$$B(n_0) = \sum_{n=n_0}^{\infty} p^{-1/\alpha}(n) < \infty. \tag{1.3}$$

It is well-known from oscillation theory that there is a significant difference in the structure of nonoscillatory (say positive) solutions between canonical and noncanonical equations. For any positive solution $\{y(n)\}$ of (1.1), it is easy to see that the first difference $\{\Delta y(n)\}$ is eventually of one sign, while the condition (1.2) ensures that this solution is eventually increasing ($\Delta y(n) > 0$). Most often, (1.1) has been studied when it is in the canonical form, see for example [1], [2], [4], [7]–[11], [13], [16] and the references cited therein. For noncanonical equations, both signs of the first difference $\{\Delta y(n)\}$ of any positive solutions $\{y(n)\}$ are possible and have to be dealt with. An approach most often in the literature for investigating such equations is to extend known results for canonical equations, see for example [3], [5], [6], [12] and the references cited therein.

Our main aim here is to study oscillatory properties of (1.1) by first transforming the noncanonical equation (1.1) into the canonical form. This approach significantly simplifies the investigation of the oscillatory behaviour of (1.1). To our best knowledge, there are no oscillation results in the literature using this method on quasi-linear difference equations. We provide some examples to demonstrate the importance of our main results. As a convenience and without loss of generality, we only deal in our proofs with positive solutions of (1.1). All functional inequalities are assumed to hold eventually, that is, for n large enough.

2. Oscillation results

Throughout the rest of the paper assume that the condition (1.3) holds. For convenience we use the following notations without further mention:

$$E(n) = p^{1/\alpha}(n)B(n)B(n+1), \quad F(n) = \sum_{s=n_1}^{n-1} 1/E(s),$$

$$\gamma = 1 + \beta - \alpha, \quad Q(n) = B(n+1)B^{\alpha-1}(n)B^{\gamma}(n-k)\eta(n)/\alpha$$

for every integer $n_1 \geqslant n_0$.

To prove our main results, we use the following lemmas.

Lemma 2.1. *Let $\{y(n)\}$ be an eventually positive solution of* (1.1). *Then one of the following two cases holds for all sufficiently large n:*

(I) $y(n) > 0$, $p(n)(\Delta y(n))^\alpha > 0$, *and* $\Delta(p(n)(\Delta y(n))^\alpha) < 0$;
(II) $y(n) > 0$, $p(n)(\Delta y(n))^\alpha < 0$, *and* $\Delta(p(n)(\Delta y(n))^\alpha) < 0$.

Proof. The proof is similar to that of Lemma 2.1 in [6] and so the details are omitted. □

The next lemma play a key role in the proof of our main results.

Lemma 2.2. *Assume that $\{y(n)\}$ is an eventually positive solution of* (1.1). *Then*

$$\Big(\frac{p^{1/\alpha}(n)\Delta y(n)}{y(n-k)}\Big)^{\alpha-1} \leqslant B^{1-\alpha}(n). \tag{2.1}$$

Proof. Let $\{y(n)\}$ be an eventually positive solution of (1.1). Then by Lemma 2.1, we see that $\Delta y(n) > 0$ or $\Delta y(n) < 0$, say for $n \geqslant n_1 \geqslant n_0$.

First assume that $\Delta y(n) < 0$ for all $n \geqslant n_1$. Since $p^{1/\alpha}(n)\Delta y(n)$ is decreasing (see (1.1))

$$y(n-k) \geqslant y(n) \geqslant \sum_{s=n}^{\infty} \frac{1}{p^{1/\alpha}(s)}(-p^{1/\alpha}(s)\Delta y(s)) \geqslant -B(n)p^{1/\alpha}(n)\Delta y(n) \geqslant 0. \tag{2.2}$$

Now (2.1) immediately follows from (2.2) and the fact that $\alpha \geqslant 1$ is the quotient of odd positive integers.

