IV. Linear boundary value problems for ordinary differential equations

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IV. *Linear boundary value problems for ordinary differential equations*

1. Preliminaries

This chapter is concerned with boundary value problems for linear nonhomogeneous vector ordinary differential equations

$$\mathbf{x}' - \mathbf{A}(t)\,\mathbf{x} = \mathbf{f}(t) \tag{1,1}$$

and the corresponding homogeneous equation

$$\mathbf{x}' - \mathbf{A}(t)\,\mathbf{x} = \mathbf{0}\,. \tag{1,2}$$

The differential equations (1,1) and (1,2) are considered in the sense of Carathéodory.

In the theory of ordinary differential equations the locution "*boundary value problems*" (BVP) refers to finding solutions to an ordinary differential equation which, in addition, satisfy some (additional) side conditions. In general, such conditions may require that the sought solution should belong to a prescribed set of functions. Very often this set is given as a set of solutions of a certain, generally nonlinear operator equation. In this chapter we restrict ourselves to the case of linear Stieltjes-integral side conditions of the form

$$\mathbf{S}\mathbf{x} = \mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{x}(t) = \mathbf{r} \tag{1,3}$$

or

$$\mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{x}(t) = \mathbf{0}\,. \tag{1,4}$$

Throughout the chapter the following hypotheses are kept to.

1.1. Assumptions. $\mathbf{A}$: $[0,1] \rightarrow L(R_n)$ *and* $\mathbf{f}$: $[0,1] \rightarrow R_n$ *are L-integrable on* $[0,1]$ $(\mathbf{f} \in L^1_n)$; $\mathbf{M}$ *and* $\mathbf{N} \in L(R_n, R_m)$, $\mathbf{r} \in R_m$, $m \geq 1$ *and* $\mathbf{K}$: $[0,1] \rightarrow L(R_n, R_m)$ *is of bounded variation on* $[0,1]$.

1.2. Definition. A function $\mathbf{x}\colon [0,1] \to R_n$ is a solution to the equation (1,1) on $[0,1]$ if it is absolutely continuous on $[0,1]$ ($\mathbf{x} \in AC_n$) and verifies $\mathbf{x}'(t) - \mathbf{A}(t)\,\mathbf{x}(t) = \mathbf{f}(t)$ a.e. on $[0,1]$.

1.3. Remark. Consequently $\mathbf{x}\colon [0,1] \to R_n$ is a solution to (1,1) on $[0,1]$ if and only if for any $t, t_0 \in [0,1]$

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{A}(s)\,\mathbf{x}(s)\,\mathrm{d}s + \int_{t_0}^{t} \mathbf{f}(s)\,\mathrm{d}s\,,$$

i.e. (1,1) is a special case of the linear generalized differential equation

$$(1,5) \qquad \mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{B}]\,\mathbf{x} + \mathrm{d}\mathbf{g} \qquad \left(\mathbf{B}(t) = \int_0^t \mathbf{A}(s)\,\mathrm{d}s\,, \quad \mathbf{g}(t) = \int_0^t \mathbf{f}(s)\,\mathrm{d}s\right).$$

1.4. Definition. A function $\mathbf{x}\colon [0,1] \to R_n$ is a solution to the *nonhomogeneous boundary value problem* (*BVP*) (1,1), (1,3) (verifies the system (1,1), (1,3)) if it is a solution of (1,1) on $[0,1]$ and satisfies (1,3). The problem of finding a solution $\mathbf{x}\colon [0,1] \to R_n$ of the homogeneous equation (1,2) on $[0,1]$ which fulfils also (1,4) is called the *homogeneous BVP* (1,2), (1,4).

1.5. Notation. Throughout the chapter $\mathbf{U}\colon [0,1] \times [0,1] \to L(R_n)$ is the fundamental matrix for the equation (1,5) defined by III.2.2 and $\mathbf{X}(t) = \mathbf{U}(t,0)$.

Let us recall that $\det \mathbf{X}(t) \neq 0$ on $[0,1]$, $\mathbf{U}(t,s) = \mathbf{X}(t)\,\mathbf{X}^{-1}(s)$ on $[0,1] \times [0,1]$,

$$(1,6) \qquad \mathbf{X}(t)\,\mathbf{X}^{-1}(s) = \mathbf{I} + \int_s^t \mathbf{A}(\tau)\,\mathbf{X}(\tau)\,\mathbf{X}^{-1}(s)\,\mathrm{d}\tau \qquad \text{for all} \quad t, s \in [0,1]$$

and

$$(1,7) \qquad \mathbf{X}(t)\,\mathbf{X}^{-1}(s) = \mathbf{I} + \int_s^t \mathbf{X}(t)\,\mathbf{X}^{-1}(\tau)\,\mathbf{A}(\tau)\,\mathrm{d}\tau \qquad \text{for all} \quad t, s \in [0,1]\,.$$

Both $\mathbf{X}(t)$ and $\mathbf{X}^{-1}(s)$ are absolutely continuous on $[0,1]$. The variation-of-constants formula reduces to

$$(1,8) \qquad \mathbf{x}(t) = \mathbf{U}(t,t_0)\,\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{U}(t,s)\,\mathbf{f}(s)\,\mathrm{d}s \qquad \text{for all} \quad t, t_0 \in [0,1]\,.$$

1.6. Remark. Since $\mathbf{A}(t)$ is supposed to be L-integrable on $[0,1]$, for any $\mathbf{x} \in AC_n$ the function $\mathbf{x}'(t) - \mathbf{A}(t)\,\mathbf{x}(t)$ is defined a.e. on $[0,1]$ and is L-integrable on $[0,1]$. Hence the operator

$$(1,9) \qquad \mathbf{L}\colon \mathbf{x} \in AC_n \to \mathbf{L}\mathbf{x}\,, \qquad (\mathbf{L}\mathbf{x})(t) = \mathbf{x}'(t) - \mathbf{A}(t)\,\mathbf{x}(t) \qquad \text{a.e. on } [0,1]$$

maps AC_n into L^1_n. Obviously it is linear and

$$\|\mathbf{L}\mathbf{x}\|_{L^1} = \int_0^1 |\mathbf{x}'(t) - \mathbf{A}(t)\,\mathbf{x}(t)|\,\mathrm{d}t \leq \int_0^1 |\mathbf{x}'(t)|\,\mathrm{d}t + \left(\int_0^1 |\mathbf{A}(t)|\,\mathrm{d}t\right) \sup_{t \in [0,1]} |\mathbf{x}(t)|$$

$$\leq \left(1 + \int_0^1 |\mathbf{A}(t)|\,\mathrm{d}t\right) \|\mathbf{x}\|_{AC}$$

for any $\mathbf{x} \in AC_n$. Moreover, for a given $\mathbf{x} \in C_n$

$$\left|\mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{x}(t)\right| \le (|\mathbf{M}| + |\mathbf{N}| + \mathrm{var}_0^1\,\mathbf{K})\,\|\mathbf{x}\|_C$$

and the operator

$$\text{(1,10)} \qquad \mathbf{S}\colon \mathbf{x} \in C_n \to \mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{x}(t) \in R_m$$

is linear and bounded. Consequently, under the assumptions 1.1

$$\mathscr{L}\colon \mathbf{x} \in AC_n \to \begin{bmatrix} \mathbf{Lx} \\ \mathbf{Sx} \end{bmatrix} \in L_n^1 \times R_m$$

is linear and bounded. The given BVP (1,1), (1,3) may be now rewritten as the linear operator equation

$$\mathscr{L}\mathbf{x} = \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix}.$$

1.7. Proposition. *Given* $\mathbf{c} \in R_n$ *and* $\mathbf{f} \in L_n^1$, *the unique solution* $\mathbf{x}$ *to* (1,1) *on* [0, 1] *such that* $\mathbf{x}(0) = \mathbf{c}$ *can be expressed in the form*

$$\mathbf{x}(t) = (\mathbf{Uc})(t) + (\mathbf{Vf})(t) \qquad \text{on } [0,1],$$

where

$$\text{(1,11)} \qquad \mathbf{U}\colon \mathbf{c} \in R_n \to \mathbf{X}(t)\,\mathbf{c} \in AC_n$$

and

$$\text{(1,12)} \qquad \mathbf{V}\colon \mathbf{f} \in L_n^1 \to \mathbf{X}(t) \int_0^t \mathbf{X}^{-1}(s)\,\mathbf{f}(s)\,\mathrm{d}s \in AC_n$$

are linear and bounded operators.

Proof. The linearity is obvious. Let $\mathbf{c} \in R_n$ and $\mathbf{f} \in L_n^1$. Then

$$\|\mathbf{Uc}\|_{AC} \le \left(1 + \int_0^1 |\mathbf{X}'(t)|\,\mathrm{d}t\right)|\mathbf{c}| = \varkappa_1 |\mathbf{c}|, \qquad \varkappa_1 < \infty$$

and

$$\|\mathbf{Vf}\|_{AC} = \int_0^1 \left|\mathbf{X}'(t)\left(\int_0^t \mathbf{X}^{-1}(s)\,\mathbf{f}(s)\,\mathrm{d}s\right) + \mathbf{f}(t)\right|\mathrm{d}t$$

$$\le \left[\left(\int_0^1 |\mathbf{X}'(t)|\,\mathrm{d}t\right)\left(\sup_{s\in[0,1]} |\mathbf{X}^{-1}(s)|\right) + 1\right]\|\mathbf{f}\|_{L^1} = \varkappa_2 \|\mathbf{f}\|_{L^1}, \qquad \varkappa_2 < \infty .$$

1.8. Remark. By the Riesz Representation Theorem an arbitrary linear bounded mapping $\mathbf{S}\colon C_n \to R_m$ may be expressed in the form (1,10), where $\mathbf{M} = \mathbf{N} = \mathbf{0}$.

If $\boldsymbol{K}(t)$ is the sum of a series of the simple jump functions of bounded variation on $[0,1]$ with the jumps $\Delta\boldsymbol{K}_j$ at $t = \tau_j \in [0,1]$ $(j = 1,2,\ldots)$, then (1,3) reduces to the *infinite point condition* (cf. I.4.23)

$$\boldsymbol{M}\,\boldsymbol{x}(0) + \sum_{j=1}^{\infty} \Delta\boldsymbol{K}_j\,\boldsymbol{x}(\tau_j) + \boldsymbol{N}\,\boldsymbol{x}(1) = \boldsymbol{r} \qquad \Big(\sum_{j=1}^{\infty} |\Delta\boldsymbol{K}_j| < \infty\Big).$$

In particular, if $\boldsymbol{K}(t)$ is a finite-step function on $[0,1]$ $(\boldsymbol{K}(t) = \boldsymbol{K}_j$ for $\tau_{j-1} \le t < \tau_j$ $(j = 1,2,\ldots,p-1)$, $\boldsymbol{K}(t) = \boldsymbol{K}_p$ for $\tau_{p-1} \le t \le 1$ where $0 = \tau_0 < \tau_1 < \ldots < \tau_p = 1)$, then (1,3) reduces to the *multipoint condition*

$$\boldsymbol{M}\,\boldsymbol{x}(0) + \sum_{j=1}^{p-1} \Delta\boldsymbol{K}_j\,\boldsymbol{x}(\tau_j) + \boldsymbol{N}\,\boldsymbol{x}(1) = \boldsymbol{r} \qquad (\Delta\boldsymbol{K}_j = \boldsymbol{K}_{j+1} - \boldsymbol{K}_j)$$

or even to the *two-point boundary conditions* (if $\Delta\boldsymbol{K}_j = \boldsymbol{0}$, $j = 1,2,\ldots,p-1$).

The problem of determining a function $\boldsymbol{x}\colon [0,1] \to R_n$ absolutely continuous on each subinterval (τ_j, τ_{j+1}) $(j = 0,1,\ldots,p-1,\ 0 = \tau_0 < \tau_1 < \ldots < \tau_p = 1)$ and such that $\boldsymbol{x}'(t) - \boldsymbol{A}(t)\,\boldsymbol{x}(t) = \boldsymbol{f}(t)$ a.e. on $[0,1]$ and

$$\boldsymbol{M}\,\boldsymbol{x}(0) + \sum_{j=0}^{p-1} [\boldsymbol{M}_j\,\boldsymbol{x}(\tau_j+) + \boldsymbol{N}_j\,\boldsymbol{x}(\tau_{j+1}-)] + \boldsymbol{N}\,\boldsymbol{x}(1) = \boldsymbol{r}$$

is called the *interface problem* and is to be dealt with separately.

1.9. Remark. If we put $\boldsymbol{K}_0(t) = \boldsymbol{K}(t+) - \boldsymbol{K}(1-)$ for $t \in [0,1)$ and $\boldsymbol{K}_0(1) = \boldsymbol{0}$, then $\boldsymbol{K} - \boldsymbol{K}_0$ is of bounded variation on $[0,1]$, $\Delta^+\boldsymbol{K}_0(t) = \boldsymbol{0}$ on $[0,1)$, $\boldsymbol{K}_0(1-) = \boldsymbol{K}_0(1) = \boldsymbol{0}$, $\boldsymbol{K}(t+) - \boldsymbol{K}_0(t+) = \boldsymbol{K}(t-) - \boldsymbol{K}_0(t-) = \boldsymbol{0}$ on $[0,1]$, $\boldsymbol{K}(1) - \boldsymbol{K}_0(1) = \boldsymbol{K}(1)$, $\boldsymbol{K}(0) - \boldsymbol{K}_0(0) = -\Delta^+\boldsymbol{K}(0) - \boldsymbol{K}(1-)$ and hence for any $\boldsymbol{x} \in C_n$ (cf. I.4.23 and I.5.5)

$$\boldsymbol{M}\,\boldsymbol{x}(0) + \boldsymbol{N}\,\boldsymbol{x}(1) + \int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\,\boldsymbol{x}(t) = \boldsymbol{M}_0\,\boldsymbol{x}(0) + \boldsymbol{N}_0\,\boldsymbol{x}(1) + \int_0^1 \mathrm{d}[\boldsymbol{K}_0(t)]\,\boldsymbol{x}(t)$$

$(\boldsymbol{M}_0 = \boldsymbol{M} - \Delta^+\boldsymbol{K}(0) - \boldsymbol{K}(1-), \qquad \boldsymbol{N}_0 = \boldsymbol{N} + \boldsymbol{K}(1).)$

Thus, without any loss of generality we may add the following hypotheses to 1.1.

1.10. Assumptions. $\boldsymbol{K}(t)$ is right-continuous on $[0,1)$, *left-continuous at* 1 *and* $\boldsymbol{K}(1) = \boldsymbol{0}$.

