14. Generating decompositions

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$k = 1$ it is an automorphism of $\mathfrak{Z}$ and for $k = 0$ an operator but not a meromorphic mapping of $\mathfrak{Z}$.

The simplest example of an automorphism of any groupoid $\mathfrak{G}$ is the identical mapping of $\mathfrak{G}$, the so-called *identical automorphism of* $\mathfrak{G}$.

13.6. Exercises

1. If any two elements of $\mathfrak{G}$ are interchangeable, then their images under every deformation of $\mathfrak{G}$ into $\mathfrak{G}^*$ are also interchangeable. The image of every Abelian groupoid is also Abelian.
2. If the product of a three-membered sequence of elements $a, b, c \in \mathfrak{G}$ consists of a single element, then the same holds for the sequence of images $\boldsymbol{d}a, \boldsymbol{d}b, \boldsymbol{d}c \in \mathfrak{G}^*$ under any deformation $\boldsymbol{d}$ of $\mathfrak{G}$ into $\mathfrak{G}^*$. The image of every associative groupoid under any deformation is also associative.
3. If $\mathfrak{G}$ is associative and has a center, then the image of the center under any deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$ lies in the center of $\mathfrak{G}^*$.
4. The inverse image of a groupoidal subset of $\mathfrak{G}^*$ under a deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$ need not be groupoidal.
5. Every meromorphic mapping of a finite groupoid $\mathfrak{G}$ is an automorphism of $\mathfrak{G}$.
6. For isomorphisms of the groupoids $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ the following statements are true: a) $\mathfrak{A} \simeq \mathfrak{A}$ (reflexivity); b) $\mathfrak{A} \simeq \mathfrak{B}$ yields $\mathfrak{B} \simeq \mathfrak{A}$ (symmetry); c) from $\mathfrak{A} \simeq \mathfrak{B}$, $\mathfrak{B} \simeq \mathfrak{C}$ there follows $\mathfrak{A} \simeq \mathfrak{C}$ (transitivity).
7. It is left to the reader to give some examples of deformation himself.

14. Generating decompositions

14.1. Basic concepts

Suppose $\mathfrak{G}$ is an arbitrary groupoid.

Definition. Any decomposition $\bar{A}$ in $\mathfrak{G}$ is called *generating* if there exists, to any two-membered sequence of the elements $\bar{a}, \bar{b} \in \bar{A}$, an element $\bar{c} \in \bar{A}$ such that $\bar{a}\bar{b} \in \bar{c}$.

As to the generating decompositions on the groupoid $\mathfrak{G}$, note that the greatest decomposition $\bar{G}_{\max}$ and the least decomposition $\bar{G}_{\min}$ are generating. On every groupoid there exist at least these two extreme generating decompositions.

The equivalence belonging to a generating decomposition (9.3) is usually called a *congruence*.

14.2. Deformation decompositions

Let $\mathfrak{G}$, $\mathfrak{G}^*$ denote arbitrary groupoids.

Suppose there exists a deformation $\boldsymbol{d}$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$. Since $\boldsymbol{d}$ is a mapping of G onto G^*, it determines a decomposition $\bar{D}$ on $\mathfrak{G}$, corresponding to $\boldsymbol{d}$; each element $\bar{a}$ of $\bar{D}$ consists of all the inverse $\boldsymbol{d}$-images of an element $a^* \in \mathfrak{G}^*$. $\bar{D}$ is called the *deformation decomposition with regard to* $\boldsymbol{d}$ or the *decomposition corresponding (belonging) to the deformation* $\boldsymbol{d}$. Since $\boldsymbol{d}$ preserves the multiplications in both groupoids, it may be expected that $\bar{D}$ is in a certain relationship with the multiplication in $\mathfrak{G}$. Consider any two elements $\bar{a}, \bar{b} \in \bar{D}$. By the definition of $\bar{D}$, there exist elements $a^*, b^* \in \mathfrak{G}^*$ such that $\bar{a}$ $(\bar{b})$ is the set of all inverse $\boldsymbol{d}$-images of a^* (b^*). Consider the product $\bar{a}\bar{b}$ of $\bar{a}$ and $\bar{b}$. Each element $c \in \bar{a}\bar{b}$ is the product of an element $a \in \bar{a}$ and an element $b \in \bar{b}$ and is, with respect to $\boldsymbol{d}c = \boldsymbol{d}ab = \boldsymbol{d}a \,.\, \boldsymbol{d}b = a^*b^*$, an $\boldsymbol{d}$-inverse image of a^*b^*. Hence c is contained in that element $\bar{c} \in \bar{D}$ which consists of the inverse images of a^*b^*. Thus we have verified that the relation $\bar{a}\bar{b} \subset \bar{c}$ is true, hence $\bar{D}$ is generating. Consequently, *the decomposition of the groupoid* $\mathfrak{G}$, *corresponding to any deformation of* $\mathfrak{G}$ *onto another groupoid is generating.*

14.3. Generating decompositions in groupoids

Let us now study the properties of generating decompositions in groupoids.

