CHAPTER IV

UNIFORM AND PROXIMITY SPACES

(Sections 23 – 25)

We have defined the concept of a uniformly continuous mapping of a semi-pseudometric space into another one. In Section 23 we shall examine the most general kind of spaces which enables one to define the concept of a uniformly continuous mapping, namely semi-uniform spaces. Section 24 concerns a particular kind of semi-uniform spaces, the uniform spaces, which are related to semi-uniform spaces similarly as pseudometrics are to semi-pseudometrics. In Section 25 the properties of semi-uniform spaces will be developed and the so-called proximity spaces will be introduced and studied. Particular attention is given to the Stone-Weierstrass theorem for proximity spaces, and to uniformly continuous extensions of bounded uniformly continuous pseudometrics and functions. The results obtained will be applied later to closure spaces.

23. SEMI-UNIFORM SPACES

In this section, which is the first of three closely related sections, we shall be concerned with defining and developing the basic properties of semi-uniform spaces and uniformly continuous mappings. The next section investigates properties of a particularly important class of semi-uniform spaces, the so-called uniform spaces. The closing section of this chapter is concerned with developing the theory of the so-called proximally coarse semi-uniformities and the related concept of a proximity. In all three sections results concerning semi-pseudometrics, proved in 18 A – 18 C, are assumed to be known.

Here we begin with the definition of a semi-uniformity and with the description of a semi-uniformity in terms of uniformly continuous semi-pseudometrics. This will help the reader to understand the extent of the generalization which is obtained by introducing the concept of a semi-uniformity instead of a collection of semi-pseudometrics uniformly equivalent to each other. In the second subsection we shall examine the relations between semi-uniformities and the induced closures. The third subsection, devoted to a discussion of the concept of a uniformly continuous mapping, is followed by an exposition of the basic constructions of new semi-uniform spaces from given ones, namely subspaces, sums and products; here the exposition parallels Section 17 dealing with the same constructions for closure spaces.

A. SEMI-UNIFORMITIES AND UNIFORM COLLECTIONS OF SEMI-PSEUDOMETRICS

It should be noted that the identity relation on a class P, denoted by J_P, and the diagonal of $P \times P$, denoted by Δ_P, are different names and symbols for the same entity, namely for the class of all pairs $\langle x, x \rangle$ such that $x \in P$.

A relation for a set P is a subset of $P \times P$. In this section we shall deal with relations for a set P containing the diagonal of $P \times P$. By 12 A.2 these relations are termed vicinities of the diagonal of $P \times P$ or vicinities on P. Given a struct $\mathscr{P}$, we want to speak about those properties of vicinities on $|\mathscr{P}|$ which depend on the structure of $\mathscr{P}$. To this end the following definition is introduced.

23 A.1. Definition. If $\mathscr{P}$ is a struct then a vicinity on $\mathscr{P}$ is defined to be a vicinity on $|\mathscr{P}|$. We shall say that "V is a vicinity of the diagonal of $\mathscr{P} \times \mathscr{P}$" meaning that V is a vicinity on $\mathscr{P}$, i.e. a vicinity of the diagonal of $|\mathscr{P}| \times |\mathscr{P}|$. It should be remarked that the sentence in quotes must be treated as an indecomposable expression (whether or not $\mathscr{P} \times \mathscr{P}$ had been defined).

23 A.2. Suppose that u is a closure operation for a set P. If $\{U_x \mid x \in P\}$ is a family such that U_x is a neighborhood of x in $\langle P, u\rangle$, then the set

$$U = \Sigma\{U_x \mid x \in P\} = \mathbf{E}\{\langle x, y\rangle \mid x \in P, y \in U_x\}$$

is a vicinity of the diagonal on P and $U[x] = U_x$ for each x in P. Let $\mathscr{U}$ be the collection of all such U. Obviously $\mathscr{U}$ is a filter on $P \times P$ consisting of vicinities of the diagonal of $P \times P$ and, for each x in P, the collection $[\mathscr{U}][x]$ (of all subsets of P of the form $U[x]$, $U \in \mathscr{U}$) is the neighborhood system at x in $\langle P, u\rangle$. Conversely, if $\mathscr{U}$ is a filter on $P \times P$ consisting of vicinities of the diagonal, then the collection $[\mathscr{U}][x]$ is a filter on P the intersection of which contains x for each x in P; by 14 B.10 there exists a unique closure operation u for P such that $[\mathscr{U}][x]$ is a local base at x in $\langle P, u\rangle$ for each x in P. This closure operation will be called the closure induced by $\mathscr{U}$. We have proved that every closure operation for P is induced by a filter on $P \times P$ consisting of vicinities of the diagonal, and conversely, every such filter induces a closure operation. It is to be observed that closures induced by different filters may coincide; for example, let $\mathscr{U}_1$ be the collection of all vicinities of the diagonal of $P \times P$ and let $\mathscr{U}_2$ be a subset of $\mathscr{U}_1$ consisting of all $U \in \mathscr{U}_1$ such that $U[x] = P$ for all x in P excepting a finite number of x's. Obviously both filters induce the discrete closure for P but $\mathscr{U}_1 \neq \mathscr{U}_2$ if P is infinite. It follows that such filters define a more restrictive structure for P than a closure operation. Now let d be a semi-pseudometric for a set P and let us consider the collection $\mathscr{U}_d$ of all vicinities of the diagonal of $P \times P$ containing a set of the form $\mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\rangle\}$, $r > 0$. Clearly $\mathscr{U}_d$ is a filter on $P \times P$ consisting of vicinities, and the closure induced by $\mathscr{U}_d$ coincides with the closure induced by d. The filter $\mathscr{U}_d$ has a significant property: it has a base consisting of symmetric vicinities, that is of vicinities U such that $U = U^{-1}$; in fact, the vicinities of the form $\mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\}$, $r > 0$, are symmetric and form a base for $\mathscr{U}_d$. Next it is apparent that two semi-pseudometrics d_1 and d_2 are uniformly equivalent (in the sense of Definition 18 B.14) if and only if $\mathscr{U}_{d_1} = \mathscr{U}_{d_2}$. Thus the notion of a uniformly continuous mapping of a semi-pseudometric space into another one depends only on the corresponding filters. This section is devoted to an investigation of "symmetric" filters on $P \times P$, consisting of vicinities of the diagonal of $P \times P$, and called semi-uniformities. As it stands, the concept of a semi-uniformity is a generalization of the concept of a semi-pseudometric; this enables one to define the notion of a uniformly continuous mapping in a most general situation.

23 A.3. Definition. A *semi-uniformity* for a set P is a filter $\mathscr{U}$ on $P \times P$ satisfying the following two conditions:

(u 1) each element of $\mathscr{U}$ contains the diagonal of $P \times P$, i.e. $\bigcap \mathscr{U} \supset \Delta_P$;
(u 2) if $U \in \mathscr{U}$, then U^{-1} contains an element of $\mathscr{U}$.

Since $\mathscr{U}$ is a filter, condition (u 2) may be replaced by the following formally stronger condition:
(u 2′) if $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$.

A *semi-uniform space* is a struct $\langle P, \mathscr{U} \rangle$ such that P is a set and $\mathscr{U}$ is a semi-uniformity for P.

A *base for a semi-uniformity* $\mathscr{U}$ is a subcollection $\mathscr{V}$ of $\mathscr{U}$ such that each element of $\mathscr{U}$ contains an element of $\mathscr{V}$; stated in other words, a base for a semi-uniformity $\mathscr{U}$ is a filter base for the filter $\mathscr{U}$. A *sub-base for a semi-uniformity* $\mathscr{U}$ is a subcollection $\mathscr{W}$ of $\mathscr{U}$ such that the collection of all finite intersections of elements of $\mathscr{W}$ is a base for $\mathscr{U}$; stated in other words, a sub-base for a semi-uniformity $\mathscr{U}$ is a filter sub-base for the filter $\mathscr{U}$.

If $\mathscr{U}$ is a semi-uniformity for a set P then $[\mathscr{U}][x] = \mathbf{E}\{U[x] \mid U \in \mathscr{U}\}$ is a filter on P and $x \in U[x]$ for each x in P. By 14 B.10 there exists a unique closure u for P such that $[\mathscr{U}][x]$ is a local base at x in $\langle P, u \rangle$ for each x in P. This closure is defined to be the *closure induced by* $\mathscr{U}$.