1.11. Definition. The side condition (1,3) $(\boldsymbol{S}\boldsymbol{x} = \boldsymbol{r})$ is *linearly dependent* if there exists $\boldsymbol{q} \in R_m$, $\boldsymbol{q} \neq \boldsymbol{0}$ such that $\boldsymbol{q}^*(\boldsymbol{S}\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in AC_n$. It is *linearly independent* if it is not linearly dependent.

1.12. Proposition. *Let* $\boldsymbol{M}, \boldsymbol{N}$ *and* $\boldsymbol{K}(t)$ *fulfil the hypotheses* 1.1 *and* 1.10. *Then the side condition* (1,3) *is linearly dependent if and only if there is* $\boldsymbol{q} \in R_m$, $\boldsymbol{q} \neq \boldsymbol{0}$ *such that*

$$\boldsymbol{q}^*\boldsymbol{M} = \boldsymbol{q}^*\boldsymbol{N} \equiv \boldsymbol{q}^*\,\boldsymbol{K}(t) \equiv \boldsymbol{0} \qquad \text{on } [0,1].$$

Proof. Let $\boldsymbol{q} \neq \boldsymbol{0}$ and let

$$(1,13) \qquad \boldsymbol{q}^*[\boldsymbol{M}\,\boldsymbol{x}(0) + \boldsymbol{N}\,\boldsymbol{x}(1) + \int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\,\boldsymbol{x}(t)] = 0 \qquad \text{for each} \quad \boldsymbol{x} \in AC_n\,.$$

Then for every $\boldsymbol{x} \in AC_n$ with $\boldsymbol{x}(0) = \boldsymbol{x}(1) = \boldsymbol{0}$ we have

$$\int_0^1 \mathrm{d}[\boldsymbol{q}^*\,\boldsymbol{K}(t)]\,\boldsymbol{x}(t) = 0\,.$$

By I.5.17 this implies $\boldsymbol{q}^*\,\boldsymbol{K}(t) = \boldsymbol{0}$ on $[0,1]$ and hence (1,13) reduces to

$$\boldsymbol{q}^*[\boldsymbol{Mc} + \boldsymbol{Nd}] = \boldsymbol{0} \qquad \text{for all} \quad \boldsymbol{c}, \boldsymbol{d} \in R_n\,.$$

Choosing $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{d} \in R_n$ arbitrary or $\boldsymbol{d} = \boldsymbol{0}$ and $\boldsymbol{c} \in R_n$ arbitrary, we obtain $\boldsymbol{q}^*\boldsymbol{N} = \boldsymbol{0}$ or $\boldsymbol{q}^*\boldsymbol{M} = \boldsymbol{0}$, respectively.

1.13. Definition. The side condition (1,3) is said to be *nonzero* if the corresponding operator $\boldsymbol{S}$ given by (1,10) is nonzero. Given $\boldsymbol{r} \in R_m$, the side condition (1,3) is *reasonable* if $\boldsymbol{q}^*\boldsymbol{r} = 0$ for any $\boldsymbol{q} \in R_m$ such that $\boldsymbol{q}^*(\boldsymbol{Sx}) = 0$ for all $\boldsymbol{x} \in AC_n$. (Obviously, given $\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in L_n^1 \times R_m$, BVP (1,1), (1,3) may be solvable only if the side condition (1,3) is reasonable.)

Given $\boldsymbol{x} \in AC_n$, let $S_j\boldsymbol{x}$ $(j = 1, 2, \dots, m)$ denote the components of the vector $\boldsymbol{Sx} \in R_m$. Then S_j: $\boldsymbol{x} \in AC_n \to S_j\boldsymbol{x} \in R$ are linear bounded functionals on AC_n and the side condition (1,3) may be rewritten as the system of equations $S_j\boldsymbol{x} = r_j$ $(j = 1, 2, \dots, m)$, where r_j are components of the vector $\boldsymbol{r}$. The side condition (1,3) is linearly dependent if and only if the functionals $S_j \in AC_n^*$ $(j = 1, 2, \dots, m)$ are linearly dependent. Since the linear subspace of AC_n spanned on $\{S_1, S_2, \dots, S_m\}$ is finite dimensional, the following assertion is obvious.

1.14. Proposition. *If the side condition (1,3) is nonzero and reasonable, then there exist a natural number l, matrices $\boldsymbol{M}_0, \boldsymbol{N}_0 \in L(R_n, R_l)$, $\boldsymbol{r}_0 \in R_l$ and a function $\boldsymbol{K}_0$: $[0,1] \to L(R_n, R_l)$ of bounded variation on $[0,1]$ such that the condition*

$$\boldsymbol{S}_0\boldsymbol{x} = \boldsymbol{M}_0\,\boldsymbol{x}(0) + \boldsymbol{N}_0\,\boldsymbol{x}(1) + \int_0^1 \mathrm{d}[\boldsymbol{K}_0(t)]\,\boldsymbol{x}(t) = \boldsymbol{r}_0$$

is linearly independent, while $\boldsymbol{Sx} = \boldsymbol{r}$ for $\boldsymbol{x} \in AC_n$ if and only if $\boldsymbol{S}_0\,\boldsymbol{x} = \boldsymbol{r}_0$.

Henceforth let us assume that the side condition (1,3) is reasonable, linearly independent and fulfils the hypotheses 1.1 and 1.10. Let us denote by p the dimension of the linear subspace spanned on the rows of $\boldsymbol{K}(t)$. If $0 < p < m$, then there exists a regular $m \times m$-matrix $\boldsymbol{\Sigma}_1$ such that

$$\boldsymbol{\Sigma}_1\,\boldsymbol{K}(t) \equiv \begin{bmatrix} \boldsymbol{0} \\ \tilde{\boldsymbol{K}}(t) \end{bmatrix} \qquad \text{on } [0,1]\,,$$

where the rows of $\tilde{\mathbf{K}}$: $[0,1] \to L(R_n, R_p)$ are linearly independent on $[0,1]$. Let us denote $m_0 = m - p$ and let the matrices $\mathbf{M}_0, \mathbf{N}_0 \in L(R_n, R_{m_0})$ and $\tilde{\mathbf{M}}, \tilde{\mathbf{N}} \in L(R_n, R_p)$ be such that

$$\Sigma_1[\mathbf{M}, \mathbf{N}, \mathbf{K}(t)] = \begin{bmatrix} \mathbf{M}_0, & \mathbf{N}_0, & \mathbf{0} \\ \tilde{\mathbf{M}}, & \tilde{\mathbf{N}}, & \tilde{\mathbf{K}}(t) \end{bmatrix}.$$

If there were $\boldsymbol{\alpha}^*[\mathbf{M}_0, \mathbf{N}_0] = \mathbf{0}$, then $\boldsymbol{\beta}^*\Sigma_1[\mathbf{M}, \mathbf{N}, \mathbf{K}(t)] \equiv \mathbf{0}$ or according to 1.12 $\boldsymbol{\beta}^*\Sigma_1 = \mathbf{0}$ should hold for $\boldsymbol{\beta}^* = (\boldsymbol{\alpha}^*, \mathbf{0}) \in R_m^*$. As Σ_1 is regular, $\boldsymbol{\beta}^*\Sigma_1 = \mathbf{0}$ implies $\boldsymbol{\beta}^* = \mathbf{0}$ and hence also $\boldsymbol{\alpha}^* = \mathbf{0}$. This means that the $m_0 \times 2n$-matrix $[\mathbf{M}_0, \mathbf{N}_0]$ has a full rank (rank $[\mathbf{M}_0, \mathbf{N}_0] = m_0$). If rank $[\mathbf{M}, \mathbf{N}] = m_0 + m_1$, i.e.

$$\operatorname{rank} \begin{bmatrix} \mathbf{M}_0, & \mathbf{N}_0 \\ \tilde{\mathbf{M}}, & \tilde{\mathbf{N}} \end{bmatrix} = m_0 + m_1 \qquad (0 \le m_1 \le p),$$

then there exists a regular $p \times p$-matrix Σ_2 such that

$$\begin{bmatrix} \mathbf{I}_{m_0}, & \mathbf{0} \\ \mathbf{0}, & \Sigma_2 \end{bmatrix} \begin{bmatrix} \mathbf{M}_0, & \mathbf{N}_0 \\ \tilde{\mathbf{M}}, & \tilde{\mathbf{N}} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_0, & \mathbf{N}_0 \\ \mathbf{M}_1, & \mathbf{N}_1 \\ \mathbf{0}, & \mathbf{0} \end{bmatrix},$$

where $\mathbf{M}_1, \mathbf{N}_1 \in L(R_n, R_{m_1})$ are such that

(1,15)
$$\operatorname{rank} \begin{bmatrix} \mathbf{M}_0, & \mathbf{N}_0 \\ \mathbf{M}_1, & \mathbf{N}_1 \end{bmatrix} = m_0 + m_1 .$$

Denoting

$$\Theta = \begin{bmatrix} \mathbf{I}_{m_0}, & \mathbf{0} \\ \mathbf{0}, & \Sigma_2 \end{bmatrix} \Sigma_1 ,$$

we obtain

(1,16)
$$\Theta[\mathbf{M}, \mathbf{N}, \mathbf{K}(t)] \equiv \begin{bmatrix} \mathbf{M}_0, & \mathbf{N}_0, & \mathbf{0} \\ \mathbf{M}_1, & \mathbf{N}_1, & \mathbf{K}_1(t) \\ \mathbf{0}, & \mathbf{0}, & \mathbf{K}_2(t) \end{bmatrix},$$

where $\mathbf{K}_1$: $[0,1] \to L(R_n, R_{m_1})$, $\mathbf{K}_2$: $[0,1] \to L(R_n, R_{m_2})$ $(m_1 + m_2 = p)$ are given by

$$\begin{bmatrix} \mathbf{K}_1(t) \\ \mathbf{K}_2(t) \end{bmatrix} \equiv \mathbf{K}^0(t) \equiv \Sigma_2 \tilde{\mathbf{K}}(t) \qquad \text{on } [0,1].$$

As Σ_2 is regular, the rows of the $p \times n$-matrix $\mathbf{K}^0(t)$ are linearly independent on $[0,1]$. Finally, let us notice that the $m \times m$-matrix Θ is regular. To summarize:

1.15. Theorem. *Any linearly independent and reasonable Stieltjes-integral side condition* (1,3) *fulfilling* 1.1 *and* 1.10 *is equivalent to the system*

(1,17)
$$\begin{aligned} \mathbf{M}_0\,\mathbf{x}(0) + \mathbf{N}_1\,\mathbf{x}(1) \qquad\qquad\qquad\qquad &= \mathbf{r}_0 , \\ \mathbf{M}_1\,\mathbf{x}(0) + \mathbf{N}_1\,\mathbf{x}(1) + \int_0^1 \mathrm{d}[\mathbf{K}_1(t)]\,\mathbf{x}(t) &= \mathbf{r}_1 , \\ \int_0^1 \mathrm{d}[\mathbf{K}_2(t)]\,\mathbf{x}(t) &= \mathbf{r}_2 , \end{aligned}$$

where $\mathbf{r}_0 \in R_{m_0}$, $\mathbf{r}_1 \in R_{m_1}$, $\mathbf{r}_2 \in R_{m_2}$ *and the* $m_0 \times n$*-matrices* $\mathbf{M}_0, \mathbf{N}_0$, *the* $m_1 \times n$*-matrices* $\mathbf{M}_1, \mathbf{N}_1, \mathbf{K}_1(t)$ *and the* $m_2 \times n$*-matrix* $\mathbf{K}_2(t)$ *are such that* (1,15) *holds and the rows of the* $(m_1 + m_2) \times n$*-matrix* $[\mathbf{K}_1^*(t), \mathbf{K}_2^*(t)]^*$ *are linearly independent on* $[0, 1]$. *There exists a regular* $m \times m$*-matrix* $\boldsymbol{\Theta}$ *such that* (1,16) *and* $\boldsymbol{\Theta}\mathbf{r} = (\mathbf{r}_0^*, \mathbf{r}_1^*, \mathbf{r}_2^*)^*$ *hold.*

1.16. Definition. The system (1,17) associated to (1,3) by 1.15 is said to be the *canonical form* of (1,3).

1.17. Remark. By I.5.16 the general form of the linear bounded operator $\mathbf{S}$: $AC_n \to R_m$ is

$$\mathbf{S}: \mathbf{x} \in AC_n \to \mathbf{M}\,\mathbf{x}(0) + \int_0^1 \mathbf{K}(t)\,\mathbf{x}'(t)\,\mathrm{d}t, \tag{1,18}$$

where $\mathbf{M} \in L(R_n, R_m)$ and $\mathbf{K}$: $[0, 1] \to L(R_n, R_m)$ is measurable and essentially bounded on $[0, 1]$. If $\mathbf{K}$ is of bounded variation on $[0, 1]$, then by integrating by parts we may easily reduce $\mathbf{S}$ to the form (1,10).

Most of the results given in this chapter may be extended to BVP with the side operator $\mathbf{S}$ of the form (1,18). Some of them are formulated and proved in the following chapter for more general BVP which include integro-differential equations, the rest is left to the reader.

2. Duality theory

Let us consider BVP (1,1), (1,3), i.e. the system

$$(1,1)\quad \mathbf{x}' - \mathbf{A}(t)\,\mathbf{x} = \mathbf{f}(t), \qquad (1,3)\quad \mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{x}(t) = \mathbf{r},$$

where $\mathbf{A}$: $[0, 1] \to L(R_n)$, $\mathbf{M}$ and $\mathbf{N} \in L(R_n, R_m)$ and $\mathbf{K}$: $[0, 1] \to L(R_n, R_m)$ fulfil 1.1 and 1.10. Moreover, we suppose that (1,3) is nonzero and reasonable (see 1.13).

Let $\mathbf{f} \in L_n^1$ and $\mathbf{r} \in R_m$. By the variation-of-constants formula 1.7 a function $\mathbf{x}$: $[0, 1] \to R_n$ is a solution to BVP (1,1), (1,3) if and only

$$\mathbf{x} = \mathbf{U}\mathbf{c} + \mathbf{V}\mathbf{f}$$

and

$$(\mathbf{S}\mathbf{U})\,\mathbf{c} = \mathbf{r} - (\mathbf{S}\mathbf{V})\,\mathbf{f}, \tag{2,1}$$

where $\mathbf{U}$: $R_n \to AC_n$ and $\mathbf{V}$: $L_n^1 \to AC_n$ are the linear bounded operators respectively given by (1,11) and (1,12),

$$\mathbf{S}\mathbf{U} = \mathbf{M}\,\mathbf{X}(0) + \mathbf{N}\,\mathbf{X}(1) + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)$$

and

$$(\mathbf{S}\mathbf{V})\,\mathbf{f} = \mathbf{N}\,\mathbf{X}(1)\int_0^1 \mathbf{X}^{-1}(t)\,\mathbf{f}(t)\,\mathrm{d}t + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\int_0^t \mathbf{X}^{-1}(s)\,\mathbf{f}(s)\,\mathrm{d}s. \tag{2,2}$$

This yields immediately the following necessary and sufficient condition for the existence of a solution to BVP (1,1), (1,3).