Next assume that $\Delta y(n) \geqslant 0$. First note that $A(n-k) + B(n-k) = B(n_0) > 0$. This, together with (1.3), implies $A(n-k) \geqslant B(n)$ for large n, say for $n \geqslant n_1$ for some $n_1 \geqslant n_0$. Then

$$y(n-k) \geqslant \sum_{s=n_0}^{n-k-1} \frac{1}{p^{1/\alpha}(s)} p^{1/\alpha}(s)\Delta y(s) \geqslant A(n-k)p^{1/\alpha}(n)\Delta y(n) \geqslant B(n)p^{1/\alpha}(n)\Delta y(n),$$

which is clearly equivalent to (2.1). The proof of the lemma is complete. □

Theorem 2.3. *Assume that the difference equation*

$$\Delta(E(n)\Delta u(n)) + Q(n)u^\gamma(n-k) = 0 \tag{2.3}$$

is oscillatory. Then (1.1) *is oscillatory.*

Proof. Assume that $\{y(n)\}$ is an eventually positive solution of (1.1). By the mean-value theorem (see [9]), it is easy to see that

$$\Delta(p(n)(\Delta y(n))^{\alpha}) \geqslant \alpha(p^{1/\alpha}(n)\Delta y(n))^{\alpha-1}\Delta(p^{1/\alpha}(n)\Delta y(n))$$

or

$$-\eta(n)y^{\beta}(n-k) \geqslant \alpha(p^{1/\alpha}(n)\Delta y(n))^{\alpha-1}\Delta(p^{1/\alpha}(n)\Delta y(n)).$$

This implies

$$\Delta(p^{1/\alpha}(n)\Delta y(n)) + \frac{1}{\alpha}(p^{1/\alpha}(n)\Delta y(n))^{\alpha-1}\eta(n)y^{\beta}(n-k) \leqslant 0. \tag{2.4}$$

Combining (2.1) and (2.4), we obtain

$$\Delta(p^{1/\alpha}(n)\Delta y(n)) + \frac{1}{\alpha}B^{\alpha-1}(n)\eta(n)y^{1+\beta-x}(n-k) \leqslant 0, \quad n \geqslant n_1 \geqslant n_0. \tag{2.5}$$

Using a method similar to the one used in the proof of Lemma 2.1 in [5], the inequality (2.5) can be rewritten in the equivalent canonical form as

$$\frac{1}{B(n+1)}\Delta\Big(p^{1/\alpha}(n)B(n)B(n+1)\Delta\Big(\frac{y(n)}{B(n)}\Big)\Big) + \frac{1}{\alpha}B^{\alpha-1}(n)\eta(n)y^{1+\beta-x}(n-k) \leqslant 0. \tag{2.6}$$

The oscillation preserving transformation $y(n) = B(n)u(n)$ reduces (2.6) to

$$\Delta(E(n)\Delta u(n)) + Q(n)u^{\gamma}(n-k) \leqslant 0. \tag{2.7}$$

But by Lemma 1 of [14], the corresponding equation (2.3) also has a positive solution. This contradiction proves the theorem. □

Lemma 2.4. *Let* $\{u(n)\}$ *be a positive solution of* (2.3)*. Then*

(i) $\{u(n)\}$ *is eventually increasing and* $E(n)\Delta u(n)$ *is eventually decreasing;*
(ii) $\{u(n)/F(n)\}$ *is eventually decreasing;*
(iii) $u(n) \geqslant F(n)E(n)\Delta u(n)$, $n \geqslant n_1$.

Proof. The proof is similar to that of Lemma 2.1 of [9] and hence omitted. □

Theorem 2.5. *Let* $\gamma > 1$*. If*

$$\sum_{n=n_0}^{\infty} \frac{F^{\gamma}(n-k)}{F^{\gamma-1}(n+1)}Q(n) = \infty \tag{2.8}$$

then (1.1) *is oscillatory.*

P r o o f. Assume that (1.1) is not oscillatory. Then by Theorem 2.3, we see that (2.3) is also not oscillatory. Let $\{u(n)\}$ be an eventually positive solution of (2.3); then there exists an integer $n_1 \geqslant n_0$ such that $u(n) > 0$ and $u(n-k) > 0$ for all $n \geqslant n_1$. By Lemma 2.4, one can see that $\{u(n)\}$ is increasing and $\{E(n)\Delta u(n)\}$ is positive and decreasing for all $n \geqslant n_2$ for some $n_2 \geqslant n_1$. Put

$$w(n) = \frac{F(n)E(n)\Delta u(n)}{u^\gamma(n)}, \quad n \geqslant n_2. \tag{2.9}$$

