8 LIFE AND WORK OF KAREL RYCHLÍK

8.1 INTRODUCTION

The monograph is devoted to the life and work of the Czech mathematician Karel Rychlík. The first chapter contains the brief overview of Rychlík's life, the survey of his scientific and pedagogical activities and the detailed description of his life story. The subsequent five chapters discuss particular groups of Rychlík's publications: *2. Algebra and Number theory, 3. Works on Mathematical Analysis, 4. Textbooks, Popularizing Papers, Translations, 5. Karel Rychlík and Bernard Bolzano, 6. Other Works on History of Mathematics.* These chapters are conceived separately, each of them is provided with the conclusion and the list of references.

The seventh chapter presents the list of Rychlík's publications, reviews and lectures at Charles University, at the Czech Technical University and in the Union of Czech Mathematicians and Physicists. The book ends with the pictorial appendix, the survey of abbreviations and the name index.

The aim of this summary is to provide the basic information on Rychlík's life and on his most important mathematical results. The third section contains the survey of Rychlík's scientific activities, including their brief evaluation. From all groups of publications only the principal works are described in separate sections, namely the papers devoted to g-adic numbers, valuation theory, algebraic number theory and abstract algebra, determinant theory. More details (in English) concerning particular groups of Rychlík's publications can be found in the couple of author's papers *Life and Work of Karel Rychlík*[1] and *Bolzano's Inheritance Research in Bohemia*[2] and on Internet pages devoted to Rychlík:

http://euler.fd.cvut.cz/publikace/HTM/Index.html.

[1] In: *Mathematics throughout the Ages*, Prometheus, Prague, 2001, 258–286.
[2] Ibid., 67–91.

8.2 LIFE OF KAREL RYCHLÍK

Karel Rychlík was born on April 16, 1885 in Benešov near Prague as the first of the three children of Barbora Srbová, married Rychlíková (1865 – 1928), and Vilém Evžen Rychlík (1857 – 1923).[3] In October 1904 he started to study mathematics and physics at the Faculty of Arts of Czech Charles-Ferdinand University in Prague (below only Charles University). He was influenced above all by Professor Karel Petr. In the school year 1907/08 Rychlík was studying at Faculté des Sciences in Paris. He was mainly interested in the lectures of Jacques Hadamard (winter semester) and Emile Picard (summer semester) called *Analyse supérieure*. Besides, Rychlík attended the lectures of Gaston Darboux, Edouard Goursat, Louis Raffy, Paul Painlevé and Marie Curie at the same faculty and the lectures on number theory at Collège de France, read by Georges Humbert. During his stay in Paris Rychlík was also working on his dissertation. On December 16, 1908 he passed the so-called "teacher examination". On March 30, 1909, on the grounds of the dissertation and rigorosum examinations of mathematics and philosophy, he was awarded the degree Doctor of Philosophy.

From 1909 till 1913 Rychlík worked as an assistant of the mathematical seminar at the Faculty of Arts of Charles University. In 1912 he was appointed associate professor (Docent). As a "private associate professor" (this position was not paid in general) Rychlík lectured at the university till 1938. In 1919 the board of professors decided on his appointment adjunct professor, in addition to the present chairs, but their suggestion remained in the ministry and was not put into practice (the financial situation of the school system was not very

[3]Karel Rychlík had a younger brother Vilém (1887 – 1913), who was a brilliant mathematician, too. Karel used to say his brother had been much cleverer than him. It is a stumper because Vilém died very young, at the age of 26. He had just finished the study of mathematics and physics at the Faculty of Arts of Charles University, received the degree Doctor of Philosophy, become an assistant at the Czech Technical University in Prague and he had written several treatises. He is told being very lively, loving women and smoking 40 cigarettes a day, which became fateful for him. One day he caught a cold somewhere and within three days he died (that was called a fast consumption).

Their younger sister Jana (1888 – 1969) studied, as an adjunct student, mathematics and biology at the Faculty of Arts and became a biology teacher. In 1918 she married Václav Špála, later the famous Czech painter, and gave precedence to her husband and children.

good). In the end Rychlík became a professor (adjunct: November 27, 1920; full: March 12, 1924) at the Czech Technical University in Prague (below only Technical University), where he had been working as an assistant since 1913. In October 1914 he undertook the duties of Professor František Velísek who had enlisted and died in the war. Rychlík began to read base lectures alternately for students of the first and second year of study, the lecture on probability theory and the lecture on vector analysis.

