1.10. Exercises

1. $A \cup \emptyset = A$; $A \cup A = A$; $A \cap \emptyset = \emptyset$; $A \cap A = A$.
2. $A \cup (A \cap B) = A$; $A \cap (A \cup B) = A$.
3. If $A \subset B$, then $A \cup B = B$, $A \cap B = A$; conversely, if either of these equalities is correct, then $A \subset B$.
4. $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$.
5. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$; $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
6. A set of a finite number n ($\geqq 0$) of elements has 2^n subsets.
7. The Cartesian product of a set of m ($\geqq 0$) elements and a set of n ($\geqq 0$) elements consists of $m \cdot n$ elements.
8. A part of the Cartesian product $A \times B$ is not necessarily the Cartesian product of a subset of A and a subset of B.

2. Decompositions (partitions) in sets

2.1. Decompositions in a set

Let G stand, throughout the book, for an arbitrary nonempty set.

A *decomposition* (*partition*) *in* G is a nonempty system of nonempty and mutually disjoint subsets of G.

This notion is one of the most important in this book and is, in fact, essential to the theory of groupoids and groups we intend to develop in the following chapters.

Every decomposition in G has therefore at least one element, each of its elements is a nonempty subset of G and, of course, the intersection of any two of its elements is empty.

A simple example of a decomposition in, let us say, the set of all positive integers is the system consisting of one single element, namely of the set of all even positive integers. More generally: the system consisting of a single element which is a nonempty subset of G is a decomposition in G. The system of sets [4] in part 1.1 is an example of a decomposition in the set of all positive integers $\geqq 2$.

2.2. Decompositions on a set

Let $\bar{A}$ be an arbitary decomposition in G. Any point of G may lie at most in one element of $\bar{A}$, since every two elements of $\bar{A}$ are disjoint; it may, of course, happen that it does not lie in any element of $\bar{A}$.

If the decomposition $\bar{A}$ is such that each point of G lies in some element of $\bar{A}$ so that $\bar{A}$ covers G, then $\bar{A}$ is called a decomposition *on* G or a decomposition *of* G. The last of the above examples is a decomposition on the set of all positive integers $\geqq 2$; in fact, every natural number $\geqq 2$ is either a prime number or a product of several prime numbers and therefore lies in some element of this decomposition.

Important examples of decompositions on G are the two so-called *extreme decompositions of* G, namely the *greatest* and the *least decomposition of* G. The greatest decomposition of G, denoted by $\bar{G}_{\max}$, consists of a single element, G. The least decomposition of G, $\bar{G}_{\min}$, is the system of all one-point sets $\{a\}$, $a \in G$.

For example, the set whose only element consists of all positive integers is the greatest decomposition of the set of all positive integers, whereas the system of all one-point sets each of which consists of a single positive integer is its least decomposition.

Note that an arbitrary decomposition $\bar{A}$ *in* G is a decomposition *on* the set $\mathbf{s}\bar{A}$. Consequently, the results concerning the properties of decompositions on sets can often be employed to describe the characteristics of decompositions in sets.

2.3. Closures and intersections

Let $\bar{A}$ stand for an arbitrary decomposition in G and B for a subset of G.

The *closure of the subset* B *in the decomposition* $\bar{A}$ is the set of all the elements of $\bar{A}$ that are incident with B. Notation: $B \sqsubset \bar{A}$ or $\bar{A} \sqsupset B$. Since every element $\bar{a} \in \bar{A}$ that is incident with B simultaneously cuts the subset $B \cap \mathbf{s}\bar{A}$ and conversely, there holds $B \sqsubset \bar{A} = (B \cap \mathbf{s}\bar{A}) \sqsubset \bar{A}$, We observe that the closure $B \sqsubset \bar{A}$ is a part of the decomposition $\bar{A}$ which may coincide with $\bar{A}$ or may even be empty. The first case, $B \sqsubset \bar{A} = \bar{A}$, occurs if and only if each element of $\bar{A}$ is incident with B. The second case, $B \sqsubset \bar{A} = \emptyset$, occurs if and only if no element of $\bar{A}$ is incident with B; it is characterized by the equality $B \cap \mathbf{s}\bar{A} = \emptyset$. When the closure $B \sqsubset \bar{A}$ is not empty, it is a decomposition in G.