23 A.4. Theorem. *Conditions* (u 1) *and* (u 2) *are necessary and sufficient for a filter base on* $P \times P$ *to be a base for a semi-uniformity for* P. *Conditions* (u 1) *and* (u 2) *are sufficient* (*but not necessary*) *for a filter sub-base on* $P \times P$ *to be a sub-base for a semi-uniformity for* P. – The proof is straightforward and may be left to the reader.

Corollary. *If* $\{\mathscr{U}_a\}$ *is a non-void family of semi-uniformities for a set* P, *then the union of* $\{\mathscr{U}_a\}$ *is a sub-base for a semi-uniformity for the set* P.

23 A.5. *A collection* $\mathscr{W}$ *of sets is a sub-base for a semi-uniformity for a set* P *if and only if* $\mathscr{W} \neq \emptyset$, *each element of* $\mathscr{W}$ *is a vicinity of the diagonal of* $P \times P$, *and if* $W \in \mathscr{W}$ *then* W^{-1} *contains a finite intersection of elements of* $\mathscr{W}$.

Proof. Let us consider the collection $\mathscr{V}$ consisting of all finite intersections of elements of $\mathscr{W}$. If $\mathscr{W}$ is a sub-base for a semi-uniformity, then $\mathscr{V}$ is a base and therefore, by 23 A.4, if $V \in \mathscr{V}$ then $V' \subset V^{-1}$ for some V' in $\mathscr{V}$; it follows that for each U in $\mathscr{W}$ the set U^{-1} contains a finite intersection of elements of $\mathscr{W}$; evidently $\mathscr{W} \neq \emptyset$ and each element of $\mathscr{W}$ contains the diagonal. Conversely, assuming that $\mathscr{W} \neq \emptyset$, $\bigcap \mathscr{W} \supset \Delta_P$, and if $U \in \mathscr{W}$ then U^{-1} contains a finite intersection of its elements, one can show without difficulty that $\mathscr{V}$ is a filter base satisfying conditions (u 1) and (u 2); now by 23 A.4 $\mathscr{V}$ is a base for a semi-uniformity and finally, by definition, $\mathscr{W}$ is a sub-base for a semi-uniformity.

23 A.6. Remarks. (a) A semi-uniformity $\mathscr{U}$ is a semi-uniformity for exactly one set P, namely $P = \mathbf{D}U = \mathbf{E}U$ for any U in $\mathscr{U}$. Thus the relation $\{\langle P, \mathscr{U} \rangle \to \to \mathscr{U} \mid \langle P, \mathscr{U} \rangle$ semi-uniform space$\}$ is one-to-one and ranges on the class of all semi-uniformities.

(b) The collection of all symmetric elements of a given semi-uniformity $\mathscr{U}$ is a base for $\mathscr{U}$; actually, if $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$ by (u 2') and thus $(U \cap U^{-1}) \in \mathscr{U}$. But $U \cap U^{-1}$ is symmetric and is contained in U.

(c) Suppose that $\mathscr{U}$ is a semi-uniformity for a set P and u is the closure induced by $\mathscr{U}$. If $\mathscr{V}$ is a base (a sub-base) for $\mathscr{U}$, then $[\mathscr{V}][x]$ is a local base (a local sub-base) at x in $\langle P, u\rangle$ for each x in P. It follows that if $\mathscr{U}$ has a base of cardinal $\mathfrak{m}$, then the local character of $\langle P, u\rangle$ is at most $\mathfrak{m}$.

23 A.7. Examples. (a) The collection $\mathscr{U}$ of all subsets of $P \times P$ containing the diagonal is clearly a semi-uniformity for the set P. The collection consisting of only one element, namely the diagonal of $P \times P$, is a base for $\mathscr{U}$. Clearly $\mathscr{U}$ is the largest semi-uniformity for P, that is, if $\mathscr{V}$ is a semi-uniformity for P, then $\mathscr{V} \subset \mathscr{U}$. Evidently, $\mathscr{U}$ induces the discrete closure. Let $\mathscr{U}_1$ be the collection of all subsets $U \subset P \times P$ of the form $\bigcup\{X_i \times X_i\}$, where $\{X_i\}$ is a finite cover of P. Obviously $\mathscr{U}_1$ is a filter base and fulfils conditions (u 1), (u 2). Thus $\mathscr{U}_1$ is a base for some semi-uniformity $\mathscr{V}$ for P. Clearly $\mathscr{V}$ induces the discrete closure operation for P. If P is infinite, then the diagonal of $P \times P$ does not belong to $\mathscr{V}$ and hence $\mathscr{V} \neq \mathscr{U}$. Thus, if P is infinite, then $\mathscr{U}$ and $\mathscr{V}$ are distinct semi-uniformities inducing the same closure operation. The smallest semi-uniformity for P consists of exactly one element, namely $P \times P$; the induced closure is accrete and $(P \times P)$ is the only semi-uniformity for P inducing the accrete closure for P.

(b) If d is a semi-pseudometric for a set P, then the collection of all sets of the form $U_r = \mathbf{E}\{\langle y, x\rangle \mid d\langle y, x\rangle < r\}$, $r > 0$, is a filter base on $P \times P$ satisfying conditions (u 1) and (u 2)) ((u 1) follows from $d\langle y, y\rangle = 0$ and (u 2) from the symmetry of d). By 23 A.4 this collection is a base for a semi-uniformity $\mathscr{U}$ which will be said to be *induced* by d. The semi-pseudometric d induces a closure for P. It is almost self-evident that these closures coincide; indeed, given an x in P, $\{U_r[x] \mid r > 0\}$ is a local base at x with respect to the closure induced by the semi-uniformity (23 A.6 (c)) and the same family is a local base at x with respect to the semi-pseudometric closure because $U_r[x]$ is the open r-sphere about x.

(c) Two semi-pseudometrics are uniformly equivalent (in the sense of definition 18 B.14) if and only if they induce the same semi-uniformity.

(d) The metric $\{\langle x, y\rangle \to |x - y|\}$ of the metric space R of reals induces a semi-uniformity by (b). Unless the contrary is explicitly stated, if R is considered as a semi-uniform space it is to be understood that the semi-uniformity is that just described.

23 A.8. Theorem. *A semi-uniformity $\mathscr{U}$ is semi-pseudometrizable (i.e. induced by a semi-pseudometric) if and only if it has a countable base.*

Proof. I. If $\mathscr{U}$ is induced by a semi-pseudometric d, and M is a set of positive reals the infimum of which is zero, then evidently the collection of all $\mathbf{E}\{\langle x, y\rangle \mid \mid d\langle x, y\rangle < r\}$, $r \in M$, is a base for $\mathscr{U}$. Since M can be taken countable, the "only if" part follows. — II. Conversely, let $\{U_n \mid n \in \mathsf{N}\}$ be a base for $\mathscr{U}$. Without loss of generality we may and shall assume that $U_0 = P \times P$ and $U_n = U_n^{-1} \supset U_{n+1}$ for each n.

Putting $d\langle x, y\rangle = 2^{-n}$ if and only if $\langle x, y\rangle \in U_n - U_{n+1}$ and $d\langle x, y\rangle = 0$ otherwise (i.e. if $\langle x, y\rangle \in \bigcap\{U_n\}$), we obtain a semi-pseudometric d for P which induces $\mathscr{U}$.

23 A.9. Definition. A *semi-pseudometric* d for a semi-uniform space $\langle P, \mathscr{U}\rangle$ is said to be *uniformly continuous* if the semi-uniformity induced by d is contained in $\mathscr{U}$, i.e. $\mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\} \in \mathscr{U}$ for each positive real r. A *uniform collection of semi-pseudometrics* is the collection of all uniformly continuous pseudometrics for a semi-uniform space.

23 A.10. Theorem. *A collection $\mathscr{M}$ of semi-pseudometrics is a uniform collection of semi-pseudometrics if and only if $\mathscr{M}$ is non-void, all elements of $\mathscr{M}$ are semi-pseudometrics for the same set, say P, and the following two conditions are fulfilled:*

(a) $d_1 \in \mathscr{M}$, $d_2 \in \mathscr{M}$ *imply* $d_1 + d_2 \in \mathscr{M}$;

(b) *if d is a semi-pseudometric for P and if for each $r > 0$ there exists a d' in $\mathscr{M}$ and an $s > 0$ such that $d'\langle x, y\rangle < s$ implies $d\langle x, y\rangle < r$, then $d \in \mathscr{M}$.*

Proof. First suppose that $\mathscr{M}$ is the collection of all uniformly continuous semi-pseudometrics for a semi-uniform space $\langle P, \mathscr{U}\rangle$. Clearly, $\{\langle x, y\rangle \to 0 \mid \langle x, y\rangle \in P \times P\} \in \mathscr{M}$ and hence $\mathscr{M} \neq \emptyset$. Evidently every $d \in \mathscr{M}$ is a semi-pseudometric for P and hence all the $d \in \mathscr{M}$ are for the same set. If $d_1, d_2 \in \mathscr{M}$, $d = d_1 + d_2$, r is a positive real and $0 < s < 2^{-1} . r$, then

$$\mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\}) \supset (\mathbf{E}\{\langle x, y\rangle \mid d_1\langle x, y\rangle < s\} \cap \\ \cap \mathbf{E}\{\langle x, y\rangle \mid d_2\langle x, y\rangle < s\}) \in \mathscr{U},$$

which shows that d is a uniformly continuous semi-pseudometric for $\langle P, \mathscr{U}\rangle$, i.e. $d \in \mathscr{M}$. Condition (b) is an immediate consequence of the definition of uniformly continuous semi-pseudometrics. The second part of the proof is an immediate consequence of the proposition which follows.