2.1. Theorem. *BVP* (1,1), (1,3) *has a solution if and only if*

$$\lambda^*(\mathbf{SU}) = \mathbf{0} \tag{2,3}$$

implies

$$\lambda^*(\mathbf{SV})\,\mathbf{f} = \lambda^*\mathbf{r}\,. \tag{2,4}$$

2.2. Remark. Applying the Dirichlet formula I.4.32 to (2,2) we obtain for any $\mathbf{f} \in L_n^1$

$$(\mathbf{SV})\,\mathbf{f} = \int_0^1 \mathbf{F}(t)\,\mathbf{f}(t)\,\mathrm{d}t\,, \tag{2,5}$$

where

$$\mathbf{F}(t) = \left(\mathbf{N}\,\mathbf{X}(1) + \int_t^1 \mathrm{d}[\mathbf{K}(s)]\,\mathbf{X}(s)\right)\mathbf{X}^{-1}(t) \qquad \text{on } [0,1]\,. \tag{2,6}$$

Hence the condition (2,4) may be rewritten as

$$\int_0^1 \lambda^*\,\mathbf{F}(t)\,\mathbf{f}(t)\,\mathrm{d}t = \lambda^*\mathbf{r}\,.$$

By (III. 4,8) the n-vector valued function $\mathbf{y}^*(t) = \lambda^*\,\mathbf{F}(t)$ is for any $\lambda^* \in R_m^*$ a unique solution of the initial value problem

$$\mathrm{d}\mathbf{y}^* = -\mathbf{y}^*\,\mathrm{d}[\mathbf{B}] - \mathrm{d}(\lambda^*\mathbf{K}) \quad \text{on } [0,1] \quad \left(\mathbf{B}(t) = \int_0^t \mathbf{A}(s)\,\mathrm{d}s\right), \qquad \mathbf{y}^*(1) = \lambda^*\mathbf{N}\,. \tag{2,7}$$

(In fact, if $\mathbf{h} \in BV_n$ is right-continuous on $[0,1)$ and left-continuous at 1, then integrating by parts (cf. I.4.33) we reduce the variation-of-constants formula for the initial value problem

$$\mathrm{d}\mathbf{y}^* = -\mathbf{y}^*\,\mathrm{d}[\mathbf{B}] - \mathrm{d}\mathbf{h}^*\,, \qquad \mathbf{y}^*(1) = \mathbf{y}_1^*$$

to the form

$$\mathbf{y}^*(t) = \left(\mathbf{y}_1^*\,\mathbf{X}(1) + \int_t^1 \mathrm{d}[\mathbf{h}^*(s)]\,\mathbf{X}(s)\right)\mathbf{X}^{-1}(t) \qquad \text{on } [0,1]\,.) \tag{2,7a}$$

Furthermore, if $\lambda^*(\mathbf{SU}) = \mathbf{0}$, then

$$\mathbf{y}^*(0) = \lambda^*\left(\mathbf{N}\,\mathbf{X}(1) + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{X}(t)\right) = -\lambda^*\mathbf{M}\,.$$

On the other hand, it follows from the variation-of-constants formula that if $\mathbf{y}^*\colon [0,1] \to R_n^*$ and $\lambda^* \in R_m^*$ solve (2,7) on $[0,1]$ and

$$\mathbf{y}^*(0) + \lambda^*\mathbf{M} = \mathbf{0}\,, \qquad \mathbf{y}^*(1) - \lambda^*\mathbf{N} = \mathbf{0}\,, \tag{2,8}$$

then

(2,9) $\quad \mathbf{y}^*(t) = \boldsymbol{\lambda}^* \mathbf{F}(t) \quad$ on $[0, 1]$ and $\boldsymbol{\lambda}^*(\mathbf{SU}) = -\mathbf{y}^*(0) + \mathbf{y}^*(0) = \mathbf{0}$.

This completes the proof of the following theorem.

2.3. Theorem. *BVP (1,1), (1,3) has a solution if and only if*

$$\int_0^1 \mathbf{y}^*(t)\, \mathbf{f}(t)\, \mathrm{d}t = \boldsymbol{\lambda}^* \mathbf{r} \tag{2,10}$$

for any solution $(\mathbf{y}^*, \boldsymbol{\lambda}^*)$ *of the system (2,7), (2,8).*

2.4. Definition. The system (2,7), (2,8) of equations for $\mathbf{y}^*\colon [0, 1] \to R_n^*$ and $\boldsymbol{\lambda}^* \in R_m^*$ is called the *adjoint boundary value problem to BVP* (1,1), (1,3) (or (1,2), (1,4)).

The following assertion provides the necessary and sufficient condition for the existence of a solution to the nonhomogeneous BVP corresponding to BVP (2,7), (2,8).

2.5. Theorem. *Let* $\mathbf{p}, \mathbf{q} \in R_n$ *and let* $\mathbf{g} \in BV_n$ *be right-continuous on* $[0, 1)$ *and left-continuous at* 1. *Then the system*

$$\mathrm{d}\mathbf{y}^* = \mathbf{y}^*\, \mathrm{d}[-\mathbf{B}] - \mathrm{d}(\boldsymbol{\lambda}^* \mathbf{K}) + \mathrm{d}\mathbf{g}^* \quad \text{on } [0, 1], \tag{2,11}$$

$$\mathbf{y}^*(0) + \boldsymbol{\lambda}^* \mathbf{M} = \mathbf{p}^*, \qquad \mathbf{y}^*(1) - \boldsymbol{\lambda}^* \mathbf{N} = \mathbf{q}^* \tag{2,12}$$

has a solution if and only if

$$\int_0^1 \mathrm{d}[\mathbf{g}^*(t)]\, \mathbf{x}(t) = \mathbf{q}^* \mathbf{x}(1) - \mathbf{p}^* \mathbf{x}(0)$$

for any solution $\mathbf{x}$ *of the homogeneous BVP (1,2), (1,4).*

Proof. Inserting (2,7a), where $\mathbf{h}^*(t) = \boldsymbol{\lambda}^* \mathbf{K}(t) - \mathbf{g}^*(t)$ into (2,12) we easily obtain that $\mathbf{y}^*\colon [0, 1] \to R_n^*$ and $\boldsymbol{\lambda}^* \in R_m^*$ verify (2,11), (2,12) if and only if

$$\mathbf{y}^*(t) = \boldsymbol{\lambda}^* \mathbf{F}(t) + \mathbf{q}^* \mathbf{X}(1)\, \mathbf{X}^{-1}(t) - \int_t^1 \mathrm{d}[\mathbf{g}^*(s)]\, \mathbf{X}(s)\, \mathbf{X}^{-1}(t) \quad \text{on } [0, 1] \tag{2,13}$$

($\mathbf{F}(t)$ given by (2,6)) and

$$\boldsymbol{\lambda}^*(\mathbf{SU}) = \mathbf{p}^* \mathbf{X}(0) - \mathbf{q}^* \mathbf{X}(1) + \int_0^1 \mathrm{d}[\mathbf{g}^*(t)]\, \mathbf{X}(t).$$

Since all the solutions of BVP (1,2), (1,4) are of the form $\mathbf{X}(t)\, \mathbf{c}$ where $(\mathbf{SU})\, \mathbf{c} = \mathbf{0}$, the theorem follows immediately.

2.6. Remark. Let us notice that under our assumptions all the solutions $\mathbf{y}^*\colon [0, 1] \to R_n$ of (2,11) on $[0, 1]$ are of bounded variation on $[0, 1]$, right-continuous on $[0, 1)$ and left-continuous at 1.

2.7. Theorem. *The homogeneous problems* (1,2), (1,4) *and* (2,7), (2,8) *possess exactly* $k = n - \operatorname{rank}(\boldsymbol{SU})$ *and* $k^* = m - \operatorname{rank}(\boldsymbol{SU})$ *linearly independent solutions, respectively.*

Proof. The homogeneous algebraic equation

$$(\boldsymbol{SU})\,\boldsymbol{c} = \boldsymbol{0} \tag{2,14}$$

has exactly $k = n - \operatorname{rank}(\boldsymbol{SU})$ linearly independent solutions. Let $\boldsymbol{C}_0$ be an arbitrary $n \times k$-matrix whose columns form a basis in the space of all solutions to (2,14). ($(\boldsymbol{SU})\,\boldsymbol{C}_0 = \boldsymbol{0}$ and $\operatorname{rank}(\boldsymbol{C}_0) = k$.) This obviously implies that the columns of the $n \times k$-matrix valued function

$$\boldsymbol{X}_0(t) = \boldsymbol{X}(t)\,\boldsymbol{C}_0 \quad \text{on } [0,1]$$

form a basis in the space of all solutions of BVP (1,2), (1,4).

The latter assertion follows from the fact that $\boldsymbol{y}^*\colon [0,1] \to R_n^*$ and $\boldsymbol{\lambda}^* \in R_m^*$ verify the system (2,7), (2,8) if and only if $\boldsymbol{y}^*(t) = \boldsymbol{\lambda}^* \boldsymbol{F}(t)$ on $[0,1]$ and (2,3) holds (cf. 2.3 and its proof). In fact, since (2,3) has exactly $k^* = m - \operatorname{rank}(\boldsymbol{SU})$ linearly independent solutions, BVP (2,7), (2,8) has also exactly k^* linearly independent solutions on $[0,1]$. In particular, given an arbitrary $\Lambda_0 \in L(R_n, R_{k^*})$ whose rows form a basis in the space of all solutions to (2,3), the rows of $(\Lambda_0\,\boldsymbol{F}(t), \Lambda_0)$ form a basis in the space of all solutions to BVP (2,7), (2,8).

2.8. Remark. From the proof of 2.7 it follows that all the solutions to BVP (1,2), (1,4) or BVP (2,7), (2,8) are of the form

$$\boldsymbol{x}(t) = \boldsymbol{X}_0(t)\,\boldsymbol{d}\,, \quad \boldsymbol{d} \in R_k \quad \text{or} \quad (\boldsymbol{y}^*(t), \boldsymbol{\lambda}^*) = \boldsymbol{\delta}^*(\Lambda_0\,\boldsymbol{F}(t), \Lambda_0)\,, \quad \boldsymbol{\delta}^* \in R_{k^*}^*\,,$$

respectively. Furthermore, by the definition of $\boldsymbol{X}_0(t)$, Λ_0, $\boldsymbol{F}(t)$

$$\operatorname{rank}(\boldsymbol{X}_0(t)) \equiv k \quad \text{and} \quad \operatorname{rank}(\Lambda_0\,\boldsymbol{F}(t), \Lambda_0) \equiv k^* \quad \text{on } [0,1]\,.$$

2.9. Remark. The number $k^* - k = m - n$ is called the *index of BVP* (1,2), (1,4).

2.10. Remark. If we added one zero row to the matrices $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{K}(t)$ in (1,4), we should obtain the equivalent problem. Let us assume that it has exactly k linearly independent solutions. Then by 2.8 its adjoint should have exactly both $k + m - n$ and $k + (m + 1) - n$ linearly independent solutions. This seems to be confusing. But we must take into account that while in the former case the adjoint problem has solutions $(\boldsymbol{y}^*, \boldsymbol{\lambda}^*)$ with $\boldsymbol{\lambda}^* \in R_m^*$, in the latter case the adjoint has solutions $(\boldsymbol{y}^*, \boldsymbol{\mu}^*)$, where $\boldsymbol{\mu}^*$ is an $(m + 1)$-vector, with an arbitrary last component. Nevertheless it can be seen that it is reasonable to remove from (1,4) all the linearly dependent rows and to consider the given BVP with linearly independent side conditions.

2.11. Remark. Given $\boldsymbol{x} \in AC_n$, $\boldsymbol{y}^* \in L_n^\infty$ and $\boldsymbol{\lambda}^* \in R_m^*$, we have by I.4.33

$$(2,15)\quad \int_0^1 \boldsymbol{y}^*(t)\,[\boldsymbol{x}'(t) - \boldsymbol{A}(t)\,\boldsymbol{x}(t)]\,\mathrm{d}t - \boldsymbol{\lambda}^*\left[\boldsymbol{M}\,\boldsymbol{x}(0) + \boldsymbol{N}\,\boldsymbol{x}(1) + \int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\,\boldsymbol{x}(t)\right]$$
$$= \int_0^1 \left[\boldsymbol{y}^*(t) + \int_t^1 \boldsymbol{y}^*(s)\,\boldsymbol{A}(s)\,\mathrm{d}s + \boldsymbol{\lambda}^*(\boldsymbol{K}(t) - \boldsymbol{N})\right]\boldsymbol{x}'(t)\,\mathrm{d}t$$
$$+ \left[\int_0^1 \boldsymbol{y}^*(s)\,\boldsymbol{A}(s)\,\mathrm{d}s - \boldsymbol{\lambda}^*(\boldsymbol{M} + \boldsymbol{N} - \boldsymbol{K}(0))\right]\boldsymbol{x}(0)\,.$$

In particular, applying again I.4.33 to the right-hand side of (2,15), we obtain that

$$(2,16)\quad \int_0^1 \boldsymbol{y}^*(t)\,[\boldsymbol{x}'(t) - \boldsymbol{A}(t)\,\boldsymbol{x}(t)]\,\mathrm{d}t - \boldsymbol{\lambda}^*\left[\boldsymbol{M}\,\boldsymbol{x}(0) + \boldsymbol{N}\,\boldsymbol{x}(1) + \int_0^1 \mathrm{d}[\boldsymbol{K}(t)]\,\boldsymbol{x}(t)\right]$$
$$= \int_0^1 \mathrm{d}\left[-\boldsymbol{y}^*(t) + \boldsymbol{y}^*(1) - \int_t^1 \boldsymbol{y}^*(s)\,\boldsymbol{A}(s)\,\mathrm{d}s - \boldsymbol{\lambda}^*\,\boldsymbol{K}(t)\right]\boldsymbol{x}(t)$$
$$- [\boldsymbol{y}^*(0) + \boldsymbol{\lambda}^*\boldsymbol{M}]\,\boldsymbol{x}(0) + [\boldsymbol{y}^*(1) - \boldsymbol{\lambda}^*\boldsymbol{N}]\,\boldsymbol{x}(1) \quad \text{for all } \boldsymbol{x} \in AC_n,\ \ \boldsymbol{y} \in BV_n,\ \boldsymbol{\lambda}^* \in R_m^*\,.$$