The *intersection of the decomposition* $\bar{A}$ *and the subset* B or *of the subset* B *and the decomposition* $\bar{A}$ is the set of all nonempty intersections of the elements of $\bar{A}$ and the subset B. Notation: $\bar{A} \sqcap B$ or $B \sqcap \bar{A}$. It is obvious that even the intersection $\bar{A} \sqcap B$ may be empty. That occurs if and only if no element of $\bar{A}$ is incident with B; this case is, as we have mentioned above, characterized by the equality $B \cap \mathbf{s}\bar{A} = \emptyset$. If the intersection $\bar{A} \sqcap B$ is not empty, it is a decomposition in the set G and even in the set B. Note that $A \sqcap \bar{B}$ is, at the same time, a decomposition on the set $B \cap \mathbf{s}\bar{A}$. There obviously holds: $\mathbf{s}(\bar{A} \sqcap B) = B \cap \mathbf{s}\bar{A}$.

To sum up: The closure $B \sqsubset \bar{A}$ and the intersection $B \sqcap \bar{A}$ are simultaneously either nonempty or empty systems of sets according as there holds $B \cap \mathbf{s}\bar{A} \neq \emptyset$ or $B \cap \mathbf{s}\bar{A} = \emptyset$. If $B \neq \emptyset$ and $\bar{A}$ covers G, then $B \sqsubset \bar{A}$ and $B \sqcap \bar{A}$ are nonempty systems, the former being a part of $\bar{A}$ and the latter a decomposition on B. Every decomposition

$\bar{A}$ on G and a nonempty set B in G therefore determine: 1) a certain nonempty subset of $\bar{A}$, namely, the closure $B \sqsubset \bar{A}$, 2) a certain decomposition on B, namely, the intersection $A \sqcap \bar{B}$.

The notions of a closure and an intersection described above will now be extended to the case when the subset $B \subset G$ is replaced by a decomposition in G. Thus it will be a question of the closure of a decomposition in a decomposition and of the intersection of two decompositions.

Let $\bar{A}$, $\bar{B}$ denote decompositions in G.

The *closure of the decomposition* $\bar{B}$ *in* $\bar{A}$ is the set of all elements $\bar{a} \in \bar{A}$ such that $\bar{a}$ is incident with some element of $\bar{B}$. Notation: $\bar{B} \sqsubset \bar{A}$ or $\bar{A} \sqsupset \bar{B}$. Since every element $\bar{a} \in \bar{A}$ incident with an element of $\bar{B}$ is simultaneously incident with $\mathbf{s}\bar{B}$, and vice versa, there holds: $\bar{B} \sqsubset \bar{A} = \mathbf{s}\bar{B} \sqsubset \bar{A}$. This relation reduces the new concept of a closure to the notion of the closure of a subset in a decomposition. If the decomposition $\bar{B}$ consists of one single element B, then of course: $\bar{B} \sqsubset \bar{A} = B \sqsubset \bar{A}$.