1. *The sum of the elements of a generating decomposition.* Let $\bar{A}$ denote a generating decomposition in $\mathfrak{G}$.

The subset $\mathbf{s}\bar{A} \subset \mathfrak{G}$, that is to say, the subset of $\mathfrak{G}$, consisting of all the elements contained in some element of $\bar{A}$, *is groupoidal.* Indeed, to any elements $a, b \in \mathbf{s}\bar{A}$ there correspond elements $\bar{a}, \bar{b}, \bar{c} \in \bar{A}$ such that $a \in \bar{a}, b \in \bar{b}, \bar{a}\bar{b} \subset \bar{c}$ whence $ab \in \bar{a}\bar{b} \subset \bar{c} \subset \mathbf{s}\bar{A}$; thus ab is an element of $\mathbf{s}\bar{A}$. The corresponding subgroupoid of $\mathfrak{G}$ is denoted by $\mathbf{s}\bar{\mathfrak{A}}$. It is evident that $\bar{A}$ is a generating decomposition on $\mathbf{s}\bar{\mathfrak{A}}$.

2. *Closures and intersections.* Let B denote a groupoidal subset and $\bar{A}, \bar{C}$ be generating decompositions in $\mathfrak{G}$.

If $B \cap \mathbf{s}\bar{C} \neq \emptyset$, *then the closure* $B \sqsubset \bar{C}$ *and the intersection* $B \sqcap \bar{C}$ *are generating decompositions in* $\mathfrak{G}$. More generally: *if* $\mathbf{s}\bar{A} \cap \mathbf{s}\bar{C} \neq \emptyset$, *then the closure* $\bar{A} \sqsubset \bar{C}$ *and the intersection* $\bar{A} \sqcap \bar{C}$ *are generating decompositions in* $\mathfrak{G}$.

Proof. The decomposition $\bar{B}_{\max}$ consisting of a single element B is obviously a generating decomposition in $\mathfrak{G}$. If $B \cap \mathbf{s}\bar{C} \neq \emptyset$, then $\mathbf{s}\bar{B}_{\max} \cap \mathbf{s}\bar{C} \neq \emptyset$ and, furthermore, $B \sqsubset \bar{C} = \bar{B}_{\max} \sqsubset \bar{C}$, $B \sqcap \bar{C} = \bar{B}_{\max} \sqcap \bar{C}$. Consequently, the second part of the above statement is, in fact, a generalization of the first part and so it is only the latter we have to prove.

a) As there holds $\bar{A} \sqsubset \bar{C} = s\bar{A} \sqsubset C$, it is sufficient to show that the decomposition $s\bar{A} \sqsubset \bar{C}$ is generating. Consider any two elements $\bar{c}_1, \bar{c}_2 \in s\bar{A} \sqsubset \bar{C}$. Since the decomposition $\bar{C}$ is generating, there exists an element $\bar{c} \in \bar{C}$ such that $\bar{c}_1 \bar{c}_2 \subset \bar{c}$. Choose two arbitrary points $x \in s\bar{A} \cap \bar{c}_1$, $y \in s\bar{A} \cap \bar{c}_2$. Then we have $xy \in s\bar{A} \,.\, s\bar{A} \cap \bar{c}_1\bar{c}_2 \subset s\bar{A} \cap \bar{c}$ whence $s\bar{A} \cap \bar{c} \neq \emptyset$. There follows $\bar{c} \in s\bar{A} \sqsubset \bar{C}$.

b) Let $\bar{x}, \bar{y} \in \bar{A} \sqcap \bar{C}$ be arbitrary elements. By the definition of $\bar{A} \sqcap \bar{C}$ there exist elements $\bar{a}_1, \bar{a}_2 \in \bar{A}$; $\bar{c}_1, \bar{c}_2 \in \bar{C}$ such that $\bar{x} = \bar{a}_1 \cap \bar{c}_1$, $\bar{y} = \bar{a}_2 \cap \bar{c}_2$. Since the decomposition $\bar{A}$ ($\bar{C}$) is generating, there exists an element $\bar{a} \in \bar{A}$ ($\bar{c} \in \bar{C}$) such that $\bar{a}_1\bar{a}_2 \subset \bar{a}$ ($\bar{c}_1\bar{c}_2 \subset \bar{c}$). So we have

$$\bar{x}\bar{y} \subset \bar{a}_1\bar{a}_2 \cap \bar{c}_1\bar{c}_2 \subset \bar{a} \cap \bar{c} \in \bar{A} \sqcap \bar{C}$$

and the proof is accomplished.