23 A.11. *Let $\mathscr{M}$ be a non-void collection of semi-pseudometrics for a set P and let $\mathscr{V}$ be the set of all sets of the form $\mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\}$, $d \in \mathscr{M}$, $r > 0$. Then $\mathscr{V}$ is a sub-base for a semi-uniformity and if $\mathscr{M}$ fulfils condition* (a) *of* 23 A.10, *then $\mathscr{V}$ is a base for a semi-uniformity. If $\mathscr{V}$ is a base for a semi-uniformity $\mathscr{U}$ and $\mathscr{M}$ fulfils condition* (b) *of* 23 A.10, *then $\mathscr{M}$ is the set of all uniformly continuous semi-pseudometrics for $\langle P, \mathscr{U}\rangle$.*

Proof. Every element of $\mathscr{V}$ is a symmetric vicinity of the diagonal of $P \times P$ and therefore, by 23 A.4 $\mathscr{V}$ is a sub-base for a semi-uniformity. Now suppose that $d_1 + d_2 \in \mathscr{M}$ whenever $d_1, d_2 \in \mathscr{M}$; it will be shown that $\mathscr{V}$ is a filter base. If $V_1, V_2 \in \mathscr{V}$, $V_i = \mathbf{E}\{\langle x, y\rangle \mid d_i\langle x, y\rangle < r_i\}$, $i = 1, 2$, where $d_i \in \mathscr{M}$ and $r_i > 0$, then $V_1 \cap V_2$ contains the vicinity $\mathbf{E}\{\langle x, y\rangle \mid (d_1 + d_2)\langle x, y\rangle < r\}$, where $r = \min(r_1, r_2)$. Finally, if $\mathscr{V}$ is a base for a semi-uniformity $\mathscr{U}$ and if d is a uniformly continuous pseudometric for $\langle P, \mathscr{U}\rangle$ then clearly d fulfils the assumptions of condition (b) of 23 A.10; thus if $\mathcal{M}$ fulfils (b), then every uniformly continuous semi-pseudometric for $\langle P, \mathcal{U} \rangle$ belongs to $\mathcal{M}$.

23 A.12. Definition. If $\mathcal{M}$ is a non-void collection of semi-pseudometrics for a set P, then by 23 A.11 the set of all $\mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\}$, $d \in \mathcal{M}$, $r > 0$ is a sub-base for a semi-uniformity which is defined to be the *semi-uniformity generated by* $\mathcal{M}$.

23 A.13. Theorem. *If a semi-uniformity $\mathcal{U}$ is generated by a non-void collection $\mathcal{M}$ of semi-pseudometrics for a set P, then $U \in \mathcal{U}$ if and only if $U \subset P \times P$ and there exists a finite sequence $\{d_i \mid i \leqq n\}$ in $\mathcal{M}$ and a positive real r such that $\Sigma\{d_i \langle x, y\rangle \mid i \leqq n\} < r$ implies $\langle x, y\rangle \in U$.*

Proof. The set $\mathcal{M}_1$ of all finite sums of semi-pseudometrics from $\mathcal{M}$ contains with each d_1 and d_2 their sum $d_1 + d_2$. Now the statement follows from 23 A.11.

Let $\mathcal{U}$ be a semi-uniformity for a set P, $\mathcal{M}$ be the set of all uniformly continuous semi-pseudometrics for $\langle P, \mathcal{U}\rangle$ and let $\mathcal{V}$ be the semi-uniformity induced by $\mathcal{M}$. Obviously $\mathcal{V}$ is contained in $\mathcal{U}$. Now we shall prove that $\mathcal{U} = \mathcal{V}$.

23 A.14. *If $\mathcal{U}$ is a semi-uniformity for a set P, then $\mathcal{U}$ is generated by the set $\mathcal{M}$ of all uniformly continuous semi-pseudometrics for $\langle P, \mathcal{U}\rangle$ which assume only two values,* 0 *and* 1.

Proof. If U is a symmetric element of $\mathcal{U}$ and if $d\langle x, y\rangle = 0$ for $\langle x, y\rangle \in U$ and $d\langle x, y\rangle = 1$ otherwise, then clearly $d = \{\langle x, y\rangle \to d\langle x, y\rangle \mid \langle x, y\rangle \in P \times P\}$ is a uniformly continuous semi-pseudometric for $\langle P, \mathcal{U}\rangle$.

As a corollary we obtain the following result which shows that a semi-uniform space if uniquely determined by the collection of all uniformly continuous semi-pseudometrics, and that a semi-uniformity $\mathcal{U}$ is the smallest semi-uniformity containing every semi-uniformity induced by a uniformly continuous semi-pseudometric for $\langle P, \mathcal{U}\rangle$.

23 A.15. Theorem. *If $\langle P, \mathcal{U}\rangle$ is a semi-uniform space then $U \in \mathcal{U}$ if and only if $U \subset P \times P$ and there exists a uniformly continuous semi-pseudometric d for $\langle P, \mathcal{P}\rangle$ such that $d\langle x, y\rangle < 1$ implies $\langle x, y\rangle \in U$.*

B. SEMI-UNIFORM CLOSURE OPERATIONS

By definition 23 A.3, if $\mathcal{U}$ is a semi-uniformity for a set P and u is the closure induced by $\mathcal{U}$, then $[\mathcal{U}][x]$ is the neighborhood system at x in $\langle P, u\rangle$ for each $x \in P$. This subsection is concerned with various descriptions of the closure induced by a semi-uniformity.

23 B.1. Definition. A *continuous semi-uniformity for a space* $\langle P, u\rangle$ is a semi-uniformity for P such that the closure induced by $\mathcal{U}$ is coarser than u. A closure operation u will be called *semi-uniformizable* if u is induced by a semi-uniformity.

Recall that if P is a closure space then a semi-neighborhood of the diagonal of the product space $P \times P$ is a neighborhood of the diagonal in ind $(P \times P)$, i.e., a subset U of $P \times P$ such that $U[x] \cap U^{-1}[x]$ is a neighborhood of x in P for each $x \in P$.

23 B.2. *If $\mathcal{U}$ is a continuous semi-uniformity for a closure space $\langle P, u \rangle$ then each element of $\mathcal{U}$ is a semi-neighborhood of the diagonal in $\langle P, u \rangle \times \langle P, u \rangle$. The set of all semi-neighborhoods of the diagonal of $\langle P, u \rangle \times \langle P, u \rangle$ is a continuous semi-uniformity for $\langle P, u \rangle$.*

Proof. Let v be the closure induced by $\mathcal{U}$. If $U \in \mathcal{U}$, then $U[x]$ is a neighborhood of x in $\langle P, v \rangle$ for each x in P, and v being coarser than u, $U[x]$ is also a neighborhood of x in $\langle P, u \rangle$. Since U^{-1} belongs to $\mathcal{U}$, $U^{-1}[x]$ is also a neighborhood of x in $\langle P, u \rangle$. Thus U is a semi-neighborhood of the diagonal of $\langle P, u \rangle \times \langle P, u \rangle$. Now let $\mathcal{W}$ be the set of all semi-neighborhoods of the diagonal of $\langle P, u \rangle \times \langle P, u \rangle$. Since $\mathcal{W}$ is the neighborhood system of the diagonal in ind $(\langle P, u \rangle \times \langle P, u \rangle)$, $\mathcal{W}$ is a filter consisting of vicinities of the diagonal, and clearly $U \in \mathcal{W}$ implies $U^{-1} \in \in \mathcal{W}$; thus $\mathcal{W}$ is a semi-uniformity which is, evidently, continuous.