The *intersection of the decomposition* $\bar{A}$ *and the decomposition* $\bar{B}$ is the set of all nonempty intersections of the individual elements of $\bar{A}$ and the elements of $\bar{B}$. Notation: $\bar{A} \sqcap \bar{B}$. It is obvious that $\bar{A} \sqcap \bar{B} = \bar{B} \sqcap \bar{A}$; with respect to this symmetry, we also speak about the intersection of the decompositions $\bar{A}$ and $\bar{B}$. Since, for every two elements $\bar{a} \in \bar{A}, \bar{b} \in \bar{B}$, there holds $\bar{a} \cap \bar{b} = (\bar{a} \cap \mathbf{s}\bar{B}) \cap (\bar{b} \cap \mathbf{s}\bar{A})$, we have: $\bar{A} \sqcap \bar{B} = (\bar{A} \sqcap \mathbf{s}\bar{B}) \sqcap (\bar{B} \sqcap \mathbf{s}\bar{A})$. Either system $\bar{A} \sqcap \mathbf{s}\bar{B}$, $\bar{B} \sqcap \mathbf{s}\bar{A}$ is a decomposition on $\mathbf{s}\bar{A} \cap \mathbf{s}\bar{B}$ or is empty according as there holds $\mathbf{s}\bar{A} \cap \mathbf{s}\bar{B} \neq \emptyset$ or $= \emptyset$. We see that, in case of $\mathbf{s}\bar{A} \cap \mathbf{s}\bar{B} \neq \emptyset$, the intersection of the decompositions $\bar{A}$ and $\bar{B}$ coincides with the intersection of the decompositions $\bar{A} \sqcap \mathbf{s}\bar{B}$ and $\bar{B} \sqcap \mathbf{s}\bar{A}$ which are on $\mathbf{s}\bar{A} \cap \mathbf{s}\bar{B}$, whereas, in case of $\mathbf{s}\bar{A} \cap \mathbf{s}\bar{B} = \emptyset$, it is empty. Note that the intersection of any two decompositions lying on the same set is always a decomposition of the latter. If $\bar{B}$ consists of one single element B, then of course, $\bar{A} \sqcap \bar{B} = \bar{A} \sqcap B$.

2.4. Coverings and refinements of a decomposition

Let $\bar{A}$, $\bar{B}$, $\bar{C}$ be decompositions in G.

The decomposition $\bar{A}$ ($\bar{B}$) is called a *covering* (*refinement*) *of the decomposition* $\bar{B}$ ($\bar{A}$) if every element of $\bar{B}$ is a part of some element of $\bar{A}$. Then we write $\bar{A} \geqq \bar{B}$ or $\bar{B} \leqq \bar{A}$. For example, the greatest decomposition of G is a covering of $\bar{A}$ and the least decomposition of $\mathbf{s}\bar{A}$ is a refinement of $\bar{A}$: $\bar{G}_{\max} \geqq \bar{A}$, $\mathbf{s}\bar{A}_{\min} \leqq \bar{A}$. In particular, ($A = B$), the decomposition $\bar{A}$ is its own covering and refinement. If $\bar{A} \geqq \bar{B}$ and, at the same time, $\bar{A} \neq \bar{B}$, then $\bar{A}$ ($\bar{B}$) is said to be a *proper covering* (*refinement*) *of* $\bar{B}$ ($\bar{A}$). Notation: $\bar{A} > \bar{B}$ or $\bar{B} < \bar{A}$.

From the definition of the meaning of the symbol $\geqq$ *there follows*:

a) $\bar{A} \geqq \bar{A}$;
b) *if* $\bar{A} \geqq \bar{B}$, $\bar{B} \geqq \bar{C}$, *then* $\bar{A} \geqq \bar{C}$;
c) *if* $\bar{A} \geqq \bar{B}$, $\bar{B} \geqq \bar{A}$, *then* $A = \bar{B}$.

The statements a) and b) are evidently true. To prove c), we proceed as follows: Suppose there holds $\bar{A} \geqq \bar{B}$, $\bar{B} \geqq \bar{A}$. Let $\bar{b} \in \bar{B}$ stand for an arbitrary element. Then there exist elements $\bar{a} \in \bar{A}$, $\bar{b}' \in \bar{B}$ such that $\bar{b}' \supset \bar{a} \supset \bar{b}$. Consequently, the sets $\bar{b}'$, $\bar{b}$ are incident and therefore, as they are elements of the same decomposition $\bar{B}$, identical. So we have $\bar{b}' = \bar{a} = \bar{b}$ and, in fact, $\bar{b} \in \bar{A}$. Accordingly, $\bar{B}$ is a part of $\bar{A}$, i.e., $\bar{B} \subset \bar{A}$ and, analogously, $\bar{A}$ is a part of $\bar{B}$, $\bar{A} \subset \bar{B}$. Hence $\bar{A} = \bar{B}$.