Now let us add the following remarks:

If $\bar{C}$ lies on $\mathfrak{G}$, then the above assumption: $B \cap s\bar{C} \neq \emptyset$ is satisfied because $s\bar{C} = G \supset B$ and we have $B \cap s\bar{C} = B \neq \emptyset$; the decomposition $B \sqcap \bar{C}$ then lies on B. Hence every generating decomposition $\bar{C}$ on $\mathfrak{G}$ and a groupoidal subset B of $\mathfrak{G}$ uniquely determine two generating decompositions in $\mathfrak{G}$: $B \sqsubset \bar{C}$, $\bar{C} \sqcap B$; the former is a subset of $\bar{C}$, the latter a decomposition on B.

In a similar way, every pair of generating decompositions $\bar{A}$, $\bar{C}$ in $\mathfrak{G}$ of which, e.g., $\bar{C}$ lies on $\mathfrak{G}$ determines two generating decompositions in $\mathfrak{G}$: $\bar{A} \sqsubset \bar{C}$, $\bar{A} \sqcap \bar{C}$; the former is a part of $\bar{C}$, the latter a decomposition on $s\bar{A}$.

Finally, if both $\bar{A}$ and $\bar{C}$ lie on $\mathfrak{G}$, then $\bar{A} \sqcap \bar{C} = (\bar{A}, \bar{C})$ (3.5). We see that *the greatest common refinement of two generating decompositions lying on $\mathfrak{G}$ is again generating* (14.4.3).

3. *Enforced coverings.* Let again $\bar{A}$, $\bar{C}$ stand for generating decompositions in $\mathfrak{G}$. Suppose $\underline{\bar{A}} = \bar{C} \sqsubset \bar{A}$, $\underline{\bar{C}} = \bar{A} \sqsubset \bar{C}$ and let $\bar{B}$ denote a common covering of $\bar{A} \sqcap s\bar{C}$, and $\bar{C} \sqcap s\bar{A}$; these decompositions obviously lie on the set $s\bar{A} \cap s\bar{C}$. Let us, moreover, consider the coverings $\mathring{A}, \mathring{C}$ of $\bar{A}$, $\bar{C}$, enforced by $\bar{B}$ (4.1). $\mathring{A}$ and $\mathring{C}$ are coupled and $\bar{B}$ is their intersection: $\mathring{A} \sqcap \mathring{C} \Rightarrow \bar{B}$.

We shall prove that *if $\bar{B}$ is generating, then $\mathring{A}$ and $\mathring{C}$ are generating as well.*

Proof. Suppose $\bar{B}$ is generating and show that, e.g., $\mathring{A}$ has the same property. To simplify the notation, put $A = s\bar{A}$, $C = s\bar{C}$.

Let $\bigcup_1\bar{a}_1$, $\bigcup_2\bar{a}_2 \in \mathring{A}$ so that $\bar{a}_1$, $\bar{a}_2$ are elements of $\bar{A}$ and $\bigcup_1(\bar{a}_1 \cap C)$, $\bigcup_2(\bar{a}_2 \cap C)$ elements of $\bar{B}$. Since $\bar{A}$ is generating, there exists, to every product $\bar{a}_1\bar{a}_2$, an element $\bar{a}_{12} \in \bar{A}$ such that $\bar{a}_1\bar{a}_2 \subset \bar{a}_{12}$ whence even $(\bar{a}_1 \cap C)$ $(\bar{a}_2 \cap C) \subset \bar{a}_{12} \cap C$. As $\bar{B}$ is generating as well, there exists an element $\bigcup_3(\bar{a}_3 \cap C) \in \bar{B}$ such that

$$\bigcup_1(\bar{a}_1 \cap C) \,.\, \bigcup_2(\bar{a}_2 \cap C) = \bigcup_1\bigcup_2(\bar{a}_1 \cap C)\,(\bar{a}_2 \cap C) \subset \bigcup_3(\bar{a}_3 \cap C),$$

where $\bar{a}_3$ denotes elements of $\bar{A}$ characterized by $\bigcup_3\bar{a}_3 \in \mathring{A}$. For each element $\bar{a}_1$ ($\bar{a}_2$) to which the symbol $\bigcup_1$ ($\bigcup_2$) applies we then have:

$$(\bar{a}_1 \cap C)\,(\bar{a}_2 \cap C) \subset (\bar{a}_{12} \cap C) \in \bigcup_3(\bar{a}_3 \cap C).$$