The formulas (2,15) and (2,16) will be called the *Green formulas.*

The adjoint BVP (2,7), (2,8) is a system of equations for an n-vector valued function $\boldsymbol{y}^*(t)$ of bounded variation on $[0, 1]$ and an m-vector parameter. Our wish is now to disclose the relationship between $\boldsymbol{y}^*$ and $\boldsymbol{\lambda}^*$ if $(\boldsymbol{y}^*, \boldsymbol{\lambda}^*)$ solves BVP (2,7), (2,8). To this end it appears to be convenient to consider BVP (1,1), (1,3) with the side condition in its canonical form (see 1.16)

$$(2,17)\qquad \begin{aligned} \boldsymbol{M}_0\,\boldsymbol{x}(0) + \boldsymbol{N}_0\,\boldsymbol{x}(1) \qquad\qquad\qquad\qquad &= \boldsymbol{r}_0\,,\\ \boldsymbol{M}_1\,\boldsymbol{x}(0) + \boldsymbol{N}_1\,\boldsymbol{x}(1) + \int_0^1 \mathrm{d}[\boldsymbol{K}_1(t)]\,\boldsymbol{x}(t) &= \boldsymbol{r}_1\,,\\ \int_0^1 \mathrm{d}[\boldsymbol{K}_2(t)]\,\boldsymbol{x}(t) &= \boldsymbol{r}_2\,. \end{aligned}$$

In this case the adjoint BVP (2,7), (2,8) reduces to the system of equations for $\boldsymbol{y} \in BV_n$, $\boldsymbol{\varkappa}_0^* \in R_{m_0}^*$, $\boldsymbol{\varkappa}_1^* \in R_{m_1}^*$ and $\boldsymbol{\varkappa}^* \in R_{m_2}^*$

$$(2,18)\qquad \mathrm{d}\boldsymbol{y}^* = \boldsymbol{y}^*\,\mathrm{d}[-\boldsymbol{B}] - \mathrm{d}(\boldsymbol{\varkappa}_1^*\boldsymbol{K}_1 + \boldsymbol{\varkappa}^*\boldsymbol{K}_2) \qquad \text{on } [0, 1]\,,$$

$$(2,19)\qquad \boldsymbol{y}^*(0) + \boldsymbol{\varkappa}_0^*\boldsymbol{M}_0 + \boldsymbol{\varkappa}_1^*\boldsymbol{M}_1 = \boldsymbol{0}\,, \qquad \boldsymbol{y}^*(1) - \boldsymbol{\varkappa}_0^*\boldsymbol{N}_0 - \boldsymbol{\varkappa}_1^*\boldsymbol{N}_1 = \boldsymbol{0}\,.$$

2.12. Remark. Let $\boldsymbol{\Theta}$ be a regular $m \times m$-matrix such that

$$\boldsymbol{\Theta}[\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{K}(t)] \equiv \begin{bmatrix} \boldsymbol{M}_0, & \boldsymbol{N}_0, & \boldsymbol{0} \\ \boldsymbol{M}_1, & \boldsymbol{N}_1, & \boldsymbol{K}_1(t) \\ \boldsymbol{0}, & \boldsymbol{0}, & \boldsymbol{K}_2(t) \end{bmatrix} \qquad \text{on } [0, 1]\,.$$

Given $\boldsymbol{\lambda}^* \in R_m^*$, let $\boldsymbol{\varkappa}_0^* \in R_{m_0}^*$, $\boldsymbol{\varkappa}_1^* \in R_{m_1}^*$ and $\boldsymbol{\varkappa}^* \in R_{m_2}^*$ be such that $\boldsymbol{\lambda}^* = (\boldsymbol{\varkappa}_0^*, \boldsymbol{\varkappa}_1^*, \boldsymbol{\varkappa}^*)\,\boldsymbol{\Theta}$.

Then

$$\boldsymbol{\lambda}^* \boldsymbol{M} = \boldsymbol{\varkappa}_0^* \boldsymbol{M}_0 + \boldsymbol{\varkappa}_1^* \boldsymbol{M}_1, \qquad \boldsymbol{\lambda}^* \boldsymbol{N} = \boldsymbol{\varkappa}_0^* \boldsymbol{N}_0 + \boldsymbol{\varkappa}_1^* \boldsymbol{N}_1$$

and

$$\boldsymbol{\lambda}^* \boldsymbol{K}(t) \equiv \boldsymbol{\varkappa}_1^* \boldsymbol{K}_1(t) + \boldsymbol{\varkappa}^* \boldsymbol{K}_2(t) \qquad \text{on } [0,1].$$

It follows that $\boldsymbol{y}^*: [0,1] \to R_n^*$ and $\boldsymbol{\lambda}^* \in R_m^*$ satisfy (2,7), (2,8) if and only if $(\boldsymbol{y}^*, \boldsymbol{\varkappa}_0^*, \boldsymbol{\varkappa}_1^*, \boldsymbol{\varkappa}^*)$, where $\boldsymbol{\varkappa}_0^* \in R_{m_0}^*$, $\boldsymbol{\varkappa}_0^* \in R_{m_1}^*$ and $\boldsymbol{\varkappa}^* \in R_{m_2}^*$ are such that $\boldsymbol{\lambda}^* = (\boldsymbol{\varkappa}_0^*, \boldsymbol{\varkappa}_1^*, \boldsymbol{\varkappa}^*)\,\boldsymbol{\Theta}$, satisfy (2,18), (2,19).

2.13. Notation. In the following $\boldsymbol{C}$ and $\boldsymbol{D}$ denote the $l \times n$-matrices such that $\boldsymbol{C}^* = [\boldsymbol{M}_0^*, \boldsymbol{M}_1^*]$ and $\boldsymbol{D}^* = [\boldsymbol{N}_0^*, \boldsymbol{N}_1^*]$ $(l = m_0 + m_1)$, $\boldsymbol{M}^c$ and $\boldsymbol{N}^c$ being arbitrary $(2n - l) \times n$-matrices complementary to $[\boldsymbol{C}, \boldsymbol{D}]$ (cf. III.5.11). Let $\boldsymbol{P}, \boldsymbol{Q} \in L(R_{2n-l}, R_n)$ and $\boldsymbol{P}^c, \boldsymbol{Q}^c \in L(R_l, R_n)$ be associated to $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{M}^c, \boldsymbol{N}^c$ by III.5.12.

Furthermore, let $\boldsymbol{P}_0^c, \boldsymbol{Q}_0^c \in L(R_{m_0}, R_n)$ and $\boldsymbol{P}_1^c, \boldsymbol{Q}_1^c \in L(R_{m_1}, R_n)$ be such that $\boldsymbol{P}^c = [\boldsymbol{P}_0^c, \boldsymbol{P}_1^c]$ and $\boldsymbol{Q}^c = [\boldsymbol{Q}_0^c, \boldsymbol{Q}_1^c]$. (By 1.16 rank$[\boldsymbol{C}, \boldsymbol{D}] = l$.)

Let us recall that according to III.5.12

$$\begin{bmatrix} -\boldsymbol{M}_0, & \boldsymbol{N}_0 \\ -\boldsymbol{M}_1, & \boldsymbol{N}_1 \\ -\boldsymbol{M}^c, & \boldsymbol{N}^c \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_0^c, & \boldsymbol{P}_1^c, & \boldsymbol{P} \\ \boldsymbol{Q}_0^c, & \boldsymbol{Q}_1^c, & \boldsymbol{Q} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{m_0}, & \boldsymbol{0}, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{I}_{m_1}, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{0}, & \boldsymbol{I}_{2n-l} \end{bmatrix} \tag{2,20}$$

and

$$\begin{bmatrix} \boldsymbol{P}_0^c, & \boldsymbol{P}_1^c, & \boldsymbol{P} \\ \boldsymbol{Q}_0^c, & \boldsymbol{Q}_1^c, & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{M}_0, & \boldsymbol{N}_0 \\ \boldsymbol{M}_1, & \boldsymbol{N}_1 \\ \boldsymbol{M}^c, & \boldsymbol{N}^c \end{bmatrix} = \begin{bmatrix} -\boldsymbol{I}_n, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{I}_n \end{bmatrix}. \tag{2,21}$$

Analogously as in III.5.17 it is easy to show on the basis of (2,20) that (2,19) holds if and only if

$$\begin{aligned} \boldsymbol{y}^*(0)\boldsymbol{P} + \boldsymbol{y}^*(1)\boldsymbol{Q} &= \boldsymbol{0}, \\ \boldsymbol{y}^*(0)\boldsymbol{P}_0^c + \boldsymbol{y}^*(1)\boldsymbol{Q}_0^c &= \boldsymbol{\varkappa}_0^*, \\ \boldsymbol{y}^*(0)\boldsymbol{P}_1^c + \boldsymbol{y}^*(1)\boldsymbol{Q}_1^c &= \boldsymbol{\varkappa}_1^*. \end{aligned} \tag{2,22}$$

This implies that BVP (2,18), (2,19) is equivalent to the problem of determining $\boldsymbol{y}^*: [0,1] \to R_n^*$ and $\boldsymbol{\varkappa}^* \in R_{m_2}^*$ such that $\boldsymbol{y}^*$ is a solution to

$$\mathrm{d}\boldsymbol{y}^* = \boldsymbol{y}^*\,\mathrm{d}[-\boldsymbol{B}] - \mathrm{d}[(\boldsymbol{y}^*(0)\boldsymbol{P}_1^c + \boldsymbol{y}^*(1)\boldsymbol{Q}_1^c)\,\boldsymbol{K}_1 + \boldsymbol{\varkappa}^*\boldsymbol{K}_2] \qquad \text{on } [0,1] \tag{2,23}$$

and

$$\boldsymbol{y}^*(0)\boldsymbol{P} + \boldsymbol{y}^*(1)\boldsymbol{Q} = \boldsymbol{0}. \tag{2,24}$$

In particular, if $(\boldsymbol{y}^*, \boldsymbol{\varkappa}^*)$ is a solution to (2,23), (2,24) and $\boldsymbol{\varkappa}_0^* \in R_{m_0}^*$ and $\boldsymbol{\varkappa}_1^* \in R_{m_1}^*$ are given by (2,22), then $(\boldsymbol{y}^*, \boldsymbol{\varkappa}_0^*, \boldsymbol{\varkappa}_1^*, \boldsymbol{\varkappa}^*)$ is a solution to (2,18), (2,19). On the other hand, if $(\boldsymbol{y}^*, \boldsymbol{\varkappa}_0^*, \boldsymbol{\varkappa}_1^*, \boldsymbol{\varkappa}^*)$ is a solution to (2,18), (2,19), then $(\boldsymbol{y}^*, \boldsymbol{\varkappa}^*)$ solves (2,23), (2,24) and $\boldsymbol{\varkappa}_0^* \in R_{m_0}^*$ and $\boldsymbol{\varkappa}_1^* \in R_{m_1}^*$ are given by (2,22).

2.14. Corollary. *BVP* (1,1), (2,17) *has a solution if and only if*

$$\int_0^1 \boldsymbol{y}^*(t)\,\boldsymbol{f}(t)\,\mathrm{d}t = (\boldsymbol{y}^*(0)\,\boldsymbol{P}_0^c + \boldsymbol{y}^*(1)\,\boldsymbol{Q}_0^c)\,\boldsymbol{r}_0 + (\boldsymbol{y}^*(0)\,\boldsymbol{P}_1^c + \boldsymbol{y}^*(1)\,\boldsymbol{Q}_1^c)\,\boldsymbol{r}_1 + \boldsymbol{\varkappa}^*\boldsymbol{r}_2$$

for any solution $(\boldsymbol{y}^*, \boldsymbol{\varkappa}^*)$ *of BVP* (2,23), (2,24).

Proof follows immediately from 2.3 and (2,22).

2.15. Remark. It is easy to see that BVP (2,23), (2,24) also possesses exactly $k^* = m - \operatorname{rank}(\boldsymbol{SU})$ linearly independent solutions.

2.16. Remark. The $m_2 \times m_2$-matrix

$$\boldsymbol{T} = \int_0^1 \boldsymbol{K}_2(t)\,\boldsymbol{K}_2^*(t)\,\mathrm{d}t$$

is regular. In fact, if there were $\boldsymbol{d}^*\boldsymbol{T} = \boldsymbol{0}$ for some $\boldsymbol{d}^* \in R_{m_2}^*$, then we should have also $\boldsymbol{d}^*\boldsymbol{T}\boldsymbol{d} = 0$, i.e.

$$0 = \int_0^1 \boldsymbol{h}^*(t)\,\boldsymbol{h}(t)\,\mathrm{d}t = \sum_{j=1}^{n} \int_0^1 (h_j(t))^2\,\mathrm{d}t\,,$$

where $\boldsymbol{h}^*(t) = (h_1(t), h_2(t), \ldots, h_n(t)) = \boldsymbol{d}^*\,\boldsymbol{K}_2(t)$ is of bounded variation on $[0, 1]$. This may happen if and only if $\boldsymbol{h}^*(t) = \boldsymbol{d}^*\,\boldsymbol{K}_2(t) = \boldsymbol{0}$ a.e. on $[0, 1]$. Since by the assumption $\boldsymbol{K}_2$ is right-continuous on $[0, 1)$ and left-continuous at 1, we have even $\boldsymbol{d}^*\,\boldsymbol{K}_2(t) \equiv \boldsymbol{0}$ on $[0, 1]$ and in virtue of the linear independence on $[0, 1]$ of the rows in $\boldsymbol{K}_2(t)$, it is $\boldsymbol{d}^* = \boldsymbol{0}$.