If $\bar{A} \geqq \bar{B}$, then every element of $\bar{B}$ is a subset of some element of $\bar{A}$; but $\bar{A}$ may contain elements in which there does not lie any element of $\bar{B}$. If such elements of $\bar{A}$ do not exist, i.e., if every element of $\bar{A}$ contains, as a subset, some element of $\bar{B}$, then the decomposition $\bar{A}$ $(\bar{B})$ is said to be a *normal covering* (*refinement*) *of* $\bar{B}$ $(\bar{A})$; if, moreover, every element of $\bar{A}$ is the sum of some elements of $\bar{B}$, then $\bar{A}$ $(\bar{B})$ is called a *pure covering* (*refinement*) *of* $\bar{B}$ $(\bar{A})$.

The decomposition $\bar{A}$ $(\bar{B})$ is a pure covering (pure refinement) of $\bar{B}$ $(\bar{A})$ only if both the decompositions $\bar{A}$, $\bar{B}$ lie on the same set $\mathbf{s}\bar{A} = \mathbf{s}\bar{B}$; conversely, the relations $\bar{A} \geqq \bar{B}$, $\mathbf{s}\bar{A} = \mathbf{s}\bar{B}$ express that $\bar{A}$ $(\bar{B})$ is a pure covering (pure refinement) of $\bar{B}$ $(\bar{A})$. Note that *if* $\bar{A}$, $\bar{B}$ *lie on* G *and* $\bar{A} \geqq \bar{B}$, *then* $\bar{A}$ $(\bar{B})$ *is a pure covering* (*pure refinement*) *of* $\bar{B}$ $(\bar{A})$.

Let us now assume the decomposition $\bar{A}$ $(\bar{B})$ to be a pure covering (pure refinement) of $\bar{B}$ $(\bar{A})$, hence $\bar{A} \geqq \bar{B}$, $\mathbf{s}\bar{A} = \mathbf{s}\bar{B}$. Then every element $\bar{a} \in \bar{A}$ is the sum of some elements of $\bar{B}$ and, evidently, the system of these element is a decomposition of $\bar{a}$. It is also clear that the system of all subsets of $\bar{B}$, each of which consists of all the elements of $\bar{B}$ that are parts of the same element of $\bar{A}$, is a certain decomposition $\bar{\bar{B}}$ of $\bar{B}$. The decomposition $\bar{\bar{B}}$, on the other hand, determines the decomposition $\bar{A}$; the latter is formed by summing all the elements of $\bar{B}$ that lie in the same element of $\bar{\bar{B}}$. The decomposition $\bar{\bar{B}}$ is said to *enforce* the covering $\bar{A}$. So we can say that $\bar{B}$ is obtained from $\bar{A}$ if every element of $\bar{A}$ is replaced by its own convenient decomposition; conversely, $\bar{A}$ is obtained from $\bar{B}$ by choosing a suitable decomposition of $\bar{B}$, i.e. $\bar{\bar{B}}$, and summing all the elements of $\bar{B}$ that lie in the same element of $\bar{\bar{B}}$.

2.5. Chains of decompositions

Let $A \supset B$ be nonempty subsets of G.

A *chain of decompositions from* A *to* B *in* G, briefly: a *chain from* A *to* B, is an α-membered ($\alpha \geqq 1$) sequence of decompositions $\bar{K}_1, \ldots, \bar{K}_\alpha$ in G with the following properties: 1) $\bar{K}_1$ lies on A; 2) $\bar{K}_{\gamma+1}$ lies on an element of $\bar{K}_\gamma$ for $1 \leqq \gamma \leqq \alpha - 1$; 3) $B \in \bar{K}_\alpha$. Notation:

$$\bar{K}_1 \to \ldots \to \bar{K}_\alpha, \text{ briefly, } [\bar{K}].$$

The set $\mathbf{s}\bar{K}_\gamma$ is, for $1 \leqq \gamma \leqq \alpha$, denoted by $\bar{a}_\gamma$; so we have, in particular, $\bar{a}_1 = A$. Moreover, we put $\bar{a}_{\alpha+1} = B$. From the definition of a chain, we have $\bar{a}_2 \in \bar{K}_1, \ldots,$

$\bar{a}_{\alpha+1} \in \bar{K}_\alpha$ and, furthermore: $A = \bar{a}_1 \supset \cdots \supset \bar{a}_{\alpha+1} = B$. The sets A, B are called the *ends of the chain* $[\bar{K}]$. We see that any element of the decomposition $\bar{K}_\alpha$ may be an end of $[\bar{K}]$. $\bar{K}_1, \ldots \bar{K}_\alpha$ are called *members of the chain* $[\bar{K}]$; $\bar{K}_1$ and $\bar{K}_\alpha$ are the *initial* and the *final member of* $[\bar{K}]$, respectively. By the *length of* $[\bar{K}]$ we understand the number α of the members of $[\bar{K}]$.