But the intersections $\bar{a}_{12} \cap C$, $\bar{a}_3 \cap C$ are elements of $\bar{A} \sqcap C$ lying on $A \cap C$. Consequently, among the elements $\bar{a}_3$ to which $\bigcup_3$ applies there exists an element $\bar{a}_3$ such that $\bar{a}_{12} \cap C = \bar{a}_3 \cap C$ and we have $\bar{a}_{12} = \bar{a}_3$. Hence there holds $\bigcup_1 \bar{a}_1 \bigcup_2 \bar{a}_2 \subset \bigcup_1 \bigcup_2 \bar{a}_{12} \subset \bigcup_3 \bar{a}_3 \in \mathring{A}$ and the proof is complete.

14.4. Generating decompositions on groupoids

Now we shall deal with generating decompositions on groupoids. The results will be useful even in case of generating decompositions in groupoids because every generating decomposition $\bar{A}$ in the groupoid $\mathfrak{G}$ is simultaneously a generating decomposition on the subgroupoid $\mathbf{s}\bar{\mathfrak{A}}$.

1. *Local properties of coverings and refinements.* Let $\bar{A} \geqq \bar{B}$ denote any two generating decompositions on $\mathfrak{G}$.

Consider two arbitrary elements $\bar{a}_1, \bar{a}_2 \in \bar{A}$. Since $\bar{A}$ is generating, there exists an element $\bar{a}_3 \in \bar{A}$ such that $\bar{a}_1\bar{a}_2 \subset \bar{a}_3$. Next, consider the decompositions in $\mathfrak{G}$: $\bar{a}_1 \sqcap \bar{B}$, $\bar{a}_2 \sqcap \bar{B}$, $\bar{a}_3 \sqcap \bar{B}$. The latter represent, with regard to $\bar{A} \geqq \bar{B}$, nonempty parts of $\bar{B}$. As $\bar{B}$ is generating, there exists, to any pair of elements $\bar{x} \in \bar{a}_1 \sqcap \bar{B}, \bar{y} \in \bar{a}_2 \sqcap \bar{B}$, an element $\bar{z} \in \bar{B}$ such that $\bar{x}\bar{y} \subset \bar{z}$.

We shall show that $\bar{z}$ *is an element of* $\bar{a}_3 \sqcap \bar{B}$, hence $\bar{z} \in \bar{a}_3 \sqcap \bar{B}$.

Indeed, from $\bar{x} \subset \bar{a}_1, \bar{y} \subset \bar{a}_2, \bar{a}_1\bar{a}_2 \subset \bar{a}_3$ there follows $\bar{x}\bar{y} \subset \bar{a}_3$. So we have $\bar{x}\bar{y} \subset \bar{z} \cap \bar{a}_3$ whence, with respect to $\bar{B} \leqq \bar{A}$, there follows $\bar{z} \subset \bar{a}_3$ (3.2) and, consequently, $\bar{z} \in \bar{a}_3 \sqcap \bar{B}$.

We observe, in particular, that if the subset $\bar{a}_1 \subset \mathfrak{G}$ is groupoidal, then $\bar{a}_1 \sqcap \bar{B}$ is a generating decomposition (14.3.2).

2. *The least common covering.* Let $\bar{A}$, $\bar{B}$ stand for arbitrary generating decompositions on $\mathfrak{G}$.

We shall show that *their least common covering* $[\bar{A}, \bar{B}]$ *is generating as well.*

To that purpose we shall consider an arbitrary ordered pair of elements $\bar{u}, \bar{v} \in [\bar{A}, \bar{B}]$. We are to verify that there exists an element $\bar{w} \in [\bar{A}, \bar{B}]$ such that $\bar{u}\bar{v} \subset \bar{w}$.