Corollary. *Let $\langle P, u \rangle$ be a closure space and let $\mathcal{U}$ be the set of all semi-neighborhoods of the diagonal of $\langle P, u \rangle \times \langle P, u \rangle$. Then $\mathcal{U}$ is the largest continuous semi-uniformity for $\langle P, u \rangle$ and the closure induced by $\mathcal{U}$ is the finest semi-uniformizable closure coarser than u. Finally, d is a continuous semi-pseudometric for $\langle P, u \rangle$ if and only if d is a uniformly continuous semi-pseudometric for $\langle P, \mathcal{U} \rangle$.*

23 B.3. Theorem. *In order that a closure operation u for a set P be semi-uniformizable it is necessary and sufficient that $x \in u(y)$ imply $y \in u(x)$, i.e. if x belongs to the closure of a one-point set (y), then y belongs to the closure of (x).*

Proof. I. Suppose that u is induced by a semi-uniformity $\mathcal{U}$ and let $\mathcal{V}$ be the set of all symmetric elements of $\mathcal{U}$. Since $\mathcal{V}$ is a base for $\mathcal{U}$ (23 A.6 (b)), $x \in uX$ if and only if $V[x] \cap X \neq \emptyset$ for each V in $\mathcal{V}$. Now, if $x \in u(y)$, then $y \in V[x]$ for each V in $\mathcal{V}$, and each $V \in \mathcal{V}$ being symmetric, we obtain $x \in V[y]$ for each V in $\mathcal{V}$, which means that $y \in u(x)$. – II. Conversely assume the condition and consider the largest continuous semi-uniformity $\mathcal{U}$ for $\langle P, u \rangle$. We shall prove that $\mathcal{U}$ induces u. It is sufficient to show that, for each $x \in P$ and each neighborhood W of x, there exists a U in $\mathcal{U}$ such that $U[x] \subset W$. Choose a family $\{V_y \mid y \in P\}$ such that V_y is a neighborhood of y in $\langle P, u \rangle$ for each y, $V_x \subset W$, and if $y \notin u(x)$ then $x \in (P - V_y)$. Put $V = \Sigma\{V_y \mid y \in P\}$, $U = V \cup V^{-1}$. Obviously U is a semi-neighborhood of the diagonal and hence $U \in \mathcal{U}$. It will be shown that $U[x] = V_x$ $(\subset W)$ and hence that U is the required element of $\mathcal{U}$. Clearly $U[x] \supset V_x$. If $y \in (U[x] - V_x)$, then $y \in V^{-1}[x]$ (because $V[x] = V_x$) and hence $x \in V[y] = V_y$; thus by construction $y \in u(x)$ and by our condition $x \in u(y)$; hence $y \in V_x$ because V_x is a neighborhood of x. But this contradicts our assumption $y \notin V_x$.

Before proceeding on we shall prove an important characterization of semi-neighborhoods of the diagonal.

23 B.4. Theorem. *Let P be a closure space. In order that a symmetric subset U of $P \times P$ be a semi-neighborhood of the diagonal of $P \times P$ it is necessary and sufficient that $\overline{X} \subset U[X]$ for each subset X of P.*

Proof. I. First suppose that U is a semi-neighborhood of the diagonal and let $X \subset P$. If $x \in \overline{X}$, then $U[x] \cap X \neq \emptyset$, so that $y \in U[x]$ for some y in X; U being symmetric, we obtain $x \in U[y]$. Thus $\overline{X} \subset U[X]$. – II. Now suppose that $\overline{X} \subset U[X]$ for each $X \subset P$. Since U is symmetric, to show that U is a semi-neighborhood of the diagonal it is sufficient to prove that $U[x]$ is a neighborhood of x in P for each $x \in P$. But by our condition $\overline{P - U[x]} \subset U[P - U[x]] = P - (x)$ and hence $U[x]$ is indeed a neighborhood of x.

Suppose that a closure u for a set P is induced by a semi-pseudometric d and let $U_r = \mathsf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\}$ for $r > 0$. For each $X \subset P$ the set $U_r[X]$ is the open r-sphere about the set X in $\langle P, d\rangle$ and therefore $uX \subset U_r[X]$. Furthermore $uX = \bigcap\{U_r[X] \mid r > 0\}$ since uX is the set of all $x \in P$ which have zero distance from X. Now we shall prove that the same formula is true for every semi-uniformity inducing the closure u.

23 B.5. Theorem. *Suppose that a closure u for a set P is induced by a semi-uniformity $\mathcal{U}$, and $\mathcal{V}$ is a base of $\mathcal{U}$. Then*

$$uX = \bigcap\{U[X] \mid U \in \mathcal{U}\} = \bigcap\{U[X] \mid U \in \mathcal{V}\}$$

for each $X \subset P$.

Proof. Each element of $\mathcal{U}$ is a semi-neighborhood of the diagonal of $\langle P, u\rangle \times \langle P, u\rangle$ (by 23 B.2) and therefore, by 23 B.4, $uX \subset U[X]$ for each symmetric U in $\mathcal{U}$ and hence each U in $\mathcal{U}$; this establishes the inclusion $\subset$. If $x \in (P - uX)$, then $V[x] \cap X = \emptyset$ for some V in $\mathcal{V}$; selecting any element V_1 of $\mathcal{V}$ contained in $V \cap V^{-1}$ we obtain $x \notin V_1[X]$ which establishes the inverse inclusion and completes the proof.

The theorem just proved gives a direct description of semi-uniform closures. Now we shall prove an interesting and perhaps a little surprising description of the product $u \times u$ where u is a semi-uniform closure.

23 B.6. Theorem. *Suppose that a closure operation u for a set P is induced by a semi-uniformity $\mathcal{U}$ and $\langle P \times P, u \times u\rangle$ is the product space $\langle P, u\rangle \times \langle P, u\rangle$. Then*

$$(u \times u) X = \bigcap\{U \circ X \circ U \mid U \in \mathcal{U}\}$$

for each subset X of $P \times P$.

The proof is based upon the following lemma which will often be used in the sequel.

23 B.7. Lemma. *If U and X are relations for a set P, then*

(*) $U \circ X \circ U = \bigcup\{U^{-1}[x] \times U[y] \mid \langle x, y\rangle \in X\}$, *and if U is symmetric (i.e. $U = U^{-1}$), then*

(**) $U \circ X \circ U = \bigcup\{U[x] \times U[y] \mid \langle x, y\rangle \in X\}$.

Proof. Formula (**) follows immediately from (*). To prove (*) it is sufficient to observe that the left side of (*) is the set of all pairs $\langle z, t\rangle$ such that $\langle z, x\rangle \in U$ and $\langle y, t\rangle \in U$ for some $\langle x, y\rangle \in X$, i.e. the set $\mathbf{E}\{\langle z, t\rangle \mid z \in U^{-1}[x],\ t \in U[y]$ for some $\langle x, y\rangle$ in $X\}$ which is, evidently, the set on the right side of (*).

Proof of 23 B.6. Let $\mathscr{V}$ be the collection of all symmetric elements of $\mathscr{U}$. Thus $\mathscr{V}$ is a base of $\mathscr{U}$ and $[\mathscr{V}][x]$ is a local base at x in $\langle P, u\rangle$ for each $x \in P$. As a consequence, the collection consisting of all sets $V[x] \times V[y]$, $V \in \mathscr{V}$, is a local base at $\langle x, y\rangle$ in $\langle P \times P, u \times u\rangle$. Since the relations V are symmetric we have $\langle z, t\rangle \in V[x] \times V[y]$ if and only if $\langle x, y\rangle \in V[z] \times V[t]$. But $\langle z, t\rangle \in (u \times u) X$ if and only if $X \cap (V[z] \times V[t]) \neq \emptyset$ for each V in $\mathscr{V}$, i.e. for each V in $\mathscr{V}$ there exists a pair $\langle x, y\rangle$ in X such that $\langle z, t\rangle \in V[x] \times V[y]$. By virtue of formula (**) of 23 B.7 we obtain $\langle z, t\rangle \in (u \times u)X$ if and only if $\langle z, t\rangle \in V \circ X \circ V$ for each $V \in \mathscr{V}$. Theorem 23 B.6 follows.

In concluding we shall describe semi-uniform closures in terms of uniformly continuous semi-pseudometrics.