An important type of chains are the so-called elementary chains over a decomposition.

Let $\bar{A}$ be a decomposition in G and B an element of $\bar{A}$ so that $B \in \bar{A}$. Denote $A = s\bar{A}$.

An *elementary chain of decompositions from A to B over $\bar{A}$*, briefly: an *elementary chain over $\bar{A}$*, is a chain of decompositions from A to B,

$$([\mathring{K}] =)\quad \mathring{K}_1 \to \cdots \to \mathring{K}_\alpha$$

such that, for $1 \leqq \gamma \leqq \alpha$, the member $\mathring{K}_\gamma$ is a covering of the decomposition $\bar{A}_\gamma = \bar{A} \sqcap \bar{a}_\gamma$, where $\bar{a}_\gamma = s\mathring{K}_\gamma$ ($\bar{a}_1 = A$).

In such a chain, first of all, the member $\mathring{K}_1$ is a covering of the decomposition $\bar{A}_1$ $(= \bar{A})$. The set $\bar{a}_2$, determined by $B \subset \bar{a}_2 \in \mathring{K}_1$, is a subset of $\bar{a}_1$ and, on $\bar{a}_2$, there is the decomposition $(\bar{A}_2 =)$ $\bar{A} \sqcap \bar{a}_2$. There holds: $B \in \bar{A}_2 \subset \bar{A}_1$. Furthermore, $\mathring{K}_2$ is a covering of $\bar{A}_2$. The set $\bar{a}_3$, determined by $B \subset \bar{a}_3 \in \mathring{K}_2$, is a subset of $\bar{a}_2$ and, on $\bar{a}_3$, there is the decomposition $(\bar{A}_3 =)$ $\bar{A} \sqcap \bar{a}_3$. There holds: $B \in \bar{A}_3 \subset \bar{A}_2$. Next, $\mathring{K}_3$ is a covering of $\bar{A}_3$, etc. Finally, the member $\mathring{K}_\alpha$ is a covering of $(\bar{A}_\alpha =)$ $\bar{A} \sqcap \bar{a}_\alpha$ and there holds $B \in \mathring{K}_\alpha$.

For example, the chain consisting of a single decomposition $\bar{A}$ is an elementary chain from A to B over $\bar{A}$, of length 1. Putting, before or after $\bar{A}$, an arbitrary finite number of greatest decompositions of A or of B, respectively, we again obtain an elementary chain of decompositions from A to B over $\bar{A}$.

Let us now consider a chain $([\bar{K}] =)$ $\bar{K}_1 \to \cdots \to \bar{K}_\alpha$ from A to B using the above notation.