Suppose $\bar{a} \in \bar{A}$ and $\bar{b} \in \bar{A}$ are arbitrary elements lying in $\bar{u}$ and $\bar{v}$, respectively, and so $\bar{a} \subset \bar{u}, \bar{b} \subset \bar{v}$. Since $\bar{A}$ is generating, there exists an element $\bar{c} \in \bar{A}$ such that $\bar{a}\bar{b} \subset \bar{c}$. The element $\bar{c}$ lies in a certain element $\bar{w} \in [\bar{A}, \bar{B}]$ and we have $\bar{c} \subset \bar{w}$.

Every element $p \in \bar{u}$ lies in a certain element $\bar{p} \in \bar{A}$ which is a part of $\bar{u}$; similarly, every element $q \in \bar{v}$ lies in a certain element $\bar{q} \in \bar{A}$ which is a part of $\bar{v}$. Moreover, the set $\bar{p}\bar{q}$ is a part of a certain element $\bar{r} \in \bar{A}$ and so $pq \in \bar{p}\bar{q} \subset \bar{r}$. From this we see that all we need to prove that $\bar{u}\bar{v} \subset \bar{w}$ applies is to verify that the element $\bar{r} \in \bar{A}$ comprising the set $\bar{p}\bar{q}$ is, for any two elements $\bar{p}, \bar{q} \in \bar{A}$, $\bar{p} \subset \bar{u}, \bar{q} \subset \bar{v}$, a part of $\bar{w}$, i.e., $\bar{r} \subset \bar{w}$.

Now, let $\bar{p}, \bar{q} \in \bar{A}$, $\bar{p} \subset \bar{u}$, $\bar{q} \subset \bar{v}$ denote arbitrary elements.

Taking account of the definition of the decomposition $\bar{A}$, $\bar{B}$ and of the fact that the elements $\bar{a}$ and $\bar{b}$ lie in $\bar{u}$ and $\bar{v}$, respectively, we conclude that there exists a binding $\{\bar{A}, \bar{B}\}$ from $\bar{a}$ to $\bar{p}$,

$$\bar{a}_1, \ldots, \bar{a}_\alpha \qquad (\text{where } \bar{a}_1 = \bar{a}, \bar{a}_\alpha = \bar{p}), \tag{1}$$

and, similarly, a binding $\{\bar{A}, \bar{B}\}$ from $\bar{b}$ to $\bar{q}$,

$$\bar{b}_1, \ldots, \bar{b}_\beta \qquad (\text{where } \bar{b}_1 = \bar{b}, \bar{b}_\beta = \bar{q}). \tag{2}$$

We may assume that $\beta = \alpha$ because if, for example, $\beta < \alpha$, then it is sufficient to denote the element $\bar{b}_\beta$ by the further symbols: $\bar{b}_{\beta+1}, \ldots, \bar{b}_\alpha$.

Since $\bar{A}$ is generating, there exist elements of $\bar{A}$

$$\bar{c}_1, \ldots, \bar{c}_\alpha \qquad (\text{where } \bar{c}_1 = \bar{c}, \bar{c}_\alpha = \bar{r}) \tag{3}$$

such that $\bar{a}_1\bar{b}_1 \subset \bar{c}_1, \ldots, \bar{a}_\alpha\bar{b}_\alpha \subset \bar{c}_\alpha$. With respect to the definition of $[\bar{A}, \bar{B}]$ and to the fact that the element $\bar{c}$ lies in $\bar{w}$, the relation $\bar{r} \subset \bar{w}$ will be proved by verifying that the sequence (3) is a binding $\{\bar{A}, \bar{B}\}$ from $\bar{c}$ to $\bar{r}$.

Since (1) and (2) are bindings $\{\bar{A}, \bar{B}\}$, there exists to every two elements $\bar{a}_\nu, \bar{a}_{\nu+1}$ and, similarly, to every two elements $\bar{b}_\nu, \bar{b}_{\nu+1}$ an element $\bar{x}_\nu \in \bar{B}$ and an element $\bar{y}_\nu \in \bar{B}$ ($\nu = 1, \ldots, \alpha - 1$), respectively, incident with both. As $\bar{B}$ is generating, there exists a certain element $\bar{z}_\nu \in \bar{B}$ for which $\bar{x}_\nu\bar{y}_\nu \subset \bar{z}_\nu$. Since $\bar{x}_\nu$ and $\bar{y}_\nu$ are incident with $\bar{a}_\nu$ and $\bar{b}_\nu$, respectively, the set $\bar{x}_\nu\bar{y}_\nu$ is incident with $\bar{a}_\nu\bar{b}_\nu$; consequently, $\bar{z}_\nu$ is incident with $\bar{a}_\nu\bar{b}_\nu$ and therefore also with $\bar{c}_\nu$. Analogously, we observe that $\bar{z}_\nu$ is incident with $\bar{c}_{\nu+1}$. Hence every two elements $\bar{c}_\nu, \bar{c}_{\nu+1}$ are incident with a certain element $\bar{z}_\nu \in \bar{B}$ and, consequently, the sequence (3) is a binding $\{\bar{A}, \bar{B}\}$ from $\bar{c}$ to $\bar{r}$.