23 B.8. Theorem. *Suppose that a closure u for a set P is induced by a semi-uniformity $\mathscr{U}$ and $\mathscr{U}$ is generated by a collection $\mathscr{M}$ of semi-pseudometrics. Finally, let $\mathscr{M}_1$ be the set of all finite sums of semi-pseudometrics from $\mathscr{M}$. Then*

(a) *$x \in uX$ if and only if the distance from x to X is zero in $\langle P, d\rangle$ for each d in $\mathscr{M}_1$.*

(b) *A subset U of P is a neighborhood of $x \in P$ in $\langle P, u\rangle$ if and only if U contains an open r-sphere about x in $\langle P, d\rangle$ for some d in $\mathscr{M}_1$.*

(c) *A net $\{x_a\}$ converges to x in $\langle P, u\rangle$ if and only if the net $\{d\langle x_a, x\rangle\}$ converges to zero in R for each d in $\mathscr{M}$.*

Proof. Statements (a) and (b) are evident (see 23 A.12 and 23 A.13). Statement (c), with $\mathscr{M}$ replaced by $\mathscr{M}_1$, is also evident (e.g. one can use (b)). It remains to notice that if the net $\{d\langle x_a, x\rangle\}$ converges to zero in R for each d in $\mathscr{M}$, then this net converges to zero for each d in $\mathscr{M}_1$.

Remark. In (a) and (b) one cannot replace $\mathscr{M}_1$ by $\mathscr{M}$.

C. UNIFORMLY CONTINUOUS MAPPINGS

By Definition 18 B.14 a mapping f of a semi-pseudometric space $\langle P_1, d_1\rangle$ into another one $\langle P_2, d_2\rangle$ is said to be uniformly continuous if for each $r > 0$ there exists an $s > 0$ such that $d_1\langle x, y\rangle < s$ implies $d_2\langle fx, fy\rangle < r$, stated in other words, if $\mathscr{U}_i$ is the semi-uniformity induced by d_i, then for each U_2 in $\mathscr{U}_2$ there exists a U_1 in $\mathscr{U}_1$ such that $\langle x, y\rangle \in U_1$ implies $\langle fx, fy\rangle \in U_2$, i.e., that $(\operatorname{gr} f \times \operatorname{gr} f)[U_1] \subset U_2$ holds.

23 C.1. Definition. A mapping f of a semi-uniform space $\langle P, \mathscr{U}\rangle$ into a semi-uniform space $\langle Q, \mathscr{V}\rangle$ is said to be *uniformly continuous* if for each V in $\mathscr{V}$ there exists a U in $\mathscr{U}$ such that $\langle x, y\rangle \in U$ implies $\langle fx, fy\rangle \in V$. A semi-uniformity $\mathscr{U}$ is said to be *uniformly finer* than a semi-uniformity $\mathscr{V}$, and $\mathscr{V}$ is said to be *uniformly coarser* than $\mathscr{U}$, if they are for the same set, say P, and the identity mapping of $\langle P, \mathscr{U}\rangle$ onto $\langle P, \mathscr{V}\rangle$ is uniformly continuous. Finally a *uniform homeomorphism* is a one-to-one mapping of a semi-uniform space $\langle P, \mathscr{U}\rangle$ onto a semi-uniform space $\langle Q, \mathscr{V}\rangle$ such that both f and f^{-1} are uniformly continuous.

Thus a mapping $f: \langle P_1, d_1\rangle \to \langle P_2, d_2\rangle$ for semi-pseudometric spaces is uniformly continuous (in the sense of Definition 18 B.14) if and only if $f: \langle P_1, \mathscr{U}_1\rangle \to \langle P_2, \mathscr{U}_2\rangle$ is uniformly continuous, where $\mathscr{U}_i$ is the semi-uniformity induced by d_i.

Before proceeding we shall prove various characterizations of uniform continuity which will usually be employed without any reference.

23 C.2. Theorem. *Suppose that f is a mapping of a semi-uniform space $\langle P, \mathscr{U}\rangle$ into a semi-uniform space $\langle Q, \mathscr{V}\rangle$, $\mathscr{U}'$ is a base for $\mathscr{U}$ and $\mathscr{V}'$ is a sub-base for $\mathscr{V}$. Each of the following conditions is equivalent to the uniform continuity of f:*

(a) *for each V in $\mathscr{V}$ there exists a U in $\mathscr{U}$ such that* $(\operatorname{gr} f \times \operatorname{gr} f)[U] \subset V$;

(b) $(\operatorname{gr} f \times \operatorname{gr} f)^{-1}[V] \in \mathscr{U}$ *for each V in $\mathscr{V}$;*

(c) $(\operatorname{gr} f \times \operatorname{gr} f)^{-1}[V] \in \mathscr{U}$ *for each V in $\mathscr{V}'$;*

(d) *for each V in $\mathscr{V}'$ there exists a U in $\mathscr{U}'$ such that* $(\operatorname{gr} f \times \operatorname{gr} f)[U] \subset V$, *i.e.* $f[U[x]] \subset V[fx]$ *for each x in P.*

Proof. For brevity let h stand for the relation $\operatorname{gr} f \times \operatorname{gr} f$. Thus $\mathbf{D}h = P \times P$, $\mathbf{E}h \subset Q \times Q$ and $h\langle x, y\rangle = \langle fx, fy\rangle$. – I. Since the implication $(\langle x, y\rangle \in U \Rightarrow \langle fx, fy\rangle \in V)$ is equivalent to $h[U] \subset V$, conditions (a) is merely a restatement of the definition. – II. Since $\mathscr{U}$ is a filter on $P \times P$ and $h[U] \subset V$ if and only if $h^{-1}[V] \supset U$ (because $\mathbf{D}h = P \times P$), condition (b) is equivalent to condition (a). – III. Obviously (b) implies (c). If (c) is fulfilled and V is an element of $\mathscr{V}$, then there exists a finite family $\{V_i \mid i \leqq n\}$ in $\mathscr{V}'$ such that $\bigcap\{V_i\} \subset V$; by (c) $h^{-1}[V_i] \in \mathscr{U}$ for each i, hence $\bigcap\{h^{-1}[V_i]\} \in \mathscr{U}$ ($\mathscr{U}$ is a filter) and finally $h^{-1}[V]$ belongs to $\mathscr{U}$ because $\mathscr{U}$ is a filter on $P \times P$ and $h^{-1}[U] \supset h^{-1}[\bigcap\{V_i\}] = \bigcap\{h^{-1}[V_i]\}$. – IV. Clearly (a) implies (d), for if $h[U] \subset V$ for some $U \in \mathscr{U}$, then we can choose a U' in $\mathscr{U}'$ with $U' \subset U$; clearly $h[U'] \subset V$). Assuming (d), if V is any element of $\mathscr{V}$, we can choose finite families $\{V_i\}$ in $\mathscr{V}'$ and $\{U_i\}$ in $\mathscr{U}'$ such that $\bigcap\{V_i\} \in V$ and $h[U_i] \subset V_i$ for each i; clearly $U = \bigcap\{U_i\} \in \mathscr{U}$ and $h[U] \subset V$, which establishes (d) $\Rightarrow$ (a).

23 C.3. Theorem. *A semi-uniformity $\mathscr{V}$ is uniformly coarser than a semi-uniformity $\mathscr{U}$ if and only if $\mathscr{V} \subset \mathscr{U}$.*

23 C.4. Theorem. *The composite of two uniformly continuous mappings is a uniformly continuous mapping; more precisely, if $f: \langle P, \mathscr{U}\rangle \to \langle Q, \mathscr{V}\rangle$ and $g: \langle Q, \mathscr{V}\rangle \to \langle R, \mathscr{W}\rangle$ are uniformly continuous mappings, then $g \circ f: \langle P, \mathscr{U}\rangle \to \langle R, \mathscr{W}\rangle$ is also a uniformly continuous mapping.*

Proof. Put $h = g \circ f$. If $W \in \mathscr{W}$, then $V = (\operatorname{gr} g \times \operatorname{gr} g)^{-1}[W] \in \mathscr{V}$ because g is uniformly continuous, and $U = (\operatorname{gr} f \times \operatorname{gr} f)^{-1}[V] \in \mathscr{U}$ because f is uniformly continuous. But clearly $U = (\operatorname{gr} h \times \operatorname{gr} h)^{-1}[W]$, which establishes that h is uniformly continuous.

23 C.5. Theorem. *The identity mapping of a semi-uniform space onto itself is a uniform homeomorphism. If f is a uniform homeomorphism then f^{-1} is also a uniform homeomorphism. If f and g are uniform homeomorphisms and $\mathbf{E}^*f = \mathbf{D}^*g$, then $g \circ f$ is also a uniform homeomorphism. It follows that the relation $\mathbf{E}\{\langle P, Q\rangle \mid$ there exists a uniform homeomorphism of P onto $Q\}$ is an equivalence on the class of all semi-uniform spaces.*

Proof. The first two statements are obvious, and to prove the third one it is sufficient to observe that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ and to apply 23 C.4 to both $g \circ f$ and $f^{-1} \circ g^{-1}$.