If $\bar{K}_\gamma$ $(\gamma = 1, 2, \ldots, \alpha)$ is not the greatest decomposition on $\bar{a}_\gamma$, i.e., if $\bar{a}_{\gamma+1}$ is a proper subset of $\bar{a}_\gamma$, then $\bar{K}_\gamma$ is called an *essential member of the chain* $[\bar{K}]$. In the opposite case, $\bar{K}_\gamma$ is an *inessential member of* $[\bar{K}]$. If the chain $[\bar{K}]$ contains at least one inessential member $\bar{K}_\gamma$, then $[\bar{K}]$ is called a *chain with iteration*, since $\bar{a}_{\gamma+1} = \bar{a}_\gamma$. If all the members of $[\bar{K}]$ are essential, then $[\bar{K}]$ is said to be a *chain without iteration*. The number α' of essential members of $[K]$ is the so-called *reduced length of* $[\bar{K}]$. There evidently holds $0 \leqq \alpha' \leqq \alpha$; the equality $\alpha' = \alpha$ is characteristic of a chain without iteration. If $A = B$, then all the members of $[\bar{K}]$ are inessential so that $\alpha' = 0$ and conversely. If $A \neq B$, then $[\bar{K}]$ may, by omitting all its inessential members, be *reduced*, i.e., shortened to a certain chain $[\bar{K}']$ without iteration. The length of the reduced chain $[\bar{K}']$ equals the reduced length α' of $[\bar{K}]$. The chain $[\bar{K}]$ may, on the other hand, be *lengthened* by way of inserting, between arbitrary members $\bar{K}_\gamma, \bar{K}_{\gamma+1}$ or before the initial member $\bar{K}_1$ (after the final member $\bar{K}_\alpha$) of $[\bar{K}]$, the greatest decomposition of the set $\bar{a}_{\gamma+1}$, or $\bar{a}_1$ $(\bar{a}_{\alpha+1})$ or an arbitrary finite member of such decompositions. Every shortening or lengthening of the chain $[\bar{K}]$ may be realized by gradually omitting or adding, respectively, one greatest decomposition lying on some of the subsets $\bar{a}_1, \ldots, \bar{a}_{\alpha+1}$. It is clear that every chain formed by shortening or lengthening of $[\bar{K}]$ has the same reduced length as $[\bar{K}]$.

A *refinement* $[\mathring{K}]$ *of* $[\bar{K}]$ is a chain of decompositions in G, with arbitrary ends, A_0, B_0, satisfying the relations $A_0 \supset A \supset B \supset B_0$, i.e., a chain of the following type:

$$\begin{aligned} &\bar{K}_{0,0} \to \bar{K}_{1,1} \to \cdots \to \bar{K}_{1,\beta_1-1} \to \bar{K}_{1,\beta_1} \to \bar{K}_{2.1} \\ &\quad \to \cdots \to \bar{K}_{2,\beta_2-1} \to \bar{K}_{2,\beta_2} \to \cdots \to \bar{K}_{\alpha,\beta_\alpha} \to \bar{K}_{\alpha+1,1} \\ &\quad \to \cdots \to \bar{K}_{\alpha+1,\beta_\alpha+1-1} \to \bar{K}_{\alpha+1,\beta_\alpha+1} \to \bar{K}_{\alpha+2,1} \\ &\quad \to \cdots \to \bar{K}_{\alpha+2,\beta_\alpha+2-1}. \end{aligned}$$

In this formula, first, $\beta_1, \beta_2, \ldots, \beta_{\alpha+2}$ stand for positive integers. Furthermore:

$$\begin{aligned} &([\bar{K}'] =)\ \bar{K}_{\delta,\beta\delta} \to \ldots \to \bar{K}_{\delta+1,\beta\delta+1-1} \\ &(\delta = 0, \ldots, \alpha + 1;\ \beta_0 = 0) \end{aligned}$$

is a chain of decompositions in G (if $\beta_{\delta+1} = 1$, read only the initial member $\bar{K}_{\delta,\beta_\delta}$) that varies according to the value of δ: For $\delta = 0$, it is a chain from $\bar{a}_0$ $(= A_0)$ to $\bar{a}_1$ $(= A)$ which need not occur if $A_0 = A$; for $\delta = 1, \ldots, \alpha$, an elementary chain from $\bar{a}_\delta$ to $\bar{a}_{\delta+1}$ over the decomposition $\bar{K}_\delta$; for $\delta = \alpha + 1$, a chain from $\bar{a}_{\alpha+1}(= B)$ to $\bar{a}_{\alpha+2}$ $(= B_0)$ which need not occur if $B = B_0$.

We observe that any refinement of the chain $[\bar{K}]$ is obtained by replacing every member $\bar{K}_\gamma$ $(\gamma = 1, \ldots, \alpha)$ of $[\bar{K}]$ by an elementary chain from $\bar{a}_\gamma$ to $\bar{a}_{\gamma+1}$ over $\bar{K}_\gamma$ and, if convenient, adding before the initial member $\bar{K}_1$ or after the final member $\bar{K}_\alpha$ of $[\bar{K}]$ an arbitrary chain with the last member A or the first member B, respectively.