3. *The greatest common refinement.* Let again $\bar{A}$, $\bar{B}$ denote arbitrary generating decompositions on $\mathfrak{G}$.

Theorem. *The greatest common refinement* $(\bar{A}, \bar{B})$ *of the decompositions* $\bar{A}$, $\bar{B}$ *is also generating.*

This theorem has already been proved (in 14.3.2) on the ground of $(\bar{A}, \bar{B}) = \bar{A} \sqcap \bar{B}$ by verifying that the intersection $\bar{A} \sqcap \bar{B}$ of the generating decompositions $\bar{A}$, $\bar{B}$ is also generating.

14.5. Exercises

1. If an element $\bar{a} \in \bar{A}$ of a generating decomposition $\bar{A}$ in the groupoid $\mathfrak{G}$ contains a groupoidal subset $X \subset \mathfrak{G}$ so that $X \subset \bar{a}$, then the element $\bar{a}$ is groupoidal as well.
2. Let $\mathfrak{G}$ denote the groupoid whose field consists of all positive integers and whose multiplication is defined as follows: the product ab ($a, b \in \mathfrak{G}$) is the number $a_1 \ldots a_\alpha b_1 \ldots b_\beta$, where the numbers $a_1, \ldots, a_\alpha$ and $b_1, \ldots, b_\beta$ are the digits of a and b, respectively, in the decimal system. Thus, for example, $14.23 = 1423$. Show that: a) the groupoid $\mathfrak{G}$ is asso-

ciative; b) the decomposition of $\mathfrak{G}$, the elements of which are the sets of all the numbers in $\mathfrak{G}$ expressed, in the decimal system, by symbols containing the same number of digits, is generating.

3. The groupoid $\mathfrak{G}$, whose field is an arbitrary set and the multiplication given by $ab = a$ ($ab = b$) for $a, b \in \mathfrak{G}$, is associative and all its decompositions are generating.

15. Factoroids

The notion of a factoroid we shall now be concerned with plays an important part throughout the following theory.

15.1. Basic concepts

Let again $\bar{A}$ denote an arbitrary generating decomposition in $\mathfrak{G}$. With $\bar{A}$ we can uniquely associate a groupoid denoted $\bar{\mathfrak{A}}$ and defined as follows: The field of $\bar{\mathfrak{A}}$ is the decomposition $\bar{A}$ and the multiplication is defined in the following way: the product of any element $\bar{a} \in \bar{A}$ and any element $\bar{b} \in \bar{A}$ is the element $\bar{c} \in \bar{A}$ for which $\bar{a}\bar{b} \subset \bar{c}$. Then we generally write

$$\bar{a} \circ \bar{b} = \bar{c},$$

and we have $\bar{a}\bar{b} \subset \bar{a} \circ \bar{b} \in \bar{\mathfrak{A}}$. We employ the symbol $\circ$ to denote the products in $\bar{\mathfrak{A}}$ in the same way as we use the symbol . to denote the products in $\mathfrak{G}$.

$\bar{\mathfrak{A}}$ is called a *factoroid in* $\mathfrak{G}$; if $\bar{A}$ is *on* $\mathfrak{G}$, then it is a *factoroid on* $\mathfrak{G}$ or a *factoroid of* $\mathfrak{G}$. Every generating decomposition in $\mathfrak{G}$ uniquely determines a certain factoroid in $\mathfrak{G}$, namely the one whose field it is; we say that to every generating decomposition in $\mathfrak{G}$ there *corresponds* or *belongs* a certain factoroid in $\mathfrak{G}$.

Note that on $\mathfrak{G}$ there exist at least two factoroids, namely the so-called *greatest factoroid*, $\bar{\mathfrak{G}}_{\max}$, belonging to the greatest generating decomposition $\bar{G}_{\max}$ and the *least factoroid*, $\bar{\mathfrak{G}}_{\min}$, belonging to the least generating decomposition $\bar{G}_{\min}$ of the groupoid $\mathfrak{G}$. These extreme factoroids on $\mathfrak{G}$ are either different from each other or coincide according as $\mathfrak{G}$ contains more than one or precisely one element.