Recall that if we say that a semi-pseudometric space $\langle P, d\rangle$ has a property for closure spaces it is to be understood that the induced closure space $\langle P, u\rangle$ has this property, and if a mapping f for semi-pseudometric spaces has a property defined for closure spaces it is to be understood that f transposed (7 B.6) to a mapping for closure spaces has this property.

23 C.6. Conventions. If we say that a semi-uniform space $\langle P, \mathscr{U}\rangle$ has a property defined for closure spaces it is to be understood that the induced closure space has this property, e.g. a semi-uniform space $\langle P, \mathscr{U}\rangle$ is discrete means that the induced closure space is discrete. Similarly, a semi-uniformity $\mathscr{U}$ is finer than a semi-uniformity $\mathscr{V}$ means that the closure induced by $\mathscr{U}$ is finer than the closure induced by $\mathscr{V}$. If f is a mapping of a semi-uniform space $\langle P_1, \mathscr{U}_1\rangle$ into a semi-uniform space $\langle P_2, \mathscr{U}_2\rangle$, then the mapping $f: \langle P_1, u_1\rangle \to \langle P_2, u_2\rangle$, where u_i is the closure induced by $\mathscr{U}_i$, is termed f transposed to a mapping for closure spaces, and if we say that a mapping f for semi-uniform spaces has a property defined for mappings for closure spaces, it is to be understood that f transposed to a mapping for closure spaces has this property; e.g. $f: \langle P_1, \mathscr{U}_1\rangle \to \langle P_2, \mathscr{U}_2\rangle$ is continuous means that $f: \langle P_1, u_1\rangle \to \langle P_2, u_2\rangle$ is continuous. Finally, if we say that a semi-pseudometric space has a property defined for semi-uniform spaces it is to be understood that the induced semi-uniform space has this property, and a similar convention is used for mappings.

23 C.7. Theorem. *Every uniformly continuous mapping is continuous and every uniform homeomorphism is a homeomorphism.*

Corollary. *If a semi-uniformity $\mathscr{U}$ is uniformly finer than a semi-uniformity $\mathscr{V}$, then $\mathscr{U}$ is finer than $\mathscr{V}$.*

Proof. It is sufficient to show that every uniformly continuous mapping is continuous. Suppose that $f: \langle P, \mathscr{U}\rangle \to \langle Q, \mathscr{V}\rangle$ is uniformly continuous; we have to show that the mapping $f: \langle P, u\rangle \to \langle P, v\rangle$ is continuous, where u and v are the induced closures. By the uniform continuity of f, $U = (f \times f)^{-1}[V] \in \mathscr{U}$ for each V in $\mathscr{V}$, but clearly $U[x] = f^{-1}[V[fx]]$ for each x in P. Since the sets of the form

$V[fx]$, $V \in \mathscr{V}$, form a neighborhood system at x in $\langle P, v \rangle$, and the sets $U[x]$, $U \in \mathscr{U}$, form a neighborhood system at x in $\langle P, u \rangle$ (we only need that $U[x]$ are neighborhoods), f is continuous by 16 A.4.

If $\mathscr{U}$ and $\mathscr{V}$ are distinct semi-uniformities inducing the same closure u for a set P, then the identity mapping $\mathsf{J} : \langle P, \mathscr{U} \rangle \to \langle P, \mathscr{V} \rangle$ is a homeomorphism but either $\mathsf{J} : \langle P, \mathscr{U} \rangle \to \langle P, \mathscr{V} \rangle$ or its inverse $\mathsf{J}: \langle P, \mathscr{V} \rangle \to \langle P, \mathscr{U} \rangle$ is not uniformly continuous. Thus a homeomorphism need not be uniformly continuous.

23 C.8. *Suppose that f is a continuous mapping of a closure space $\langle P, u \rangle$ into a closure space $\langle Q, v \rangle$, $\mathscr{U}$ is the largest (= uniformly finest) continuous semi-uniformity for $\langle P, u \rangle$ and $\mathscr{V}$ is a continuous semi-uniformity for $\langle Q, v \rangle$. Then $f: \langle P, \mathscr{U} \rangle \to \langle Q, \mathscr{V} \rangle$ is uniformly continuous.*

Proof. If $V \in \mathscr{V}$, then V is a semi-neighborhood of the diagonal of $\langle Q, v \rangle \times \times \langle Q, v \rangle$ by 23 B.2, and f being continuous, $(f \times f)^{-1} [U]$ is a semi-neighborhood of the diagonal in $\langle P, u \rangle \times \langle P, u \rangle$; hence, by 23 B.2, U belongs to $\mathscr{U}$.

23 C.9. It is often rather difficult to decide whether or not two semi-uniform spaces are uniform homeomorphs of each other. A uniform property is a property is a property $\mathfrak{P}$ such that if P possesses $\mathfrak{P}$ then each uniform homeomorph of P also possesses $\mathfrak{P}$. To show that two semi-uniform spaces are not uniformly homemorphic it is sufficient to find a uniform property which is possessed by one space but not by the other. For example, a semi-uniformity for a set P is said to be *uniformly discrete* if it contains the diagonal of $P \times P$. Clearly "to be uniformly discrete" is a uniform property. Thus a uniformly discrete semi-uniform space is uniformly homeomorphic to no semi-uniform space which is not uniformly discrete. Next, it has already been shown (23 A.2) that a discrete semi-uniformity for an infinite set need not be uniformly discrete. Thus there exist two discrete semi-uniformities for an infinite set P which are not uniformly homeomorphic. A less trivial example may be in place. A semi-uniformity $\mathscr{U}$ for a set P is said to be *totally bounded* if for each U in $\mathscr{U}$ there exists a finite subset X of P such that $U[X] = P$. Obviously "to be totally bounded" is a uniform property. It may be shown that R is not totally bounded but every bounded subset of R endowed with the metric semi-uniformity is totally bounded (25 B.16). Thus, for example, R and $]\!] -1, 1 [\![$ are not uniformly homeomorphic but they are homeomorphic (e.g. $\{x \to x \,.\, (1 + |x|)^{-1} \mid x \in \mathsf{R}\} : \mathsf{R} \to \to]\!] 1, 1 [\![$ is a homeomorphism).

The following theorem describes uniform continuity in terms of uniformly continuous functions. The simple proof is left to the reader.

23 C.10. Theorem. *Suppose that f is a mapping of a semi-uniform space $\langle \mathscr{P}_1, \mathscr{U}_1 \rangle$ into another one $\langle P_2, \mathscr{U}_2 \rangle$, $\mathscr{M}_i$ is the set of all uniformly continuous semi-pseudometrics for $\langle P_i, \mathscr{U}_i \rangle$ and $\mathscr{U}_2$ is generated by a collection $\mathscr{M}'_2$ of pseudometrics. Then each of the following two conditions is equivalent to the uniform continuity of f:*

(a) *if $d \in \mathscr{M}_2$, then $d \circ (\operatorname{gr} f \times \operatorname{gr} f) \in \mathscr{M}_1$;*

(b) *if $d \in \mathscr{M}'_2$, then $d \circ (\operatorname{gr} f \times \operatorname{gr} f) \in \mathscr{M}_1$.*

23 C.11. Definition. The class of all semi-uniformities ordered by the relation $\{\mathscr{U}$ is uniformly finer than $\mathscr{V}\}$ will be denoted by $\mathbf{U}$. Given a set P, the ordered subset $\mathbf{U}$ consisting of all semi-uniformities for P will be denoted by $\mathbf{U}(P)$. The set of all uniformly continuous mappings of a semi-uniform space P into another one Q will be denoted by $\mathbf{U}(P, Q)$. Occasionally we will use the letter $\mathbf{U}$ to denote the class of all semi-uniform spaces.