From today's view, it was a pity that Rychlík remained only private associate professor at Charles University. The main subject of his research was algebra and number theory. It was possible, even necessary, to read such topics at Charles University. In fact, Rychlík was the first who introduced methods and concepts of "modern" abstract algebra in our country – by means of his published treatises as well as university lectures. Besides, as a professor there he would have had a stronger influence on the young generation of Czech mathematicians. But Rychlík spent most of his time (and energy) at the Technical University where he had to adapt his lectures for future engineers. Nevertheless, he approached his work seriously there. In addition to the usual teaching activities, he was a member of many committees, such as organization committee, inceptive committees, etc.

In 1904 Rychlík became a member of the Union of Czech Mathematicians and Physicists (below only the Union) and until World War II he was also a member of its committee. Almost the whole of his life Rychlík lectured in the Union and his lectures were very closely related to his scientific research. He was also a member of the Royal Bohemian Society of Sciences (elected on January 11, 1922), the Czech Academy of Sciences and Arts (May 23, 1924) and the Czechoslovak National Research Council under the Academy (May 19, 1925).

Besides, Rychlík took part in several congresses: *5th Congress of Czech Naturalists and Physicians* in Prague (1914; contribution [R11]),[4] *International Congress of Mathematicians* in Strasbourg (1920), *6th Congress of Czechoslovak Naturalists, Physicians and Engineers in Prague* (1928), *International Congress of Mathematicians* in Bologna (1928; contrib. [R28]), *Congress of Mathematicians of Slavonic Countries* in Warszawa (1929; contrib. [R33]) and *Second Congress of Mathematicians of Slavonic Countries* in Prague (1934; contrib. [R43]).

In 1939 all Czech universities were closed, after the war Rychlík retired. In the last period of his life Rychlík invested his energy to the history of mathematics, above all to the inheritance of Bernard Bolzano, which he had been interested in since his youth, but after the retirement he was engaged in this topic fully. Karel Rychlík died on May 28, 1968 at the age of 83.

[4] References [R...] denote the publications of Karel Rychlík; their list is given in the section 7.1, pp. 242–250.

8.3 WORK OF KAREL RYCHLÍK

8.3.1 Survey of Scientific Activities

Rychlík's works can be divided into five groups:

- Algebra and Number Theory (22 works),
- Mathematical Analysis (7),
- Textbooks, Popularization Papers and Translations (16),
- Works Devoted to Bernard Bolzano (14),
- Other Works on History of Mathematics (29).

In the Czech mathematical community, Rychlík's name is mostly related to his textbooks on elementary number theory ([R37], [R46]) and on the theory of polynomials with real coefficients [R64], which are certainly very interesting and useful, but which are not "real" scientific contributions. Worth mentioning is the less known textbook [R44] (1938) on probability theory, written for students of the technical university, yet in a very topical way: Rychlík builds the probability theory using the axiomatic method introduced by A. N. Kolmogorov in his book *Grundbegriffe der Wahrscheinlichkeitsrechnung* [10] from 1933.[5] In this context, let us also mention the popularization papers [R5] and [R6] on three special cases ($n = 3, 4, 5$) of Fermat Last Theorem, which are cited in the second volume of Dickson's trilogy *History of the Theory of Numbers* [Dic1] from 1934 and in Ribenboim's book *Fermat's Last Theorem for Amateurs* [Rib3] from 1999.[6]

Not only among mathematicians and not only in Bohemia, Rychlík is widely known as the historian of mathematics, above all in the connection with Bernard Bolzano. As far as the number of citations is concerned, this domain is unequivocally in the first place. Preparing for printing Bolzano's manuscript *Functionenlehre* [R34] and two parts of *Zahlenlehre* ([R36], [R85]), Rychlík earned the place in practically all Bolzano's bibliographies. Well-known is also Rychlík's paper [R19] containing the proof of continuity and non-differentiability

[5]See p. 163.

[6]For references concerning algebra and number theory see pp. 115–121.

of so-called Bolzano's function. Besides, in the literature devoted to Bernard Bolzano we can often find citations of Rychlík's papers concerning Bolzanos logic [R68] and the theory of real numbers ([R65], [R66]) that were based on the study of Bolzano's manuscripts. It should be emphasized that Rychlík sooner than the others made many of important surprises hidden in Bolzano's hardly readable manuscripts accessible and hence contributed to Bolzano's fame in the mathematical community.