Let us put

$$\boldsymbol{L}_2(t) = -\int_0^t \boldsymbol{K}_2^*(s)\,\boldsymbol{T}^{-1}\,\mathrm{d}s \qquad \text{on } [0, 1]\,.$$

For $\boldsymbol{K}_2(1) = \boldsymbol{0}$ and $\boldsymbol{L}_2(0) = \boldsymbol{0}$, the integration-by-parts formula I.4.33 yields

$$(2,25) \qquad \int_0^1 \mathrm{d}[\boldsymbol{K}_2(t)]\,\boldsymbol{L}_2(t) = \left(\int_0^1 \boldsymbol{K}_2(t)\,\boldsymbol{K}_2^*(t)\,\mathrm{d}t\right)\boldsymbol{T}^{-1} = \boldsymbol{T}\boldsymbol{T}^{-1} = \boldsymbol{I}_{m_2}\,.$$

This enables us to express also the parameter $\boldsymbol{\varkappa}^*$ in (2,23), (2,24) in terms of $\boldsymbol{y}^*$. Let $(\boldsymbol{y}^*, \boldsymbol{\varkappa}^*)$ verify (2,23) on $[0, 1]$, then by (2,25)

$$\boldsymbol{\Phi}_2\boldsymbol{y} = \int_0^1 \mathrm{d}\left[\boldsymbol{y}^*(t) - \int_t^1 \boldsymbol{y}^*(s)\,\boldsymbol{A}(s)\,\mathrm{d}s - (\boldsymbol{y}^*(0)\,\boldsymbol{P}_1^c + \boldsymbol{y}^*(1)\,\boldsymbol{Q}_1^c)\,\boldsymbol{K}_1(t)\right]\boldsymbol{L}_2(t)$$

$$= \int_0^1 \mathrm{d}[\boldsymbol{\varkappa}^*\,\boldsymbol{K}_2(t)]\,\boldsymbol{L}_2(t) = \boldsymbol{\varkappa}^*\,.$$

The operator $\boldsymbol{y} \in BV_n \to \boldsymbol{\Phi}_2 \boldsymbol{y} \in R^*_{m_2}$ is linear and bounded. In fact, given $\boldsymbol{y} \in BV_n$,

$$|\boldsymbol{\Phi}_2 \boldsymbol{y}| \leq [\mathrm{var}_0^1 \boldsymbol{y}^* + (\sup_{t\in[0,1]} |\boldsymbol{y}^*(t)|)] \left(\int_0^1 |\boldsymbol{A}(s)| \, \mathrm{d}s + (|\boldsymbol{P}_1^c| + |\boldsymbol{Q}_1^c|) \, \mathrm{var}_0^1 \boldsymbol{K}_1\right) (\sup_{t\in[0,1]} |\boldsymbol{L}_2(t)|)$$

$$\leq \left[1 + \int_0^1 |\boldsymbol{A}(s)| \, \mathrm{d}s + (|\boldsymbol{P}_1^c| + |\boldsymbol{Q}_1^c|) \, \mathrm{var}_0^1 \boldsymbol{K}_1\right] \sup_{t\in[0,1]} |\boldsymbol{L}_2(t)| \, \|\boldsymbol{y}^*\|_{BV}.$$

The adjoint BVP (2,23), (2,24) to BVP (1,1), (2,17) may be thus written in the form

$$\mathrm{d}\boldsymbol{y}^* = \boldsymbol{y}^* \, \mathrm{d}[-\boldsymbol{B}] - \mathrm{d}[(\boldsymbol{\Phi}_1 \boldsymbol{y}) \boldsymbol{K}_1 - (\boldsymbol{\Phi}_2 \boldsymbol{y}) \boldsymbol{K}_2] \qquad \text{on } [0,1],$$

$$\boldsymbol{y}^*(0) \boldsymbol{P} + \boldsymbol{y}^*(1) \boldsymbol{Q} = \boldsymbol{0},$$

where $\boldsymbol{\Phi}_j$: $BV_n \to R^*_{m_j}$ $(j = 1, 2)$ are known linear bounded operators $(\boldsymbol{\Phi}_1 \boldsymbol{y} = \boldsymbol{y}^*(0) \boldsymbol{P}_1^c + \boldsymbol{y}^*(1) \boldsymbol{Q}_1^c)$.

3. Generalized Green's functions

Let us continue the investigation of BVP (1,1), (1,3). In addition to 1.1 we assume throughout the paragraph that 1.10 holds ($\boldsymbol{K}$ is right-continuous on $[0, 1)$ and left-continuous at 1 and $\boldsymbol{K}(1) = 0$).

Let $\mathscr{L}$ denote the linear bounded operator

$$(3,1) \qquad \mathscr{L}: \boldsymbol{x} \in AC_n \to \begin{bmatrix} \boldsymbol{x}'(t) - \boldsymbol{A}(t) \boldsymbol{x}(t) \\ \boldsymbol{M} \boldsymbol{x}(0) + \boldsymbol{N} \boldsymbol{x}(1) + \int_0^1 \mathrm{d}[\boldsymbol{K}(t)] \, \boldsymbol{x}(t) \end{bmatrix} \in L_n^1 \times R_m$$

(cf. 1.6). It may be shown from 2.3 that its range $R(\mathscr{L})$ is closed in $L_n^1 \times R_m$ and consequently $R(\mathscr{L})$ equipped with the norm of $L_n^1 \times R_m$ becomes a Banach space. We shall show this fact directly, without making use of Theorem 2.3. The symbols $\boldsymbol{U}$, $\boldsymbol{V}$ are again defined by (1,11) and (1,12).

3.1. Theorem. *The range* $R(\mathscr{L})$ *of the operator* (3,1) *is closed in* $L_n^1 \times R_m$.

Proof. A couple $\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in L_n^1 \times R_m$ belongs to $R(\mathscr{L})$ if and only if (2,1) has a solution $\boldsymbol{c} \in R_n$, i.e. if and only if $\boldsymbol{r} - (\boldsymbol{S}\boldsymbol{V}) \boldsymbol{f} \in R(\boldsymbol{S}\boldsymbol{U})$. $R(\boldsymbol{S}\boldsymbol{U})$ being finite dimensional, it is closed. Since

$$\boldsymbol{W}: \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in L_n^1 \times R_m \to \boldsymbol{r} - (\boldsymbol{S}\boldsymbol{V}) \boldsymbol{f} \in R_m$$

is a continuous operator, the set $\boldsymbol{W}_{-1}(R(\boldsymbol{S}\boldsymbol{U})) = R(\mathscr{L})$ of all $\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in L_n^1 \times R_m$ such that $\boldsymbol{W}\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in R(\boldsymbol{S}\boldsymbol{U})$ is also closed.

The concept of the generalized inverse matrix introduced in the Section I.2 and in particular theorems I.2.6 and I.2.7 enables us to give the necessary and sufficient condition for the existence of a solution to BVP (1,1), (1,3) in the following form.

3.2. Theorem. *BVP* (1,1), (1,3) *possesses a solution if and only if*

$$[\mathbf{I}_m - (\mathbf{SU})(\mathbf{SU})^{\#}]\,[\mathbf{r} - (\mathbf{SV})\,\mathbf{f}] = \mathbf{0}\,,$$

where $(\mathbf{SU})^{\#}$ *is the generalized inverse matrix to* $(\mathbf{SU})$. *If this condition is satisfied, then any solution* $\mathbf{x}$ *of BVP* (1,1), (1,3) *is of the form*

$$(3,2)\qquad \mathbf{x}(t) = \mathbf{X}(t)\,[\mathbf{I}_n - (\mathbf{SU})^{\#}\,(\mathbf{SU})]\,\mathbf{d} + \mathbf{X}(t)\,(\mathbf{SU})^{\#}\,[\mathbf{r} - (\mathbf{SV})\,\mathbf{f}] + (\mathbf{Vf})(t) \qquad on\ [0,1]\,,$$

where $\mathbf{d} \in R_n$ *may be arbitrary.*

Proof follows by I.2.6 and I.2.7 from the equivalence between BVP (1,1), (1,3) and the equation (2,1) $((\mathbf{SU})\,\mathbf{c} = \mathbf{r} - (\mathbf{SV})\,\mathbf{f})$.

3.3. Remark. By 2.3 the homogeneous BVP (1,2), (1,4) has only the trivial solution if and only if $\operatorname{rank}(\mathbf{SU}) = n$. Consequently, BVP (1,1), (1,3) is uniquely solvable for any $\begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix} \in R(\mathscr{L})$ if and only if $\operatorname{rank}(\mathbf{SU}) = n$.

On the other hand, BVP (1,1), (1,3) has a solution for any $\mathbf{f} \in L_n^1$ and $\mathbf{r} \in R_m$ if and only if $(\mathbf{SU})\,\mathbf{c} = \mathbf{q}$ is solvable for any $\mathbf{q} \in R_m$. ((2,1) has to be solvable for any $\mathbf{r} \in R_m$ and $\mathbf{f}(t) \equiv \mathbf{0}$ on $[0,1]$.) This holds if and only if (2,3) has only the trivial solution, i.e. if and only if $\operatorname{rank}(\mathbf{SU}) = m$.

In particular, BVP (1,1), (1,3) has a unique solution for any $\mathbf{f} \in L_n^1$ and $\mathbf{r} \in R_m$ if and only if $m = n$ and $\det(\mathbf{SU}) \neq 0$.

3.4. Theorem. *Let BVP* (1,1), (1,3) *have a solution. Then all its solutions are of the form*

$$(3,3)\qquad \mathbf{x}(t) = \mathbf{x}_0(t) + \mathbf{H}_0(t)\,\mathbf{r} + \int_0^1 \mathbf{G}_0(t,s)\,\mathbf{f}(s)\,\mathrm{d}s \qquad on\ [0,1]\,,$$

where $\mathbf{x}_0(t) = \mathbf{X}(t)\,[\mathbf{I} - (\mathbf{SU})^{\#}\,(\mathbf{SU})]\,\mathbf{d}$ $(\mathbf{d} \in R_n)$ *is an arbitrary solution to the homogeneous BVP* (1,2), (1,4),

$$(3,4)\qquad \mathbf{H}_0(t) = \mathbf{X}(t)\,(\mathbf{SU})^{\#} \qquad for\quad t \in [0,1]\,,$$

$$\mathbf{G}_0(t,s) = \mathbf{X}(t)\,\Delta(t,s)\,\mathbf{X}^{-1}(s) - \mathbf{X}(t)\,(\mathbf{SU})^{\#}\,\mathbf{F}(s) \qquad for\quad t,s \in [0,1]\,,$$

$$\Delta(t,s) = \mathbf{0} \qquad for\quad t < s\,, \qquad\qquad \Delta(t,s) = \mathbf{I}_n \qquad for\quad t \geq s$$

and

$$(3,5)\qquad \mathbf{F}(s) = \left[\mathbf{N}\,\mathbf{X}(1) + \int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\right]\mathbf{X}^{-1}(s) \qquad for\quad s \in [0,1]\,.$$

Proof follows immediately from 3.2 and (2,5), (2,6).

3.5. Remark. Let us notice that the representations (3,2) or (3,3) of the solutions to BVP (1,1), (1,3) are true even if the generalized inverse matrix $(\mathbf{SU})^{\#}$ to $(\mathbf{SU})$ is replaced by an arbitrary $n \times m$-matrix $\mathbf{B}$ such that $(\mathbf{SU})\, \mathbf{B}(\mathbf{SU}) = (\mathbf{SU})$ (see I.2.11).

3.6. Lemma. *The $n \times m$-matrix valued function $\mathbf{H}(t) = \mathbf{H}_0(t)$ and the $n \times n$-matrix valued function $\mathbf{G}(t, s) = \mathbf{G}_0(t, s)$ defined by (3,4) possess the following properties*

(i) $\mathbf{H}(t)$ *is absolutely continuous on* $[0, 1]$,

(ii) $\mathbf{G}(t, s)$ *is measurable in* (t, s) *on* $[0, 1] \times [0, 1]$, $\operatorname{var}_0^1 \mathbf{G}(., s) < \infty$ *for a.e.* $s \in [0, 1]$ *and* $\mathbf{G}(t, .)$ *is for any* $t \in [0, 1]$ *measurable and essentially bounded on* $[0, 1]$,

(iii) $\gamma(s) = |\mathbf{G}(0, s)| + \operatorname{var}_0^1 \mathbf{G}(., s)$ *is measurable and essentially bounded on* $[0, 1]$,

(iv) $\mathbf{G}(t, s) = \mathbf{G}_a(t, s) - \mathbf{G}_b(t, s)$ *on* $[0, 1] \times [0, 1]$, *where for any* $s \in [0, 1]$ $\mathbf{G}_a(., s)$ *is absolutely continuous on* $[0, 1]$ *and* $\mathbf{G}_b(., s)$ *is a simple jump function with the jump* $\mathbf{I}_n$ *at* $t = s$.

Proof. The assertions (i) and (ii) are obvious. Furthermore, $\mathbf{F}$ is of bounded variation on $[0, 1]$ and for any $s \in [0, 1]$

$$\gamma(s) \le |\mathbf{X}^{-1}(s)| + |(\mathbf{SU})^{\#}|\, |\mathbf{F}(s)| + (\operatorname{var}_0^1 \mathbf{X})\,(|\mathbf{X}^{-1}(s)| + |(\mathbf{SU})^{\#}|\, |\mathbf{F}(s)|)$$
$$\le (1 + \operatorname{var}_0^1 \mathbf{X}) \sup_{s \in [0,1]} (|\mathbf{X}^{-1}(s)| + |(\mathbf{SU})^{\#}|\, |\mathbf{F}(s)|) = \varkappa < \infty .$$

The last assertion is proved by putting $\mathbf{G}_a(t, s) = \mathbf{G}_0(t, s)$ if $t \ge s$, $\mathbf{G}_a(t, s) = \mathbf{G}_0(t, s) + \mathbf{I}_n$ if $t < s$ and $\mathbf{G}_b(t, s) = \mathbf{0}$ if $t \ge s$, $\mathbf{G}_b(t, s) = \mathbf{I}_n$ if $t < s$.

3.7. Remark. Let us notice that actually we have proved that $\gamma(s)$ is bounded on $[0, 1]$ and hence also $\mathbf{G}_0(t, s)$ is bounded on $[0, 1] \times [0, 1]$ ($|\mathbf{G}_0(t, s)| \le \gamma(s) \le \varkappa < \infty$ on $[0, 1] \times [0, 1]$). Moreover, by (3,4) $\operatorname{var}_0^1 \mathbf{G}_0(t, .) + \operatorname{var}_0^1 \mathbf{G}_0(., s) < \infty$ for all $t, s \in [0, 1]$.

3.8. Lemma. *Let* $\mathbf{H}: [0, 1] \to L(R_m, R_n)$ *and* $\mathbf{G}: [0, 1] \times [0, 1] \to L(R_n)$ *fulfil* (*i*)–(*iii*) *from* 3.6. *Then for any couple* $\begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in L_n^1 \times R_m$ *the n-vector valued function*

$$\mathbf{h}(t) = \mathbf{H}(t)\, \mathbf{r} + \int_0^1 \mathbf{G}(t, s)\, \mathbf{f}(s)\, \mathrm{d}s$$

is of bounded variation on $[0, 1]$ *and the linear operator*

$$\mathcal{M} : \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in L_n^1 \times R_m \to \mathbf{h} \in BV_n$$

is bounded.