Remarks. (a) It may be in place to recall that $\mathbf{C}$ denotes the class of all closure operations ordered by the relation $\{u$ is finer than $v\}$, and $\mathbf{C}(P, Q)$, where P and Q are closure spaces, denotes the set of all continuous mappings of P into Q. In accordance with earlier conventions, the symbol $\mathbf{C}(P, Q)$ is also meaningful if P and Q are semi-uniform spaces or semi-pseudometric spaces; e.g. if $\langle P, \mathscr{U}\rangle$ is a semi-uniform space and $\langle Q, d\rangle$ is a semi-pseudometric space then $\mathbf{C}(\langle P, \mathscr{U}\rangle, \langle Q, d\rangle)$ is the set of all continuous mappings of $\langle P, \mathscr{P}\rangle$ into $\langle Q, d\rangle$; if u is the closure induced by $\mathscr{U}$ and v is the closure induced by d then $\mathbf{E}\{\langle f, f : \langle P, u\rangle \to \langle Q, v\rangle\rangle \mid f \in \mathbf{C}(\langle P, \mathscr{U}\rangle, \langle Q, d\rangle)\}$ is a one-to-one relation ranging on $\mathbf{C}(\langle P, u\rangle, \langle P, v\rangle)$. Similarly, $\mathbf{U}(\langle P_1, d_1\rangle, \langle P_2, d_2\rangle)$ is meaningful if d_i are semi-pseudometrics; it denotes the set of all uniformly continuous mappings of $\langle P_1, d_1\rangle$ into $\langle P_2, d_2\rangle$. If $\mathscr{U}_i$ is the semi-uniformity induced by d_i, then

$$\mathbf{E}\{\langle f, f : \langle P_1, \mathscr{U}_1\rangle \to \langle P_2, \mathscr{U}_2\rangle\rangle \mid f \in \mathbf{U}(\langle P_1, d_1\rangle, \langle P_2, d_2\rangle)\}$$

is a one-to-one relation ranging on $\mathbf{U}(\langle P_1, \mathscr{U}_1\rangle, \langle P_2, \mathscr{U}_2\rangle)$.

(b) Theorem 23 C.4 can be restated as follows: the composition of mappings is a strongly associative partial composition on the class of all uniformly continuous mappings.

D. SUBSPACES, SUMS AND PRODUCTS

Much of the introduction to Section 17 concerning the constructions for closure spaces may serve as a motivation for the definitions given below; it is only necessary to replace the expressions closure space, closure operation, continuous mapping and "coarser than" by the corresponding expressions for semi-uniform spaces, that is, semi-uniform space, semi-uniformity, uniformly continuous mapping and "uniformly coarser than".

23 D.1. Definition. If $\langle P, \mathscr{U}\rangle$ is a semi-uniform space and $Q \subset P$, then the collection $[\mathscr{U}] \cap (Q \times Q)$ (consisting of $U \cap (Q \times Q)$, $U \in \mathscr{U}$) is obviously a semi-uniformity for Q which is called the *relativization* of $\mathscr{U}$ to Q; the corresponding semi-uniform space is said to be a *subspace* of $\langle P, \mathscr{U}\rangle$. A class of semi-uniform spaces is said to be *hereditary* if, with each space $\mathscr{P}$, it contains all subspaces of $\mathscr{P}$.

As in the case of closure spaces a subspace of a space is uniquely determined by the underlying set.

23 D.2. *Suppose that* $\langle Q, \mathscr{V}\rangle$ *is a subspace of a semi-uniform space* $\langle P, \mathscr{U}\rangle$.

Then

(a) *The closure induced by* $\mathscr{V}$ *is a relativization of the closure induced by* $\mathscr{U}$;

(b) $\mathscr{V}$ *is the unique uniformly coarsest (= smallest) semi-uniformity for Q which renders the identity mapping of Q into* $\langle P, \mathscr{U}\rangle$ *uniformly continuous (compare with* 17 A.2*)*.

(c) *If* $R \subset Q$, *then* $\langle R, \mathscr{W}\rangle$ *is a subspace of* $\langle Q, \mathscr{V}\rangle$ *if and only if* $\langle R, \mathscr{W}\rangle$ *is a subspace of* $\langle P, \mathscr{U}\rangle$.

The proof is straightforward and therefore left to the reader.

In connection with statement (a) we shall prove the following result:

23 D.3. *If* $\langle Q, v\rangle$ *is a subspace of a semi-uniformizable closure space* $\langle P, u\rangle$ *and if a semi-uniformity* $\mathscr{V}$ *induces v, then* $\mathscr{V}$ *is a relativization of a semi-uniformity inducing u.*

Proof. Let $\mathscr{U}_1$ be the largest continuous semi-uniformity for $\langle P, u\rangle$ (see 23 B.2) and put $\mathscr{U} = [\mathscr{V}] \cup [\mathscr{U}_1]$ (= the collection of all $V \cup U_1$, $V \in \mathscr{V}$, $U_1 \in \mathscr{U}_1$). It is easily seen that $\mathscr{U}$ has the required properties.

In accordance with the general description of the restriction of a mapping we introduce the following definition (compare with definition 17 A.12 of the restriction of mappings for closure spaces).

23 D.4. Definition. The *restriction of a mapping f* for semi-uniform spaces is a mapping $f : \mathscr{P} \to \mathscr{Q}$ such that $\mathscr{P}$ is a subspace of $\mathbf{D}^*f$ and $\mathscr{Q}$ is a subspace of $\mathbf{E}^*f$.

Remark. As in the case of closure spaces, the concept of a subspace was defined in such a manner that 23 D. 6 hold. A restriction of a mapping f for semi-uniform spaces is a mapping g for semi-uniform spaces such that the graph of g is a restriction of the graph of f, $\mathbf{D}^*g$ is a subspace of $\mathbf{D}^*f$ and $\mathbf{E}^*g$ is a subspace of $\mathbf{E}^*f$; if $\mathbf{D}^*g = \mathbf{D}^*f$ then g is a range-restriction, and if $\mathbf{E}^*g = \mathbf{E}^*f$ then g is a domain-restriction. In accordance with the general rule 7 B. 5, the extension of a mapping g for semi-uniform spaces is any mapping f such that g is a restriction of f. A *uniform embedding* is a mapping f for semi-uniform spaces such that the range-restriction f: $\mathbf{D}^*f \to \mathbf{E}f$ is a uniform homeomorphism (where $\mathbf{E}f$ is considered as a space).

23 D.5. *Every restriction of a uniformly continuous mapping is a uniformly continuous mapping.* — Obvious.

23 D.6. *A mapping f of a semi-uniform space P into a semi-uniform space Q is uniformly continuous if and only if the range-restriction of f to a mapping of P onto the subspace* $\mathbf{E}f$ *of Q is uniformly continuous.*

Proof. "Only if" follows from 23 D.5 and "if" is obvious.

23 D.7. *If g is a restriction of a mapping f for semi-uniform spaces and* g_1 *and* f_1 *are the transposes of f and g to mappings for closure spaces, then* g_1 *is a restriction of* f_1. — Obvious.

Remark. Sometimes we shall need the following immediate consequence of the definitions: An injective mapping of a semi-uniform space $\langle P, \mathscr{U}\rangle$ into a semi-uniform space $\langle Q, \mathscr{V}\rangle$ is a uniform embedding if and only if f is uniformly continuous and there exists a sub-base $\mathscr{U}'$ for $\mathscr{U}$ such that for each U in $\mathscr{U}'$ there exists a V in $\mathscr{V}$ with $(f \times f)[U] \supset V \cap (\mathbf{E}f \times \mathbf{E}f)$.

Now we proceed to sums of semi-uniform spaces.

23 D.8. Definition. The *sum of a family* $\{\langle P_a, \mathscr{U}_a\rangle \mid a \in A\}$ *of semi-uniform spaces*, denoted by $\Sigma\{\langle P_a, \mathscr{U}_a\rangle\}$, is the semi-uniform space $\langle P, \mathscr{U}\rangle$ where $P = \Sigma\{P_a\}$ and $\mathscr{U}$, called the sum semi-uniformity, is the collection of all subsets of $P \times P$ containing a set of the form

$$(*) \qquad \bigcup\{(\mathrm{inj}_a \times \mathrm{inj}_a)[U_a] \mid a \in A\}$$

where $U_a \in \mathscr{U}_a$ for each a in A.

Of course we must show that $\mathscr{U}$ is actually a semi-uniformity, that is, that the relations of the form $(*)$ form a base of a semi-uniformity. By virtue of **23 A.4** it is sufficient to show that each relation $(*)$ is a vicinity, that is, contains the diagonal, and U^{-1} is of the form $(*)$ whenever U is of that form. But this is almost self-evident.

23 D.9. Theorem. *Let* $\langle P, \mathscr{U}\rangle$ *be the sum of a family* $\{\langle P_a, \mathscr{U}_a\rangle \mid a \in A\}$ *of semi-uniform spaces. Then*

(a) *If* u_a *is the closure induced by* $\mathscr{U}_a$ *for each* a*, then the sum closure* $\Sigma\{u_a\}$ *is induced by* $\mathscr{U}$.