A range of other papers on the history of mathematics more or less relates to Bolzano, too, namely the works devoted to N. H. Abel [R88], A.-L. Cauchy ([R58], [R59], [R60], [R61], [R69], [R86]) and the prize of the Royal Bohemian Society of Sciences for the problem of the solution of any algebraic equation of a degree higher than four in radicals ([R81], [R82]). Some of the remaining papers are only short reports ([R45], [R54], [R55], [R62]) or loose processing of literature ([R76], [R78]), the others contain a good deal of an original work based on primary sources, namely the papers devoted to E. Galois [R63], F. Korálek [R80], M. Lerch ([R27], [R74]), E. Noether [R71], F. Rádl ([R56], [R57]), B. Tichánek ([R25], [R75], [R79]), E. W. Tschirnhaus [R77] and F. Velísek [R20]. Moreover, Rychlík adds his own views and valuable observations, which shows his wide insight and deep interest in the history of mathematics and in mathematics itself. On October 21, 1968 the Czechoslovak Academy of Sciences awarded Rychlík in memoriam a prize for the series of 13 papers on the history of mathematics published after 1957, namely [R58], [R59], [R60], [R63], [R65], [R67], [R68], [R69], [R71], [R77], [R78], [R82], [R88].

Rychlík's algebraic works are known only to a relatively narrow circle of mathematicians. But it is just this first group, in which the most important mathematical papers of Karel Rychlík are included. These works, discussed in the section 8.3.2, were published between 1914 and 1934, that is in the period of the birth and formation of "modern" abstract algebra, and they were devoted to particularly topical problems from this domain. Regrettably only three papers of Karel Rychlík were published in a generally reputable magazine – Crelle's Journal; the most of them were published in de facto local Bohemian journals. It was certainly meritorious for enlightenment in the Czech mathematical audience, but although some of the works were written in German, they were not noticed by the mathematical community abroad, even though they were referred in *Jahrbuch über die Fortschritte der Mathematik* or *Zentralblatt für Mathematik und ihre Grenzgebiete* (nevertheless, it was not only Rychlík who published mostly for the Czech audience; in fact, this situation was common at that time in the young autonomous republic). On the other hand, Rychlík's papers published in Crelle's Journal became known and they have been cited in the literature – this concerns above all the treatise *Zur Bewertungstheorie der algebraischen Körper* [R22] from 1923, thanks to which Rychlík gained a certain position in the history of valuation theory. This paper is cited for example by R. Böffgen a M. A. Reichert ([B-R1], 1987), H. Hasse ([Has1], 1926; [H-S1], 1933), A. N. Kochubei ([Koc1], 1998), W. Krull ([Kru1], 1930; [Kru2], 1932), M. Nagata ([Nag1], 1953), W. Narkiewicz ([Nar1], 1974), A. Ostrowski ([Ost3], 1933; [Ost4], 1935), P. Ribenboim ([Rib1], 1985), P. Roquette ([Roq1], 1999), O. F. G. Schilling ([Sch1], 1950), F. K. Schmidt ([H-S1], 1933; [Scm1], 1933), W. Więsław ([Wie2], 1988) and others.

In his papers Rychlík mostly came out of a certain work (see Fig. 2.1, page 62) and gave some improvements – mainly he based definitions of the main concepts or proofs of the main theorems on another base, in the spirit of abstract algebra, and in this way he generalized or simplified them. Typical features of the papers are the brevity, conciseness, topicality as well as the "modern" way of writing (from the point of view of that time). Although the amount of these publications is not very large and they are relatively short, they give evidence of Rychlík's wide horizons, of the fact that he followed the current world mathematical literature, noticed problems and possible generalizations that later proved to be substantial, aimed for correct but as simple as possible proofs.

A bit aside stand Rychlík's occasional works on mathematical analysis. The papers [R17] and [R21] devoted to continuous non-differentiable functions in p-adic number fields are closely related to the previous group of Rychlík's publications. It is interesting that this couple of papers represents one of the first works studying p-adic analysis at all.

It shall be added that even in the citation database *Web of Science*, which monitors 8440 "valuable" journals from 1980 to the present, it is possible to find eleven citations of Rychlík's publications (twice the paper on valuation theory [R22], otherwise works concerning Bernard Bolzano: five times [R85], once [R34], [R19], [R82] and [R86]). Nevertheless, the total amount of citations after the year 1980 is greater than eleven – there are other citations in books as well as in journals and proceedings that are not monitored by the database.