Clearly, $w(n) > 0$ and

$$\Delta w(n) = \frac{F(n+1)\Delta(E(n)\Delta u(n))}{u^\gamma(n+1)} + \frac{\Delta u(n)}{u^\gamma(n+1)} - \frac{F(n)E(n)\Delta u(n)}{u^\gamma(n)u^\gamma(n+1)}\Delta u^\gamma(n). \tag{2.10}$$

Since $u(n)$ is positive and increasing, we have

$$\Delta u^\gamma(n) = u^\gamma(n+1) - u^\gamma(n) > 0$$

and using this in (2.10) implies

$$\Delta w(n) \leqslant -\frac{F(n+1)Q(n)u^\gamma(n-k)}{u^\gamma(n+1)} + \frac{\Delta u(n)}{u^\gamma(n+1)}, \quad n \geqslant n_2. \tag{2.11}$$

From Lemma 2.4, we know that $u(n)/F(n)$ is eventually decreasing and hence for $n \geqslant n_3 \geqslant n_2$

$$\frac{u(n+1)}{F(n+1)} \leqslant \frac{u(n-k)}{F(n-k)}. \tag{2.12}$$

Combining (2.12) with (2.11) yields

$$\Delta w(n) \leqslant -\frac{F^\gamma(n-k)}{F^{\gamma-1}(n+1)}Q(n) + \frac{\Delta u(n)}{u^\gamma(n+1)}, \quad n \geqslant n_3. \tag{2.13}$$

Summing up (2.13) from n_3 to n, we obtain

$$\sum_{s=n_3}^{n} \frac{F^\gamma(s-k)}{F^{\gamma-1}(s+1)}Q(s) \leqslant w(n_3) + \sum_{s=n_3}^{n} \frac{\Delta u(s)}{u^\gamma(s+1)}. \tag{2.14}$$

Let $f(x) = u(n) + \Delta u(n)(x-n)$, $n \leqslant x \leqslant n+1$, $n \geqslant n_3$. Then $f(n) = u(n)$, $f(n+1) = u(n+1)$ and $f'(x) = \Delta u(n) > 0$, $n < x < n+1$, $n \geqslant n_3$. Thus, f is continuous and increasing for $x \geqslant n_3$. We then have

$$\begin{aligned} \frac{\Delta u(s)}{u^\gamma(s+1)} &= \int_s^{s+1} \frac{\Delta u(s)}{u^\gamma(s+1)}\,\mathrm{d}x = \int_s^{s+1} f^{-\gamma}(s+1)f'(x)\,\mathrm{d}x \\ &< \int_s^{s+1} f^{-\gamma}(x)f'(x)\,\mathrm{d}x = \frac{1}{1-\gamma}(f^{1-\gamma}(s+1) - f^{1-\gamma}(s)). \end{aligned}$$

This implies that

$$\sum_{s=n_3}^{n} \frac{\Delta u(s)}{u^{\gamma}(s+1)} \leqslant \frac{1}{1-\gamma}(f^{1-\gamma}(n+1) - f^{1-\gamma}(n_3)).$$

Since $\gamma > 1$ and f is an increasing function, it follows from (2.14) that

$$\sum_{s=n_3}^{n} \frac{F^{\gamma}(s-k)}{F^{\gamma-1}(s+1)} Q(s) \leqslant w(n_3) + \frac{F^{1-\gamma}(n_3)}{\gamma-1} < \infty,$$

which contradicts (2.8) as $n \to \infty$. The proof of the theorem is complete. □

Theorem 2.6. *Let* $0 < \gamma < 1$. *If*

$$\sum_{n=n_0}^{\infty} F^{\gamma}(n-k)Q(n) = \infty \tag{2.15}$$

then (1.1) *is oscillatory.*

P r o o f. Assume that (1.1) has a nonoscillatory solution; then by Theorem 2.3, equation (2.3) also has a nonoscillatory solution. Let $\{u(n)\}$ be an eventually positive solution of (2.3) and there exists an integer $n_1 \geqslant n_0$ such that $u(n) > 0$ and $u(n-k) > 0$ for all $n \geqslant n_1$. By Lemma 2.4, $u(n)$ is increasing and $E(n)\Delta u(n)$ is positive and decreasing for all $n \geqslant n_2$ for some $n_2 \geqslant n_1$. From Lemma 2.4 (iii), we have