In particular, if every member of the chain $[\bar{K}]$ is replaced by the elementary chain formed by this member only, we again obtain the chain $[\bar{K}]$. Consequently, the chain $[\bar{K}]$ is its own refinement.

The reduced length of every refinement of $[\bar{K}]$ is the sum of the reduced lengths of the single elementary chains and the mentioned added chains; it therefore equals, at least, the reduced length of $[\bar{K}]$.

2.6. Exercises

1. $\mathbf{s}A \sqsubset \bar{A} = \bar{A} = \mathbf{s}A \sqcap \bar{A}$.
2. $\mathbf{s}(B \sqsubset \bar{A}) \sqsubset \bar{A} = B \sqsubset \bar{A}$;
   $\mathbf{s}(B \sqcap \bar{A}) \sqcap \bar{A} = B \sqcap \bar{A}$;
   $\mathbf{s}(B \sqsubset \bar{A}) \sqcap \bar{A} = B \sqsubset \bar{A} = \mathbf{s}(B \sqcap \bar{A}) \sqsubset \bar{A}$.

3. If $B \sqsubset \bar{A} = B \sqcap \bar{A}$, then for every element $\bar{a} \in \bar{A}$ there holds either $\bar{a} \subset B$ or $\bar{a} \cap B = \emptyset$; and conversely.
4. $s(s\bar{A} \sqsubset \bar{C}) \sqcap \bar{A} = s\bar{C} \sqcap \bar{A}$.
5. If $B \supset C$, then there holds: a) $(C \sqsubset \bar{A}) \sqcap B = C \sqsubset (\bar{A} \sqcap B)$. With regard to this equality, the set on either side of the latter may be denoted by $C \sqsubset \bar{A} \sqcap B$. In particular, for $C = B$, we have $(B \sqsubset \bar{A}) \sqcap B = \bar{A} \sqcap B$; b) $(B \sqsubset \bar{A}) \sqcap C = \bar{A} \sqcap C$.
6. If one of the following three statements is true, then the remaining two are true as well: a) Every element of the decomposition $\bar{A}$ is incident with at least one element of the decomposition $\bar{C}$; b) $\bar{A} = \bar{C} \sqsubset \bar{A}$; c) $s\bar{A} = s(s\bar{C} \sqsubset \bar{A})$.
7. Every lengthening of an arbitrary chain of decompositions is simultaneously its refinement.
8. The number p_{n+1} of decompositions of every finite set of order $n + 1$ $(\geqq 1)$ is finite. The numbers p_{n+1} are given by the formula:

$$p_{n+1} = \sum_{\nu=1}^{n} \binom{n}{\nu} p_\nu \qquad (p_0 = 1).$$

So we have, in particular:

$$p_1 = 1, \ p_2 = 2, \ p_3 = 5, \ p_4 = 15, \ p_5 = 52, \ p_6 = 203, \ \ldots$$

3. Decompositions on sets

In this chapter we shall deal with decompositions on sets. The results are often useful (see: 2.2) when we are to describe the properties of decompositions in sets; in fact: a decomposition $\bar{A}$ in the set G is, simultaneously, a decomposition on the set $s\bar{A}$.

3.1. Bindings in decompositions

Let $\bar{A}$, $\bar{B}$ stand for decompositions of G.

Consider two arbitrary elements $\bar{a}, \bar{p} \in \bar{A}$.

A *binding from $\bar{a}$ to $\bar{p}$ in $\bar{A}$ with regard to $\bar{B}$* is a finite sequence of elements of $\bar{A}$:

$$\bar{a}_1, \ldots, \bar{a}_\alpha \quad (\alpha \geqq 2)$$

such that $\bar{a}_1 = \bar{a}$, $\bar{a}_\alpha = \bar{p}$ and that every two neighbouring members $\bar{a}_\beta$, $\bar{a}_{\beta+1}$ $(\beta = 1, \ldots, \alpha - 1)$ are incident with the same element $\bar{b}_\beta \in \bar{B}$. Such a binding is said to be *generated* by the decomposition $\bar{B}$; we speak, briefly, about the *binding $\{\bar{A}, \bar{B}\}$ from $\bar{a}$ to $\bar{p}$.*