Figure 1.2 at page 14 demonstrates the distribution of subjects of Rychlík's interest, their development and changes in the course of time, as well as connections of publications to other scientific activities. We can observe:

$\boxed{1}$ At the beginning of his career Rychlík wrote several works on algebra without a deeper relation to his later publications. We only note that [R3] concerning substitution groups was a dissertation, [R4] and [R7] on the theory of algebraic forms were inceptive works.

$\boxed{2}$ In 1910 – 1921 Rychlík was an editor of the magazine *Časopis pro pěstování matematiky a fyziky* with the responsibility for tasks for secondary school students, published in the supplement of this journal. In the same time five Rychlík's popularizing papers ([R5], [R6], [R8], [R9], [R18]) appeared in this supplement.

$\boxed{3}$ The most important mathematical papers by Karel Rychlík concern algebra and number theory and they originated mainly by the middle of the twenties. Precisely, by the year 1924 when the Bolzano Committee under the Royal Bohemian Society of Sciences was established – compare $\boxed{8}$. Rychlík was its member since the very beginning – and it should be emphasized that he was a very active member. The most of the remaining papers involved in the group $\boxed{3}$ have their origin – or at least inspiration – in those published earlier (compare Fig. 2.1, page 62). Rychlík's lectures at Charles University in the period 1912 – 1930 were closely related just to this domain, similarly with the lectures in the Union.

$\boxed{4}$ Seven papers belong to mathematical analysis. Among them there are two couples [R17], [R21] and [R41], [R42]) consisting of the Czech and German variant of almost the same text. Otherwise, the works on analysis are mutually independent.

Although the list of publications itself leads to the opinion that mathematical analysis was only a marginal domain of Rychlík's interest, the list of lectures shows that it was the main domain of his pedagogical duties at the Technical University. Consequently, in addition to the activities in the Bolzano Committee, it was another reason why Rychlík did not publish more algebraic papers.

$\boxed{5}$ An interesting example of mutual relations of publication activities and lecturing is related to probability theory. In the school year 1914/15 Rychlík started reading the "classical" lecture named *Probability calculus* at the Technical University. In the summer semester of the school year 1931/32 Rychlík lectured on *Probability calculus (the theory of Mises)* at Charles University. For the winter semester of the school year 1933/34 Rychlík had announced the lecture on linear algebra, but shortly before the beginning of the semester he changed the topic for *Probability calculus (from the axiomatic point of view)*; hence we can see that he immediately reacted to the publication of Kolmogorov's book [10], the first work where the probability theory was built axiomatically in today's sense. In 1938 Rychlík's textbook *Introduction to probability calculus* [R44] was published. Although it was intended for students of the Technical University, it was written in a very topical way, using Kolmogorov's axiomatic method. As far as we can judge by the textbook, the quality of Rychlík's lectures in the period before the World War II was outstanding. Towards the end of working on the textbook, in the winter semester of the school year 1936/37, Rychlík announced the lecture *Probability calculus from the axiomatic point of view* at the Charles University one more time. As it was outlined above, within the domain of probability theory we can observe the development from the classical lecture at the technical university over the study of modern trends and their modifications and improvements up to the publication of a well-elaborated textbook.

A close relation to the probability theory can also be found in the couple of Rychlík's papers [R41] and [R42] published in 1933, which are included in the group of publications on mathematical analysis. Rychlík came back to probability theory once again at the end of the World War II, when he started to translate Glivenko's book *Probability theory*; the translation [R47] was published in 1950.

$\boxed{6}$ World War II, postwar years and the beginning of a communist epoch were hard for Karel Rychlík. He did not return to the Technical nor to Charles University, the Bolzano Committee was abolished together with the Royal

Bohemian Society of Sciences in 1951. By the middle of the fifties only a short paper [R45] on the history of mathematics, the second edition [R46] of a former textbook [R37] on elementary number theory and four translations from Russian – see $\boxed{7}$ – had been published.