$$u(n-k) \geqslant F(n-k)E(n-k)\Delta u(n-k) \geqslant F(n-k)E(n)\Delta u(n)$$

since $E(n)\Delta u(n)$ is decreasing. Using this in (2.3), dividing the resulting inequality by $(E(n)\Delta u(n))^{\gamma}$ and summing up from n_1 to n, we obtain

$$\sum_{s=n_1}^{n} \frac{\Delta(E(s)\Delta u(s))}{(E(s)\Delta u(s))^{\gamma}} + \sum_{s=n_1}^{n} F^{\gamma}(s-k)Q(s) \leqslant 0, \quad n \geqslant n_1$$

or

$$\begin{aligned}\sum_{s=n_1}^{n} F^{\gamma}(s-k)Q(s) &\leqslant -\sum_{s=n_1}^{n} \frac{\Delta(E(s)\Delta u(s))}{(E(s)\Delta u(s))^{\gamma}} \leqslant \sum_{s=n_1}^{n} \int_{E(s+1)\Delta u(s+1)}^{E(s)\Delta u(s)} \frac{1}{x^{\gamma}}\,\mathrm{d}x \\ &< \int_0^{E(n_1)\Delta u(n_1)} \frac{1}{x^{\gamma}}\,\mathrm{d}x = \frac{E(n_1)\Delta u(n_1)^{1-\gamma}}{1-\gamma} < \infty\end{aligned}$$

since $0 < \gamma < 1$, which contradicts (2.15) as $n \to \infty$. The proof of the theorem is complete. □

3. Examples

In this section, we present two examples to illustrate our main results.

Example 3.1. Consider the second-order quasilinear retarded difference equation

$$\Delta(n^3(n+1)^3(\Delta y_n)^3)+n^4y^5(n-3)=0,\quad n\geqslant 4. \tag{3.1}$$

Here $p(n)=n^3(n+1)^3$, $\alpha=3$, $\beta=5$, $k=3$ and $\eta(n)=n^4$. By a simple calculation, we obtain $B(n)=1/n$, $E(n)=1$, $F(n)=n$, $\gamma=3$ and $Q(n)=\frac{1}{3}n^2/(n+1)(n-3)^3$. The transformed canonical equation is

$$\Delta^2u(n)+\frac{n^2}{3(n+1)(n-3)^3}u^3(n-3)=0,\quad n\geqslant 4.$$

Now, the condition (2.8) becomes

$$\sum_{n=4}^{\infty}\frac{(n-3)^3}{3(n+1)^2}\frac{n^2}{(n+1)(n-3)^3}=\sum_{n=4}^{\infty}\frac{n^2}{3(n+1)^3}=\infty,$$

that is, the condition (2.8) is satisfied. Therefore by Theorem 2.5, the equation (3.1) is oscillatory.

Example 3.2. Consider the second-order nonlinear retarded difference equation

$$\Delta(n^3(n+1)^3(\Delta y(n))^3)+n(n+1)y^{7/3}(n-2)=0,\quad n\geqslant 3. \tag{3.2}$$

Here $p(n)=n^3(n+1)^3$, $\alpha=3$, $\beta=\frac{7}{3}$, $k=2$, and $\eta(n)=n(n+1)$. By a simple computation, we see that $B(n)=1/n$, $E(n)=1$, $F(n)=n$, $\gamma=\frac{1}{3}$ and $Q(n)=1/3n(n-2)^{1/3}$. The transformed canonical equation is

$$\Delta^2u(n)+\frac{1}{3n(n-2)^{1/3}}u^{1/3}(n-2)=0,\quad n\geqslant 3.$$

The condition (2.15) becomes

$$\sum_{n=3}^{\infty}\frac{(n-2)^{1/3}}{3n(n-2)^{1/3}}=\sum_{n=3}^{\infty}\frac{1}{3n}=\infty,$$

so by Theorem 2.6 the equation (3.2) is oscillatory.

4. Conclusion

The oscillatory behavior of solutions of the equation (1.1) for the case $\alpha = \beta$ is discussed in [12] and therefore in this paper we have studied oscillation of the equation (1.1) for the case $\alpha \neq \beta$. Thus the results of this paper are new and complementary to some existing results reported in the literature. Further note that our results are applicable only to delay difference equations.

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