Proof. Given $\boldsymbol{f} \in L^1_n$, $\boldsymbol{r} \in R_m$ and a subdivision $\{0 = t_0 < t_1 < \ldots < t_p = 1\}$ of $[0, 1]$,

$$\sum_{j=1}^{p} \left| \int_0^1 \boldsymbol{G}(t_j, s)\, \boldsymbol{f}(s)\, \mathrm{d}s - \int_0^1 \boldsymbol{G}(t_{j-1}, s)\, \boldsymbol{f}(s)\, \mathrm{d}s \right| + \left| \int_0^1 \boldsymbol{G}(0, s)\, \boldsymbol{f}(s)\, \mathrm{d}s \right|$$
$$\leq \int_0^1 \left(\sum_{j=1}^{p} |\boldsymbol{G}(t_j, s) - \boldsymbol{G}(t_{j-1}, s)| + |\boldsymbol{G}(0, s)| \right) |\boldsymbol{f}(s)|\, \mathrm{d}s$$
$$\leq \int_0^1 \gamma(s)\, |\boldsymbol{f}(s)|\, \mathrm{d}s \leq \left(\sup_{s \in [0,1]} |\gamma(s)| \right) \|\boldsymbol{f}\|_{L^1} \leq \varkappa \|\boldsymbol{f}\|_{L^1} .$$

Hence also

$$\mathrm{var}_0^1 \left(\int_0^1 \boldsymbol{G}(., s)\, \boldsymbol{f}(s)\, \mathrm{d}s \right) + \left| \int_0^1 \boldsymbol{G}(0, s)\, \boldsymbol{f}(s)\, \mathrm{d}s \right| \leq \varkappa \|\boldsymbol{f}\|_{L^1}$$

and

$$\left\| \mathscr{M} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \right\|_{BV} \leq c(\|\boldsymbol{f}\|_{L^1} + |\boldsymbol{r}|) = c \left\| \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \right\|_{L^1 \times R}$$

where

$$c = \varkappa + |\boldsymbol{H}(0)| + \int_0^1 |\boldsymbol{H}'(t)|\, \mathrm{d}t < \infty .$$

3.9. Remark. Let the operator $\mathscr{L}^{\oplus}$ be defined by

$$\mathscr{L}^{\oplus}: \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in L^1_n \times R_m \to \boldsymbol{H}_0(t)\, \boldsymbol{r} + \int_0^1 \boldsymbol{G}_0(t, s)\, \boldsymbol{f}(s)\, \mathrm{d}s \in AC_n , \tag{3,6}$$

where the matrix valued functions $\boldsymbol{G}_0(t, s)$ and $\boldsymbol{H}_0(t)$ are given by (3,4) ($R(\mathscr{L}^{\oplus}) \subset AC_n$ due to 3.4). According to (3,2) and (3,4)

$$\mathscr{L}^{\oplus} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} = \boldsymbol{U}(\boldsymbol{SU})^{\#} (\boldsymbol{r} - \boldsymbol{SVf}) + \boldsymbol{Vf} = \boldsymbol{U}(\boldsymbol{SU})^{\#}\, \boldsymbol{r} + \boldsymbol{Vf} - \boldsymbol{U}(\boldsymbol{SU})^{\#} (\boldsymbol{SV})\, \boldsymbol{f}$$

for any $\boldsymbol{f} \in L^1_n$ and $\boldsymbol{r} \in R_m$. Consequently $\mathscr{L}^{\oplus}$ is linear and bounded (cf. also 3.6 and 3.8). Moreover, for any $\boldsymbol{f} \in L^1_n$ and $\boldsymbol{r} \in R_m$ such that BVP (1,1), (1,3) has a solution $\left(\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in R(\mathscr{L})\right)$ $\mathscr{L}\mathscr{L}^{\oplus} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix}$ and hence $\mathscr{L}\mathscr{L}^{\oplus}\mathscr{L}\boldsymbol{x} = \mathscr{L}\boldsymbol{x}$ for any $\boldsymbol{x} \in AC_n$, i.e. $\mathscr{L}\mathscr{L}^{\oplus}\mathscr{L} = \mathscr{L}$ ($\mathscr{L}^{\oplus}$ is a generalized inverse operator to $\mathscr{L}$).

In particular, if $m = n$ and $\mathrm{rank}\,(\boldsymbol{SU}) = n$, then by 3.3 $\mathscr{L}^{\oplus}$ becomes a bounded inverse operator to $\mathscr{L}$. In this case the functions $\boldsymbol{G}_0(t, s)$, $\boldsymbol{H}_0(t)$ are called the Green couple of BVP (1,1), (1,3) (or (1,2), (1,4)), while the function $\boldsymbol{G}_0(t, s)$ is the Green function of BVP (1,1), (1,3).

3.10. Definition. A couple $\boldsymbol{G}(t, s)$, $\boldsymbol{H}(t)$ of matrix valued functions fulfilling (i)–(iii) of 3.6 is called the *generalized Green couple* if for all $\boldsymbol{f} \in L^1_n$, $\boldsymbol{r} \in R_m$ such that BVP

(1,1), (1,3) has a solution, the function

$$\boldsymbol{x}(t) = \boldsymbol{H}(t)\,\boldsymbol{r} + \int_0^1 \boldsymbol{G}(t,s)\,\boldsymbol{f}(s)\,\mathrm{d}s$$

is also a solution to BVP (1,1), (1,3).

3.11. Remark. By 3.4 and 3.6 the couple $\boldsymbol{G}_0(t,s)$, $\boldsymbol{H}_0(t)$ given by (3,4) is a generalized Green couple of BVP (1,1), (1,3).

3.12. Theorem. *A linear bounded operator* $\mathscr{L}^+ \colon R(\mathscr{L}) \to AC_n$ *fulfils* $\mathscr{L}\mathscr{L}^+\mathscr{L} = \mathscr{L}$ *if and only if there exists a generalized Green couple* $\boldsymbol{G}(t,s)$, $\boldsymbol{H}(t)$ *such that* $\mathscr{L}^+$ *is given by*

$$(3,7) \qquad \mathscr{L}^+ \colon \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in R(\mathscr{L}) \to \boldsymbol{H}(t)\,\boldsymbol{r} + \int_0^1 \boldsymbol{G}(t,s)\,\boldsymbol{f}(s)\,\mathrm{d}s \in AC_n\,.$$

Proof. Let $\mathscr{L}\mathscr{L}^+\mathscr{L} = \mathscr{L}$ and let $\mathscr{L}^{\oplus}$ be given by (3,6). According to 3.9 $\mathscr{L}(\mathscr{L}^+ - \mathscr{L}^{\oplus})\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} = \boldsymbol{0}$, i.e. $(\mathscr{L}^+ - \mathscr{L}^{\oplus})\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in N(\mathscr{L})$ for each $\boldsymbol{f} \in L^1_n$ and $\boldsymbol{r} \in R_m$. In particular, $\mathscr{L}^+ = \mathscr{L}^{\oplus}$ on $R(\mathscr{L})$ if $N(\mathscr{L}) = \{\boldsymbol{0}\}$. If $k = \dim N(\mathscr{L}) = n - \operatorname{rank}(\boldsymbol{SU}) > 0$, let $\boldsymbol{X}_0(t)$ be defined as in the proof of 2.7. Then $\operatorname{rank}(\boldsymbol{X}_0(t)) = k$ on $[0,1]$ (cf. 2.8) and given $\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in R(\mathscr{L})$, there exists $\boldsymbol{d} \in R_k$ such that

$$(3,8) \qquad (\mathscr{L}^+ - \mathscr{L}^{\oplus})\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix}(t) \equiv \boldsymbol{X}_0(t)\,\boldsymbol{d} \qquad \text{on } [0,1]\,.$$

By I.2.6, I.2.7 and I.2.15 this is possible if and only if

$$\boldsymbol{d} \equiv \boldsymbol{X}_0^{\#}(t)\,(\mathscr{L}^+ - \mathscr{L}^{\oplus})\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix}(t) \qquad \text{on } [0,1]\,.$$

By the definition $\boldsymbol{X}_0(t) = \boldsymbol{X}(t)\,\boldsymbol{C}_0$ on $[0,1]$, where $\boldsymbol{C}_0 \in L(R_k, R_n)$ has a full rank ($\operatorname{rank}(\boldsymbol{C}_0) = k$). According to 2.16 $\boldsymbol{X}_0^{\#}(t) = \boldsymbol{C}_0^{\#}\,\boldsymbol{X}^{-1}(t)$. It follows immediately that the mapping

$$\boldsymbol{\Phi} \colon \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in R(\mathscr{L}) \to \boldsymbol{d} = \boldsymbol{X}_0^{\#}(t)\,(\mathscr{L}^+ - \mathscr{L}^{\oplus})\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix}(t) \in R_k$$

is a linear bounded vector valued functional on $R(\mathscr{L})$. Let $\boldsymbol{\Psi}$ be its arbitrary extension on the whole space $L^1_n \times R_m$. ($\boldsymbol{\Psi}$ is defined and bounded on $L^1_n \times R_m$ and $\boldsymbol{\Psi} = \boldsymbol{\Phi}$ on $R(\mathscr{L})$.) Then there exist a function $\boldsymbol{\Theta}_1 \colon [0,1] \to L(R_n, R_k)$ essentially bounded and measurable on $[0,1]$ and $\boldsymbol{\Theta}_2 \in L(R_m, R_k)$ such that

$$\boldsymbol{\Psi}\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} = \int_0^1 \boldsymbol{\Theta}_1(s)\,\boldsymbol{f}(s)\,\mathrm{d}s + \boldsymbol{\Theta}_2\boldsymbol{r} \qquad \text{for all } \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in L^1_n \times R_m\,.$$

Together with (3,6) and (3,8) this yields (3,7), where

$$(3,9)\quad \mathbf{G}(t,s) = \mathbf{G}_0(t,s) + \mathbf{X}_0(t)\,\boldsymbol{\Theta}_1(s) \qquad \text{for all} \quad t\in[0,1] \quad \text{and a.e.} \quad s\in[0,1],$$
$$\mathbf{H}(t) = \mathbf{H}_0(t) + \mathbf{X}_0(t)\,\boldsymbol{\Theta}_2 \qquad \text{on} \quad [0,1]$$

obviously fulfil the conditions (i)–(iii) of 3.6.

The proof will be completed by taking into account the obvious fact that if $\mathbf{G}(t,s)$, $\mathbf{H}(t)$ is a generalized Green couple, then the operator (3,7) fulfils $\mathscr{L}\mathscr{L}^+\mathscr{L} = \mathscr{L}$.

3.13. Proposition. *A couple* $(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in L_n^\infty \times R_m^*$ *fulfils*

$$(3,10)\qquad \int_0^1 \mathbf{z}^*(t)\,\mathbf{f}(t)\,\mathrm{d}t = \boldsymbol{\lambda}^*\mathbf{r} \qquad \textit{for all} \quad \begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix} \in R(\mathscr{L})$$

if and only if there exists $\mathbf{y}^*\colon [0,1] \to R_n^*$ *such that* $\mathbf{y}^*(t) = \mathbf{z}^*(t)$ *a.e. on* $[0,1]$ *and* $(\mathbf{y}^*, \boldsymbol{\lambda}^*)$ *is a solution of BVP* (2,7), (2,8).

Proof. Let $\mathbf{z}^* \in L_n^\infty$ and $\boldsymbol{\lambda}^* \in R_m^*$. Then by the Green formula (2,15), (3,10) holds if and only if for any $\mathbf{x} \in AC_n$

$$(3,11)\qquad \int_0^1 \left[\mathbf{z}^*(t) + \int_t^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s + \boldsymbol{\lambda}^*(\mathbf{K}(t) - \mathbf{N})\right]\mathbf{x}'(t)\,\mathrm{d}t$$
$$+ \left[\int_0^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s - \boldsymbol{\lambda}^*(\mathbf{M} + \mathbf{N} - \mathbf{K}(0))\right]\mathbf{x}(0) = 0\,.$$

In particular, if $\mathbf{x}(t) \equiv \mathbf{x}(0)$ on $[0,1]$, (3,11) means that

$$\left[\int_0^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s - \boldsymbol{\lambda}^*(\mathbf{M} + \mathbf{N} - \mathbf{K}(0))\right]\mathbf{c} = 0$$

for each $\mathbf{c} \in R_n$, i.e.

$$(3,12)\qquad \int_0^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s = \boldsymbol{\lambda}^*(\mathbf{M} + \mathbf{N} - \mathbf{K}(0))\,.$$

Consequently (3,11) holds for each $\mathbf{x} \in AC_n$ if and only if

$$\int_0^1 \left[\mathbf{z}^*(t) + \int_t^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s + \boldsymbol{\lambda}^*(\mathbf{K}(t) - \mathbf{N})\right]\mathbf{v}(t)\,\mathrm{d}t = 0 \qquad \text{for any} \quad \mathbf{v} \in L_n^1$$

or $\mathbf{z}^*(t) = \mathbf{u}^*(t)$ a.e. on $[0,1]$, where

$$(3,13)\qquad \mathbf{u}^*(t) = -\int_t^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s - \boldsymbol{\lambda}^*(\mathbf{K}(t) - \mathbf{N}) \qquad \text{on} \quad [0,1]\,.$$

Let us put $\mathbf{y}^*(t) \equiv \mathbf{u}^*(t)$ on $(0,1)$, $\mathbf{y}^*(0) = \mathbf{u}^*(0+)$ and $\mathbf{y}^*(1) = \mathbf{u}^*(1-)$. Then owing to (3,13) and (3,12)

$$\mathbf{y}^*(1) = \boldsymbol{\lambda}^*\mathbf{N} \quad \text{and} \quad \mathbf{y}^*(0) = -\boldsymbol{\lambda}^*\mathbf{M}$$

and $(\mathbf{z}^*, \boldsymbol{\lambda}^*)$ fulfils (3,10) if and only if $\mathbf{z}^*(t) = \mathbf{y}^*(t)$ a.e. on $[0, 1]$. The proof will be completed by taking into account that

$$\int_t^1 \mathbf{y}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s = \int_t^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s \qquad \text{for any} \quad t \in [0, 1]$$

and hence

$$\mathbf{y}^*(t) = \mathbf{y}^*(1) + \int_t^1 \mathbf{y}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s + \boldsymbol{\lambda}^*\,\mathbf{K}(t) \qquad \text{on } [0, 1].$$

The latter implication follows from 2.3.

The set of all generalized Green couples is characterized in the following theorem.

If $k = n - \operatorname{rank}(\mathbf{SU}) > 0$, then $\mathbf{C}_0$ is an arbitrary $n \times k$-matrix whose columns form a basis in the space of all solutions to $(\mathbf{SU})\,\mathbf{c} = \mathbf{0}$ and $\mathbf{X}_0(t) = \mathbf{X}(t)\,\mathbf{C}_0$ on $[0, 1]$.