$\boxed{8}$ The situation became less unpleasant in 1955, when the "new" Czechoslovak Academy of Sciences officially entrusted Rychlík with organizing Bolzano's manuscript inheritance (during the following years he was sometimes given a reward for a particular work). In the same year the Czech Literary Fund, which supported old scientists and their widows, started to pay him a regular remuneration. Moreover, in 1958 the Bolzano Committee was restored under the academy (nevertheless only for three years). A glimpse at the Figure 1.2 is sufficient to notice that after many unfortunate years a fertile period suddenly came.

$\boxed{9}$ Besides the manuscripts of Bernard Bolzano, Rychlík was deeply interested in the history of mathematics in general. The most of his historical papers were devoted to a certain personality; remaining four papers are included in the group denoted by $\boxed{10}$.

8.3.2 Algebra and Number Theory

Rychlík's papers on algebra and number theory can be divided as follows.

Principal Papers

g-adic Numbers .... [R11], [R12], [R17], [R21]

Valuation Theory .... [R14], [R22]

Algebraic Numbers, Abstract Algebra .... [R15], [R16], [R23], [R24], [R26]
[R31], [R32], [R33], [R39], [R40]

Determinant Theory .... [R38], [R43]

Other Works

Groups of substitutions .... [R2], [R3]

Theory of Algebraic Forms .... [R4], [R7]

Figure 2.2 at page 68 illustrates the influences in the development of the algebraic number theory. The aim of the scheme is to show Rychlík's place there; hence there are not all existing influences – the predetermination of the figure would be covered up. It just tries to show the two main streams, the *ideal theory* represented by Richard Dedekind and his continuators, and the *divisor theory* represented by Leopold Kronecker, his student Kurt Hensel, his student Helmut Hasse and other mathematicians, including Karel Rychlík.

The survey of quotations in Rychlík's principal algebraic papers (except the two papers on determinant theory that stay a little bit aside) is given in the

Figure 2.1, page 62. It is evident that Rychlík was above all influenced by Kurt Hensel.

g-adic Numbers

In the first paper [R11] of the considered group Rychlík generalizes Hensel's ideas concerning additive and multiplicative normal form of g-adic numbers, which he extends to algebraic number fields.

The second paper [R12] is devoted to the introduction and properties of the ring of g-adic numbers. While Hensel took the way analogous to the construction of the field of real numbers by means of decimal expansions, Rychlík came out – alike Cantor – from the concepts of fundamental sequence and limit. As he notes, one of the merits of this approach is, that directly from the definition, it can be immediately seen that the ring of g-adic numbers depends only on primes contained in g, not on their powers. Of course, the idea of constructing the field of p-adic numbers (for a prime p) came from Kürschák [Kür2], who introduced the concept of *valuation*. Rychlík generalized the notion of limit in a slightly different way, closer to Hensel. Moreover, he studied comprehensively rings of g-adic numbers for a composite number g. Kürschák's paper [Kür2] is cited only in the postscript that seems to be written subsequently. It is plausible he came to the idea of the generalization of Cantor's approach independently of Kürschák.[7] In the mentioned postscript Rychlík generalized Kürschák's technique for the case of the composite number g and defined what was later called *pseudo-valuation* of a ring R.[8]

In 1920 Karel Petr published in the Czech journal *Časopis pro pěstování mathematiky a fysiky* a very simple example of a continuous non-differentiable function [41].[9] Only the knowledge of the definition of continuity and derivative and a simple arithmetic theorem is necessary to understand both the construction and the proof of continuity and non-differentiability of the function.

Petr's function is defined in the interval $[0, 1]$ as follows:

$$\text{if} \quad x = \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} + \cdots; \quad a_k \in \{0, 1, \ldots, 9\},$$

$$\text{then} \quad f(x) = \frac{b_1}{2^1} \pm \frac{b_2}{2^2} \pm \frac{b_3}{2^3} \pm \frac{b_4}{2^4} \pm \cdots; \quad b_k = \begin{cases} 0 & \text{for even } a_k, \\ 1 & \text{for odd } a_k, \end{cases} \tag{8.1}$$

[7] At least since 1909, when he lectured in the Union *On Algebraic Numbers according to Kurt Hensel*, Rychlík had been involved in this topic and was trying to improve Hensel's ideas – here the solid foundation of the basic concepts was in the first place.

[8] It is almost unknown but interesting that Rychlík defined this concept 20 years before the publication of Mahler's paper [Mah1], which is usually considered as a work where the general pseudo-valuation (Pseudobewertung) was introduced. At the end of the paper [Mah2] K. Mahler himself remarked that pseudo-valuations had already appeared in the work [Deu1] of M. Deuring (chap. VI, §10, 11) published in 1935, namely for hypercomplex systems, but he had found it out after the printing of the previous paper [Mah1].