If $k^* = m - \operatorname{rank}(\mathbf{SU}) > 0$, then Λ_0 is an arbitrary $k^* \times n$-matrix whose rows form a basis in the space of all solutions to $\boldsymbol{\lambda}^*(\mathbf{SU}) = \mathbf{0}$ and $\mathbf{Y}_0(t) = \Lambda_0\,\mathbf{F}(t)$ on $[0, 1]$, where $\mathbf{F}(t)$ is given by (3,5).

3.14. Theorem. *A couple $\mathbf{G}$: $[0,1] \times [0,1] \to L(R_n)$, $\mathbf{H}$: $[0,1] \to L(R_m, R_n)$ is a generalized Green couple to BVP (1,1), (1,3) if and only if there exist a function Θ_1: $[0,1] \to L(R_n, R_k)$ essentially bounded and measurable on $[0,1]$, a function Σ: $[0,1] \to L(R_{k^*}, R_n)$ of bounded variation on $[0,1]$ and $\Theta_2 \in L(R_m, R_k)$ such that*

$$\text{(3,14)} \qquad \mathbf{G}(t,s) = \mathbf{G}_0(t,s) + \mathbf{X}_0(t)\,\Theta_1(s) + \Sigma(t)\,\mathbf{Y}_0(s)$$
$$\textit{for all} \quad t \in [0,1] \quad \textit{and a.e.} \quad s \in [0,1],$$
$$\mathbf{H}(t) = \mathbf{H}_0(t) + \mathbf{X}_0(t)\,\Theta_2 - \Sigma(t)\,\Lambda_0 \qquad \textit{on } [0,1],$$

where $\mathbf{G}_0(t,s)$ and $\mathbf{H}_0(t)$ are given by (3,4), the terms $\mathbf{X}_0(t)\,\Theta_1(s)$ and $\mathbf{X}_0(t)\,\Theta_2$ vanish if $k = 0$ and the terms $\Sigma(t)\,\mathbf{Y}_0(s)$ and $\Sigma(t)\,\Lambda_0$ vanish if $k^* = 0$.

Proof. Let us assume that $k > 0$ and $k^* > 0$.

(a) Let $\mathbf{G}(t,s)$, $\mathbf{H}(t)$ be a generalized Green couple of BVP (1,1), (1,3). Then by 3.12 and its proof there exist Θ_1: $[0,1] \to L(R_n, R_k)$ essentially bounded on $[0,1]$ and $\Theta_2 \in L(R_m, R_k)$ such that for all $\begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in R(\mathscr{L})$

$$\mathbf{H}(t)\,\mathbf{r} + \int_0^1 \mathbf{G}(t,s)\,\mathbf{f}(s)\,\mathrm{d}s$$
$$= [\mathbf{H}_0(t) + \mathbf{X}_0(t)\,\Theta_2]\,\mathbf{r} + \int_0^1 [\mathbf{G}_0(t,s) + \mathbf{X}_0(t)\,\Theta_1(s)]\,\mathbf{f}(s)\,\mathrm{d}s \qquad \text{on } [0,1].$$

By 3.13 and 2.8 this holds if and only if there exists Σ: $[0,1] \to L(R_{k^*}, R_n)$ such that (3,14) holds. According to 2.8 and I.2.15 $[\mathbf{Y}_0(s), \Lambda_0]\,[\mathbf{Y}_0(s), \Lambda_0]^{\#} = \mathbf{I}_{k^*}$ for any $s \in [0,1]$. The functions $\mathbf{P}(t,s) = \mathbf{G}(t,s) - \mathbf{G}_0(t,s) - \mathbf{X}_0(t)\,\Theta_1(s)$ and $\mathbf{Q}(t)$

$= -\mathbf{H}(t) + \mathbf{H}_0(t) + \mathbf{X}_0(t)\,\boldsymbol{\Theta}_2$ as functions of t for a.e. $s \in [0,1]$ are of bounded variation on $[0,1]$. Therefore the function $\Sigma(t) = [\mathbf{P}(t,s), \mathbf{Q}(s)]\,[\mathbf{Y}_0(s), \Lambda_0]^*$ for all $t \in [0,1]$ and a.e. $s \in [0,1]$ (cf. I.2.6 and I.2.7) has also a bounded variation on $[0,1]$.

(b) Let $\boldsymbol{\Theta}_1\colon [0,1] \to L(R_n, R_k)$ be essentially bounded on $[0,1]$, $\boldsymbol{\Theta}_2 \in L(R_m, R_k)$ and let $\Sigma\colon [0,1] \to L(R_{k^*}, R_n)$ be of bounded variation on $[0,1]$. Then the functions $\mathbf{G}(t,s)$, $\mathbf{H}(t)$ given by (3,14) are sure to fulfil (i)–(iii) from 3.6 and since by 2.3

$$\int_0^1 \mathbf{Y}_0(t)\,\mathbf{f}(t)\,\mathrm{d}t = \Lambda_0 \mathbf{r} \qquad \text{for all } \begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix} \in R(\mathscr{L}),$$

it is easy to verify that $\mathbf{G}(t,s)$, $\mathbf{H}(t)$ is a generalized Green couple.

The modification of the proof if $k = 0$ and/or $k^* = 0$ is obvious.

3.15. Theorem. *Let $\mathbf{G}_0(t,s)$ and $\mathbf{H}_0(t)$ be given by* (3,4). *Then $\mathbf{G}(t,s) = \mathbf{G}_0(t,s)$ and $\mathbf{H}(t) = \mathbf{H}_0(t)$ fulfil for any $s \in (0,1)$ the relations*

$$\mathbf{G}(t,s) - \mathbf{G}(0,s) - \int_0^t \mathbf{A}(\tau)\,\mathbf{G}(\tau,s)\,\mathrm{d}\tau = \Delta(t,s) \qquad \textit{for all} \quad t \in [0,1], \tag{3,15}$$

$$\mathbf{M}\,\mathbf{G}(0,s) + \mathbf{N}\,\mathbf{G}(1,s) + \int_0^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{G}(\tau,s) = [\mathbf{I} - (\mathbf{SU})(\mathbf{SU})^*]\,\mathbf{F}(s) \tag{3,16}$$

and

$$\mathbf{H}(t) - \mathbf{H}(0) - \int_0^t \mathbf{A}(\tau)\,\mathbf{H}(\tau)\,\mathrm{d}\tau = \mathbf{0} \qquad \textit{on } [0,1], \tag{3,17}$$

$$\mathbf{M}\,\mathbf{H}(0) + \mathbf{N}\,\mathbf{H}(1) + \int_0^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{H}(\tau) = (\mathbf{SU})(\mathbf{SU})^*. \tag{3,18}$$

Proof follows easily by inserting (3,4) into (3,15)–(3,18) and making use of (1,6) and

$$\int_0^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\,\Delta(\tau,s)\,\mathbf{X}^{-1}(s) = \int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\,\mathbf{X}^{-1}(s) + \Delta^+\mathbf{K}(s)$$

(cf. also III.2.13).

3.16. Remark. Let us notice that $\mathbf{F}(1) = \mathbf{F}(1-) = \mathbf{N}$ and by (1,7) and the Dirichlet formula I.4.32

$$\int_s^1 \mathbf{F}(\sigma)\,\mathbf{A}(\sigma)\,\mathrm{d}\sigma = \mathbf{F}(s) - \mathbf{F}(1) + \mathbf{K}(s) \qquad \text{on } [0,1]. \tag{3,19}$$

3.17. Theorem. *The functions* $\mathbf{G}(t,s) = \mathbf{G}_0(t,s)$, $\mathbf{H}(t) = \mathbf{H}_0(t)$ *given by* (3,4) *fulfil for any* $t \in (0,1)$ *the relations*

$$\mathbf{G}(t,s) - \mathbf{G}(t,1) - \int_s^1 \mathbf{G}(t,\sigma)\,\mathbf{A}(\sigma)\,\mathrm{d}\sigma - \mathbf{H}(t)\,\mathbf{K}(s) = \Delta(t,s) \quad \text{for any} \quad s \in [0,1], \tag{3,20}$$

$$\mathbf{G}(t,0) - \mathbf{H}(t)\,\mathbf{M} = \mathbf{X}(t)\,[\mathbf{I} - (\mathbf{SU})^{\#}\,(\mathbf{SU})], \qquad \mathbf{G}(t,1) + \mathbf{H}(t)\,\mathbf{N} = \mathbf{0}. \tag{3,21}$$

Proof. Given $t \in (0,1)$ and $s \in [0,1]$,

$$\mathbf{G}_0(t,s) = -\mathbf{H}_0(t)\,\mathbf{N}\,\mathbf{X}(1)\,\mathbf{X}^{-1}(s)$$
$$-\,\mathbf{H}_0(t)\int_s^1 \mathrm{d}[\mathbf{K}(\tau)]\,\mathbf{X}(\tau)\,\mathbf{X}^{-1}(s) - \int_s^1 \mathrm{d}_\sigma[\Delta(t,\sigma)]\,\mathbf{X}(\sigma)\,\mathbf{X}^{-1}(s).$$

Our assertion follows readily taking into account the variation-of-constants formula for the initial value problem $\mathrm{d}\mathbf{y}^* = -\mathbf{y}^*\,\mathrm{d}[\mathbf{B}] - \mathrm{d}\mathbf{h}^*$, $\mathbf{y}^*(1) = \mathbf{y}_1^*$ (cf. also the proof of 2.3).

On the other hand, we have

3.18. Theorem. *Let* $\mathbf{G}\colon [0,1] \times [0,1] \to L(R_n)$ *and* $\mathbf{H}\colon [0,1] \to L(R_m, R_n)$ *fulfil for any* $s \in (0,1)$ *the relations* (3,15)–(3,18) *and let* $\gamma(s) = |\mathbf{G}(0,s)| + \mathrm{var}_0^1\,\mathbf{G}(.,s) \le \gamma_0 < \infty$ *on* $[0,1]$, $\mathbf{G}$ *being measurable* $[0,1] \times [0,1]$. *Then* $\mathbf{G}(t,s)$, $\mathbf{H}(t)$ *is a generalized Green couple for BVP* (1,1), (1,3).

Proof. Let $\begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix} \in R(\mathscr{L})$ and

$$\mathbf{x}(t) = \mathbf{H}(t)\,\mathbf{r} + \int_0^1 \mathbf{G}(t,s)\,\mathbf{f}(s)\,\mathrm{d}s \qquad \text{on } [0,1].$$

$$\int_0^1 \left(\int_0^1 |\mathbf{A}(\tau)\,\mathbf{G}(\tau,s)\,\mathbf{f}(s)|\,\mathrm{d}s\right)\mathrm{d}\tau \le \int_0^1 |\mathbf{A}(\tau)| \left(\int_0^1 \gamma_0 |\mathbf{f}(s)|\,\mathrm{d}s\right)\mathrm{d}\tau$$
$$\le \left(\int_0^1 |\mathbf{A}(\tau)|\,\mathrm{d}\tau\right)\gamma_0 \|\mathbf{f}\|_{L^1} < \infty,$$

the Tonelli-Hobson theorem I.4.36 yields

$$\int_0^t \mathbf{A}(\tau)\left(\int_0^1 \mathbf{G}(\tau,s)\,\mathbf{f}(s)\,\mathrm{d}s\right)\mathrm{d}\tau = \int_0^1 \left(\int_0^t \mathbf{A}(\tau)\,\mathbf{G}(\tau,s)\,\mathrm{d}\tau\right)\mathbf{f}(s)\,\mathrm{d}s$$

for any $t \in [0,1]$. Consequently in virtue of (3,15) and (3,17)

$$\mathbf{x}(t) - \mathbf{x}(0) - \int_0^t \mathbf{A}(\tau)\,\mathbf{x}(\tau)\,\mathrm{d}\tau = \int_0^1 \Delta(t,s)\,\mathbf{f}(s)\,\mathrm{d}s = \int_0^t \mathbf{f}(\tau)\,\mathrm{d}\tau.$$

Finally, taking into account that $\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in R(\mathscr{L})$ if and only if (cf. 3.1)

$$[\boldsymbol{I} - (\boldsymbol{SU})(\boldsymbol{SU})^{\#}]\left[\boldsymbol{r} - \int_0^1 \boldsymbol{F}(s)\,\boldsymbol{f}(s)\,\mathrm{d}s\right] = \boldsymbol{0}$$

and applying I.4.38 it is not difficult to check that $\boldsymbol{x}(t)$ verifies also the side condition (1,3).

3.19. Remark. By the variation-of-constants formula III.2.13 for generalized differential equations $\boldsymbol{G}$: $[0,1] \times [0,1] \rightarrow L(R_n)$ fulfils (3,15) and (3,16) for all $s \in (0,1)$ if and only if there exists $\boldsymbol{C}$: $[0,1] \rightarrow L(R_n)$ such that

$$\boldsymbol{G}(t,s) = \boldsymbol{X}(t)\,\Delta(t,s)\,\boldsymbol{X}^{-1}(s) + \boldsymbol{X}(t)\,\boldsymbol{C}(s) \qquad \text{for all} \quad t \in [0,1] \quad \text{and} \quad s \in (0,1)$$

and

$$(\boldsymbol{SU})\,\boldsymbol{C}(s) = -(\boldsymbol{SU})(\boldsymbol{SU})^{\#}\,\boldsymbol{F}(s) \qquad \text{on } (0,1).$$

Hence according to I.2.6 $\boldsymbol{G}(t,s)$ fulfils (3,15) and (3,16) if and only if

$$\boldsymbol{G}(t,s) = \boldsymbol{G}_0(t,s) + \boldsymbol{X}_0(t)\,\boldsymbol{D}(s), \tag{3,22}$$

where $\boldsymbol{X}_0(t)$ has the same meaning as in 3.14 and vanishes if $k = n - \operatorname{rank}(\boldsymbol{SU}) = 0$ and $\boldsymbol{D}(s)$ is an arbitrary $k \times n$-matrix valued function defined on (0, 1).

Analogously $\boldsymbol{H}$: $[0,1] \rightarrow L(R_m, R_n)$ fulfils (3,17), (3,18) if and only if

$$\boldsymbol{H}(t) = \boldsymbol{H}_0(t) + \boldsymbol{X}_0(t)\,\boldsymbol{\Gamma} \qquad \text{on } [0,1], \tag{3,23}$$

where $\boldsymbol{\Gamma} \in L(R_m, R_k)$ may be arbitrary.

Since $\operatorname{rank}(\boldsymbol{X}_0(t)) \equiv k$ on $[0,1]$ (cf. 2.8), we have by I.2.6.

$$\begin{aligned} \boldsymbol{D}(s) &= \boldsymbol{X}_0^{\#}(t)\,(\boldsymbol{G}(t,s) - \boldsymbol{G}_0(t,s)) \qquad \text{for all} \quad s \in (0,1) \quad \text{and} \quad t \in [0,1], \\ \boldsymbol{\Gamma} &= \boldsymbol{X}_0^{\#}(t)\,(\boldsymbol{H}(t) - \boldsymbol{H}_0(t)) \qquad \text{for all} \quad t \in [0,1]. \end{aligned} \tag{3,24}$$

Now, let $\boldsymbol{G}(t,s)$, $\boldsymbol{H}(t)$ satisfy also (3,20), (3,21) for any $t \in (0,1)$. Then $\operatorname{var}_0^1 \boldsymbol{G}(t,.) < \infty$ for any $t \in [0,1]$. Moreover, by (3,23) and (3,24)

$$\boldsymbol{D}(0+) = \boldsymbol{X}_0^{\#}(t)\,(\boldsymbol{G}(t,0+) - \boldsymbol{G}_0(t,0+)) = \boldsymbol{X}_0^{\#}(t)\,(\boldsymbol{H}(t) - \boldsymbol{H}_0(t))\,\boldsymbol{M} = \boldsymbol{\Gamma}\boldsymbol{M} \tag{3,25}$$

and

$$\boldsymbol{D}(1-) = -\boldsymbol{\Gamma}\boldsymbol{N}. \tag{3,26}$$

Putting $\boldsymbol{D}(0) = \boldsymbol{D}(0+)$, $\boldsymbol{D}(1) = \boldsymbol{D}(1-)$, $\boldsymbol{D}$ will be of bounded variation on $[0,1]$. By (3,20)

$$\boldsymbol{D}(s) - \boldsymbol{D}(1) - \int_s^1 \boldsymbol{D}(\tau)\,\boldsymbol{A}(\tau)\,\mathrm{d}\tau - \boldsymbol{\Gamma}\boldsymbol{K}(s) = \boldsymbol{0} \qquad \text{on } [0,1].$$

This together with (3,25), (3,26) may hold if and only if there is $\boldsymbol{W} \in L(R_{k^*}, R_k)$ such that $\boldsymbol{D}(s) \equiv \boldsymbol{W}\boldsymbol{Y}_0(s)$ on $[0,1]$ and $\boldsymbol{\Gamma} = \boldsymbol{W}\Lambda_0$ (cf. 2.8). To summarize:

$$\boldsymbol{G}\text{: } [0,1] \times [0,1] \rightarrow L(R_n) \quad \text{and} \quad \boldsymbol{H}\text{: } [0,1] \rightarrow L(R_m, R_n)$$

satisfy (3,15)–(3,18) for any $s \in (0, 1)$ and (3,20), (3,21) for any $t \in (0, 1)$ if and only if

$$\mathbf{G}(t, s) = \mathbf{G}_0(t, s) + \mathbf{X}_0(t)\, \mathbf{W} \mathbf{Y}_0(s) \qquad \text{on } [0, 1] \times [0, 1],$$
$$\mathbf{H}(t) = \mathbf{H}_0(t) - \mathbf{X}_0(t)\, \mathbf{W} \Lambda_0 \qquad \text{on } [0, 1],$$

where $\mathbf{W} \in L(R_{k^*}, R_k)$ may be arbitrary.

3.20. Remark. By the definition (3,4) of $\mathbf{G}_0(t, s)$ and $\mathbf{H}_0(t)$ and 3.17

$$\begin{aligned}
(3{,}27) \qquad & \mathbf{G}_0(t, 0+) - \mathbf{H}_0(t)\, \mathbf{M} = \mathbf{X}(t)\, [\mathbf{I} - (\mathbf{SU})^{\#} (\mathbf{SU})] \qquad \text{if } t > 0, \\
& \mathbf{G}_0(0, 0+) - \mathbf{H}_0(0)\, \mathbf{M} = -(\mathbf{SU})^{\#} (\mathbf{SU}), \\
& \mathbf{G}_0(t, 1-) + \mathbf{H}_0(t)\, \mathbf{N} = \mathbf{0} \qquad \text{if } t < 1, \\
& \mathbf{G}_0(1, 1-) + \mathbf{H}_0(1)\, \mathbf{N} = \mathbf{I}.
\end{aligned}$$

In particular, for any $\mathbf{g} \in BV_n$ right-continuous on $[0, 1)$ and left-continuous at 1

$$(3{,}28) \quad \int_0^1 \mathrm{d}[\mathbf{g}^*(\tau)]\, (\mathbf{G}_0(\tau, 0+) - \mathbf{H}_0(\tau)\, \mathbf{M}) = \int_0^1 \mathrm{d}[\mathbf{g}^*(\tau)]\, \mathbf{X}(\tau)\, [\mathbf{I} - (\mathbf{SU})^{\#} (\mathbf{SU})]$$

and

$$(3{,}29) \qquad \int_0^1 \mathrm{d}[\mathbf{g}^*(\tau)]\, (\mathbf{G}_0(\tau, 1-) + \mathbf{H}_0(\tau)\, \mathbf{N}) = \mathbf{0}.$$

We shall conclude this section by proving that the couple $\mathbf{G}_0(t, s)$, $\mathbf{H}_0(t)$ has also the meaning of a generalized Green couple for the adjoint nonhomogeneous BVP (2,11), (2,12).

3.21. Theorem. *Let $\mathbf{g} \in BV_n$ be right-continuous on $[0, 1)$ and left-continuous at 1 and let $\mathbf{p}, \mathbf{q} \in R_n$. Then, if BVP (2,11), (2,12) has a solution, the couple $(\mathbf{y}^*, \boldsymbol{\lambda}^*)$ given by*

$$\begin{aligned}
(3{,}30) \quad & \mathbf{y}^*(s) = \mathbf{q}^*\, \mathbf{G}_0(1, s) - \mathbf{p}^*\, \mathbf{G}_0(0, s) - \int_0^1 \mathrm{d}[\mathbf{g}^*(\tau)]\, \mathbf{G}_0(\tau, s) \qquad \text{on } (0, 1), \\
& \mathbf{y}^*(0) = \mathbf{y}^*(0+), \qquad \mathbf{y}^*(1) = \mathbf{y}^*(1-), \\
& \boldsymbol{\lambda}^* \;\; = -\mathbf{q}^*\, \mathbf{H}_0(1) + \mathbf{p}^*\, \mathbf{H}_0(0) + \int_0^1 \mathrm{d}[\mathbf{g}^*(\tau)]\, \mathbf{H}_0(\tau)
\end{aligned}$$

is also its solution.

Proof. (a) By (3,4) $\mathbf{G}_0(1, s) = \mathbf{X}(1)\, (\mathbf{X}^{-1}(s) - (\mathbf{SU})^{\#}\, \mathbf{F}(s))$ on $[0, 1]$ and owing to (1,7) and (3,19)

$$\int_s^1 \mathbf{G}_0(1, \sigma)\, \mathbf{A}(\sigma)\, \mathrm{d}\sigma$$
$$= \mathbf{X}(1)\, \mathbf{X}^{-1}(s) - \mathbf{X}(1)\, (\mathbf{SU})^{\#}\, \mathbf{F}(s) - \mathbf{I} + \mathbf{X}(1)\, (\mathbf{SU})^{\#}\, \mathbf{F}(1) - \mathbf{X}(1)\, (\mathbf{SU})^{\#}\, \mathbf{K}(s)$$

or

$$(3{,}31) \quad \mathbf{G}_0(1, s) = \mathbf{G}_0(1, 1) + \int_s^1 \mathbf{G}_0(1, \sigma)\, \mathbf{A}(\sigma)\, \mathrm{d}\sigma + \mathbf{H}_0(1)\, \mathbf{K}(s) \qquad \text{on } [0, 1].$$

Let us notice that $\boldsymbol{G}_0(1,1-) = \boldsymbol{G}_0(1,1) = \boldsymbol{I} - \boldsymbol{X}(1)(\boldsymbol{SU})^+ \boldsymbol{F}(1)$ and $\boldsymbol{G}_0(1,0+) = \boldsymbol{G}_0(1,0) = \boldsymbol{X}(1) - \boldsymbol{X}(1)(\boldsymbol{SU})^\# \boldsymbol{F}(0)$, $\boldsymbol{F}(1) = \boldsymbol{N}$, $\boldsymbol{F}(0) = (\boldsymbol{SU}) - \boldsymbol{M}$. Furthermore,

$$\boldsymbol{G}_0(0,0) = \boldsymbol{I} - (\boldsymbol{SU})^\# \boldsymbol{F}(0) = \boldsymbol{I} - (\boldsymbol{SU})^\# (\boldsymbol{SU}) + (\boldsymbol{SU})^\# \boldsymbol{M},$$

$$\boldsymbol{G}_0(0,s) = -(\boldsymbol{SU})^\# \boldsymbol{F}(s)$$

if $s > 0$. In particular, $\boldsymbol{G}_0(0,1-) = \boldsymbol{G}_0(0,1) = -(\boldsymbol{SU})^\# \boldsymbol{F}(1) = -(\boldsymbol{SU})^\# \boldsymbol{N}$. Hence, making use of (3,19)

$$(3,32) \qquad \int_s^1 \boldsymbol{G}_0(0,\sigma)\boldsymbol{A}(\sigma)\,\mathrm{d}\sigma = \boldsymbol{G}_0(0,s) - \boldsymbol{G}_0(0,1) - \boldsymbol{H}_0(0)\boldsymbol{K}(s) \qquad \text{on } [0,1].$$

Now, if $\boldsymbol{y}^*\colon [0,1] \to R_n^*$ and $\boldsymbol{\lambda}^* \in R_m^*$ are given by (3,30), then by (3,15), (3,20), (3,31), (3,32) and I.4.32

$$\boldsymbol{y}^*(s) - \boldsymbol{y}^*(1) - \int_s^1 \boldsymbol{y}^*(\sigma)\boldsymbol{A}(\sigma)\,\mathrm{d}\sigma + \boldsymbol{\lambda}^*\boldsymbol{K}(s)$$

$$= -\int_0^1 \mathrm{d}[\boldsymbol{g}^*(\tau)]\left(\boldsymbol{G}_0(\tau,s) - \boldsymbol{G}_0(\tau,1) - \int_s^1 \boldsymbol{G}_0(\tau,\sigma)\boldsymbol{A}(\sigma)\,\mathrm{d}\sigma - \boldsymbol{H}_0(\tau)\boldsymbol{K}(s)\right)$$

$$= \boldsymbol{g}^*(s) - \boldsymbol{g}^*(1) \qquad \text{on } [0,1].$$

(Here we have also made use of the assumption $\boldsymbol{g}^*(1-) = \boldsymbol{g}^*(1)$, $\boldsymbol{g}^*(0+) = \boldsymbol{g}^*(0)$ and of the fact that $\boldsymbol{G}_0(0,s+) = \boldsymbol{G}_0(0,s)$ if $s > 0$ and $\boldsymbol{G}_0(1,s-) = \boldsymbol{G}_0(1,s)$ if $s < 1$.)

(b) By (3,21), (3,27) and (3,29)

$$\boldsymbol{y}^*(1) - \boldsymbol{\lambda}^*\boldsymbol{N} = \boldsymbol{q}^*[\boldsymbol{G}_0(1,1-) + \boldsymbol{H}_0(1)\boldsymbol{N}] - \boldsymbol{p}^*[\boldsymbol{G}_0(0,1-) + \boldsymbol{H}_0(0)\boldsymbol{N}]$$

$$- \int_0^1 \mathrm{d}[\boldsymbol{g}^*(\tau)]\,(\boldsymbol{G}_0(\tau,1-) + \boldsymbol{H}_0(\tau)\boldsymbol{N}) = \boldsymbol{q}^*.$$

(c) Finally, by (3,21), (3,27) and (3,28)

$$\boldsymbol{y}^*(0) + \boldsymbol{\lambda}^*\boldsymbol{M} = \boldsymbol{q}^*[\boldsymbol{G}_0(1,0+) - \boldsymbol{H}_0(1)\boldsymbol{M}] - \boldsymbol{p}^*[\boldsymbol{G}_0(0,0+) - \boldsymbol{H}_0(0)\boldsymbol{M}]$$

$$- \int_0^1 \mathrm{d}[\boldsymbol{g}^*(\tau)]\,(\boldsymbol{G}_0(\tau,0+) - \boldsymbol{H}_0(\tau)\boldsymbol{M})$$

$$= \boldsymbol{p}^* + \left[\boldsymbol{q}^*\boldsymbol{X}(1) - \boldsymbol{p}^* - \int_0^1 \mathrm{d}[\boldsymbol{g}^*(\tau)]\,\boldsymbol{X}(\tau)\right][\boldsymbol{I} - (\boldsymbol{SU})^\#(\boldsymbol{SU})].$$

Since $\boldsymbol{x}_0(t) = \boldsymbol{X}(t)[\boldsymbol{I} - (\boldsymbol{SU})^\#(\boldsymbol{SU})]$ is a solution to the homogeneous BVP (1,2), (1,4), the last expression reduces to $\boldsymbol{p}^*$ (cf. 2.5).

Notes

Canonical form of Stieltjes integral conditions (IV.3.15) is due to Zimmerberg [2]. Section IV.2 is based on Vejvoda, Tvrdý [1] and Tvrdý, Vejvoda [1]. In IV.2.16 the idea of Pagni [1] is utilized. For writing IV.3, the paper Brown [1] was stimulating.

Bryan [1], Cole [2], Halanay, Moro [1], Krall [1]–[4] and Tucker [1] are related references to IV.2, while Reid [2], [3], Chitwood [1], Zubov [1], [2] and Bradley [1] concern IV.3. For a historical survey of the subject and a more complete bibliography the reader is referred e.g. to Whyburn [2], Conti [2], Reid [1] and Krall [9]. More detail concerning some special questions (as e.g. two-point problems, second order and n-th order equations, selfadjointness, expansion theorems) as well as examples may be found in the monographs Coddington, Levinson [1], Reid [1], Najmark [1] and Cole [1].

The interface problems were treated in Conti [3], Krall [2], [3], Parhimovič [3], Stallard [1] and Zettl [1]. Boundary problems in the L^p-setting were dealt with in Krall [1]–[8], Brown [1], [3], Brown, Krall [1], [3]. Expansion theorems for problems with a multipoint or Stieltjes integral side conditions are to be found in Krall [5], Brown, Green, Krall [1] and Coddington, Dijksma [1]. For applications to controllability, minimization problems and splines see Brown [2], Brown, Krall [2], Halanay [1] and Marchiò [1].