[9] See page 144.

the sign before b_{k+1} is opposite than the one before b_k for $a_k \in \{1,3,5,7\}$, the same otherwise.

The graph of an approximation of Petr's function can be seen in the left part of the picture 3.5 (see page 139). To show it more graphically, a four-adic number system was used. Comparing with the graph on the right, the necessity of the exception to the rule of sign assignment awarded to the digit 9 can be understood; the result would not be a continuous function.

In the same year and in the same journal Karel Rychlík generalized Petr's function in his paper [R17]; the German version [R21] with the same content was published two years later in Crelle's journal. Rychlík carried the function from the real number field $\mathbb{R}$ to the field of p-adic numbers $\mathbb{Q}_p$:

$$\begin{aligned} x &= a_r p^r + a_{r+1} p^{r+1} + \cdots, \quad r \in \mathbb{Z}, \quad a_i \in \{0,1,\ldots,p-1\}; \\ f(x) &= a_r p^r + a_{r+2} p^{r+2} + a_{r+4} p^{r+4} + \cdots . \end{aligned} \tag{8.2}$$

The proof that the function (8.2) is continuous in $\mathbb{Q}_p$, but has a derivative nowhere in this field, is rather elementary. At the end Rychlík remarks it would be possible to follow the same considerations in any field of p-adic algebraic numbers (introduced by K. Hensel) subsistent to the algebraic number field of a finite degree over $\mathbb{Q}$.

We shall remark that this Rychlík's work was one of the first published papers dealing with p-adic continuous functions. In Hensel's [Hen12] some elementary p-adic analysis can be found, but otherwise it was developed much later.

Valuation Theory

In his paper *Über Limesbildung und allgemeine Körpertheorie* [Kür2] [10] József Kürschák introduced the concept of *valuation* (*Bewertung*) as a mapping $\|\cdot\|$ of a given field K into the set of non-negative real numbers, satisfying the following conditions:

$$\forall a \in K,\ a \neq 0 : \quad \|a\| > 0; \quad \|0\| = 0, \tag{V1}$$
$$\forall a \in K : \quad \|1 + a\| \leq 1 + \|a\|, \tag{V2}$$
$$\forall a, b \in K : \quad \|a \cdot b\| = \|a\| \cdot \|b\|, \tag{V3}$$
$$\exists a \in K : \quad \|a\| \neq 0, 1. \tag{V4}$$

The main result of Kürschák's paper is the proof of the following theorem.

THEOREM 1. Every valued field K can be extended to a complete, algebraically closed valued field.

[10] The results of this paper were first briefly outlined in Kürschák's contribution *Über Limesbildung und allgemeine Körpertheorie* at the *Fifth International Congress of Mathematicians* in Cambridge in 1912, published one year later as [Kür1].

First, Kürschák constructs the completion of K in the sense of fundamental sequences; it is not difficult to extend the valuation from K to its completion. Then he extends the valuation from the complete field to its algebraic closure. Finally he proves that the completion of the algebraic closure is algebraically closed. The most difficult step is the second one. Kürschák shows that if α is a root of a monic irreducible polynomial

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n, \quad a_i \in K \quad (a_n = \pm N\alpha), \tag{8.3}$$

it is necessary to define its value as $\|\alpha\| = \|a_n\|^{\frac{1}{n}}$. To prove that this is the valuation, the most laborious and lengthy point is the verification of the condition (V2). For this purpose Kürschák generalizes Hadamard's results concerning power series in the complex number field. Nevertheless, at the beginning of his paper Kürschák remarks that in all cases, where instead of the condition (V2) a stronger condition

$$\|a + b\| \leq \mathrm{Max}(\|a\|, \|b\|) \quad \text{for all } a, b \in K \tag{V2'}$$

holds, i.e. for *non-archimedean valuations*, it is possible to generalize Hensel's considerations concerning the decomposition of polynomials over $\mathbb{Q}_p$, especially the assertion, later called *Hensel's Lemma*:

LEMMA 2. Let $\|\cdot\|$ be a non-archimedean valuation defined on a complete field K'. If the polynomial (8.3) is irreducible and $\|a_n\| \leq 1$, then it is also $\|a_i\| \leq 1$ for all coefficients a_i, $1 \leq i \leq n-1$.

Kürschák didn't prove Hensel's Lemma for a field with a non-archimedean valuation – he wrote he had not succeeded in its generalization for all cases, it means for archimedean valuations, too. So he turned to the unified proof based on Hadamard's theorems, valid for all valuations.

Alexander Ostrowski proved in his paper [Ost2] that every field K with an archimedean valuation is isomorphic to a certain subfield of the complex number field $\mathbb{C}$ in the way that for every $a \in K$ and the corresponding $\overline{a} \in \overline{K}$ it is $\|a\| = |\overline{a}|^\rho$, where $|\cdot|$ is the usual absolute value on $\mathbb{C}$, $0 < \rho < 1$, ρ does not depend on a (such valuations are called *equivalent*). In other words, up to isomorphism, the only complete fields for an archimedean valuation are $\mathbb{R}$ and $\mathbb{C}$, where the problem of the extension of valuation is trivial. Hence it is possible to restrict the considerations only to non-archimedean valuations and use the generalization of Hensel's Lemma.

And precisely this was into full details done by Karel Rychlík in [R14] and [R22]. The second paper is the German variant of the first one written in Czech with practically the same content. But only the German work became wide known – thanks to its publication in Crelle's journal, while its Czech original was not noticed by the mathematical community abroad. The paper [R22] is cited e.g. by R. Böffgen, H. Hasse, A. N. Kochubei, W. Krull, M. Nagata, W. Narkiewicz, A. Ostrowski, P. Ribenboim, P. Roquette, O. F. G. Schilling, F. K. Schmidt and W. Więsław (see page 269). In the connection with some variant of the above lemma, Rychlík's name is mentioned also without the explicite citation of the work; see the papers of I. Kaplansky [Kap1], J. Eršov [Ers1], I. Efrat and M. Jarden [E-M1], the recent book [Rib2] of P. Ribenboim etc.

Theory of Algebraic Numbers, Abstract Algebra

The papers included in this group were published in Czech journals, in Czech or German, and remained almost unknown outside Bohemia. They are, nevertheless, very interesting and manifest Rychlík's wide horizons as well as the fact that he followed the latest development in the theory, studied the current mathematical literature, noticed problems or possible generalizations that later turned out to be important. Let us only mention that in his papers we can find the definition of divisors in algebraic number fields via a factor group and an external direct product, introduction of divisibility via the concept of a semi-group and other ideas; more details can be found in author's paper *Life and Work of Karel Rychlík* (see footnote 2).

Determinant Theory

The first of the couple of Rychlík's papers devoted to determinants [R38], published in Crelle's journal in 1931, concerns the assertion that the determinant of a matrix $A \in K^{n\times n}$, $n > 1$, where two rows or columns are identical, is zero. It can be easily proved for the case that the characteristic of the given field K is different from 2. Rychlík cites the book [Has2] of H. Hasse, where a completely general proof using the Laplace's "Entwicklungssatz" is given. Rychlík gives a simple proof of the considered assertion just for the field K of characteristic 2. He steps as follows. Consider the determinant of a matrix $X = (x_{ij})$ as a polynomial over $\mathbb{Z}$ in indeterminates x_{ij}. If a matrix X^* has two identical rows (columns), then it is $|X^*| = 0$ in a ring which arises from $\mathbb{Z}$ by adjunction of the elements of X^*; it is also $|X^*| \equiv 0 \pmod 2$. This implies $|X^*| = 0$ in a ring which arises from a prime field of K by the adjunction of the elements of X^*, hence also in a ring which arises from K by this adjunction. If A is a matrix with elements of K and with two identical rows (columns), then the determinant $|A|$ is received from $|X^*|$ by substituting the elements of A for the elements of X^*, so it is $|A| = 0$.

This Rychlík's paper didn't remain completely unknown – it was cited for example by O. Haupt in the third edition of his *Einführung in die Algebra I* [Hau1] from 1956.

The second paper [R43] published in 1934 is written in Czech and it comes out of the paper [Pet1] of K. Petr, where the determinant theory is based on the definition of a determinant as an alternating m-linear form. Rychlík generalizes Petr's considerations for the case of an arbitrary field K of an arbitrary characteristic. For this purpose it is necessary to give a suitable definition of an alternating m-linear form (equivalent to Petr's one for fields of characteristics different from 2).