(b) *The mappings* $\mathrm{inj}_a : \langle P_a, \mathscr{U}_a\rangle \to \langle P, \mathscr{U}\rangle$ *are uniform embeddings (which will be called the canonical embeddings).*

(c) $\mathscr{U}$ *is the uniformly finest semi-uniformity for* P *such that all the mappings* $\mathrm{inj}_a : \langle P_a, \mathscr{U}_a\rangle \to \langle P, \mathscr{U}\rangle$ *are uniformly continuous.*

(d) *A mapping* f *of* $\langle P, \mathscr{U}\rangle$ *into a semi-uniform space* $\langle Q, \mathscr{V}\rangle$ *is uniformly continuous if and only if all the mappings* $f \circ \mathrm{inj}_a : \langle P_a, \mathscr{U}_a\rangle \to \langle Q, \mathscr{V}\rangle$ *are uniformly continuous.*

Proof. Denote by P'_a the set $\mathrm{inj}_a[P_a] = (a) \times P_a$ and let $\mathscr{U}'_a$ be the relativization of $\mathscr{U}$ to P'_a. If $\{U_a\}$ is a family such that $U_a \in \mathscr{U}_a$ for each a and if U is the corresponding set $(*)$, then $U \cap (P'_a \times P'_a) = (\mathrm{inj}_a \times \mathrm{inj}_a)[U_a]$. It follows that $\mathrm{inj}_a :$ $: \langle P_a, \mathscr{U}_a\rangle \to \langle P'_a, \mathscr{U}'_a\rangle$ is a uniform homeomorphism for each a (which proves (b)) and $\bigcup\{P'_a \times P'_a\}$ belongs to $\mathscr{U}$. Since $\{P'_a\}$ is a disjoint family, we find that each set P'_a is simultaneously open and closed in the space $\langle P, \mathscr{U}\rangle$. It follows that the closure induced by $\mathscr{U}$ coincides with the sum closure $\Sigma\{u_a\}$.

Statement (c) is almost evident; indeed, if $\mathscr{V}$ is a semi-uniformity for P such that all mappings $\mathrm{inj}_a : \langle P_a, \mathscr{U}_a\rangle \to \langle P, \mathscr{V}\rangle$ are uniformly continuous, then necessarily $\mathscr{V}$ contains each set of the form $(*)$, but the sets of the form $(*)$ form a base for $\mathscr{U}$ and hence $\mathscr{U} \subset \mathscr{V}$. It remains to prove (d). If f is uniformly continuous then each mapping in question is uniformly continuous as the composite of two uniformly continuous mappings. Conversely suppose that each composite in question is uniformly continuous and let V be an element of $\mathscr{V}$. Since each composite is uniformly continuous we can choose a family $\{U_a\}$ such that $U_a \in \mathscr{U}_a$ and

$$(f \circ \text{inj}_a \times f \circ \text{inj}_a)[U_a] \subset V$$

for each a. If U is the set $(*)$ corresponding to $\{U_a\}$, then clearly $(f \times f)[U] \subset V$; this establishes the uniform continuity of f.

Now we shall turn to products.

23 D.10. Definition. The *product of a family* $\{\langle P_a, \mathscr{U}_a\rangle \mid a \in A\}$ *of semi-uniform spaces*, denoted by $\Pi\{\langle P_a, \mathscr{U}_a\rangle \mid a \in A\}$ is defined to be the semi-uniform space $\langle P, \mathscr{U}\rangle$ where P is the product of the family $\{P_a\}$ of the underlying sets, and $\mathscr{U}$, called the *product semi-uniformity*, is the collection of all subsets of $P \times P$ containing a set of the form

$$(*) \qquad \mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in P \times P, a \in F \Rightarrow \langle \text{pr}_a x, \text{pr}_a y\rangle \in U_a\}$$

where F is a finite subset of A and $U_a \in \mathscr{U}_a$ for each a. Sets of the form $(*)$ are then called the *canonical elements* of the product semi-uniformity.

It must be shown that the collection of all canonical elements of $\mathscr{U}$ is a base for a semi-uniformity. It is sufficient to show that the collection of all sets of the form $(*)$ with F one-point form a sub-base for a semi-uniformity; but this follows from the Corollary of 23 A.4. The main properties of products are summarized in the following.

23 D.11. Theorem. *Let* $\langle P, \mathscr{U}\rangle$ *be the product of a family* $\{\langle P_a, \mathscr{U}_a\rangle \mid a \in A\}$ *of semi-uniform spaces. Then*

(a) *The product closure is induced by* $\mathscr{U}$, *more precisely, if* u_a *is induced by* $\mathscr{U}_a$ *for each* a, *then the product closure* $\Pi\{u_a\}$ *is induced by* $\mathscr{U}$.

(b) *Each mapping* $\text{pr}_a : \langle P, \mathscr{U}\rangle \to \langle P_a, \mathscr{U}_a\rangle$ *is uniformly continuous (and called the projection of* $\langle P, \mathscr{U}\rangle$ *into* $\langle P_a, \mathscr{U}_a\rangle$*).*

(c) $\mathscr{U}$ *is the uniformly coarsest (= smallest) semi-uniformity such that all the mappings* $\text{pr}_a : \langle P, \mathscr{U}\rangle \to \langle P_a, \mathscr{U}_a\rangle$ *are uniformly continuous.*

(d) *A mapping* f *of a semi-uniform space* $\langle Q, \mathscr{V}\rangle$ *into* $\langle P, \mathscr{U}\rangle$ *is uniformly continuous if and only if all the mappings* $\text{pr}_a \circ f : \langle Q, \mathscr{V}\rangle \to \langle P_a, \mathscr{U}_a\rangle$, $a \in A$, *are uniformly continuous.*

(e) *Fix an* α *in* A. *If the projection* $\text{pr}_\alpha : \langle P, \mathscr{U}\rangle \to \langle P_\alpha, \mathscr{U}_\alpha\rangle$ *is surjective (in particular, if* $P \neq \emptyset$*), then a mapping* h *of* $\langle P_\alpha, \mathscr{U}_\alpha\rangle$ *into a semi-uniform space* $\langle R, \mathscr{W}\rangle$ *is uniformly continuous if and only if the composite* $h \circ \text{pr}_\alpha : \langle P, \mathscr{U}\rangle \to \langle R, \mathscr{W}\rangle$ *is uniformly continuous.*

Proof. I. Statement (a) will follow from the following observation: If U_a is any subset of $P_a \times P_a$ and x is any point of P then the set

$$(**) \qquad \mathbf{E}\{y \mid y \in P, \text{pr}_a y \in U_a[\text{pr}_a x]\}$$

coincides with the set

$$(***) \qquad (\mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in P \times P, \langle \text{pr}_a x, \text{pr}_a y\rangle \in U_a\})[x].$$

Indeed, given $x \in P$, the sets $(**)$ with a in A and U_a in $\mathscr{U}_a$ form a local sub-base at x in $\langle P, \Pi\{u_a\}\rangle$ (because $[\mathscr{U}_a][\text{pr}_a x]$ is a neighborhood system at $\text{pr}_a x$ in

$\langle P_a, u_a \rangle$) and the sets (***) with a in A and U_a in $\mathscr{U}_a$ form a local sub-base at x in $\langle P, u \rangle$ (because the sets $\mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in P \times P, \langle \mathrm{pr}_a x, \mathrm{pr}_a y\rangle \in \mathscr{U}_a\}$ form a sub-base for $\mathscr{U}$).

II. Let f_a be the projection of $\langle P, \mathscr{U}\rangle$ into $\langle P_a, \mathscr{U}_a\rangle$; we have $(f_a \times f_a)^{-1}[U_a] = \mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in P \times P, \langle \mathrm{pr}_a x, \mathrm{pr}_a y\rangle \in U_a\} \in \mathscr{U}$ for each U_a in $\mathscr{U}_a$, and this means that each f_a is uniformly continuous and establishes statement (b). If $\mathscr{U}'$ is any semi-uniformity such that all mappings $\mathrm{pr}_a : \langle P, \mathscr{U}'\rangle \to \langle P_a, \mathscr{U}_a\rangle$, $a \in A$, are uniformly continuous, then necessarily every set

$$\mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in P \times P, \langle \mathrm{pr}_a x, \mathrm{pr}_a y\rangle \in U_a\}$$

with a in A and U_a in $\mathscr{U}_a$ belongs to $\mathscr{U}'$; but these sets form a sub-base for $\mathscr{U}$ and hence $\mathscr{U} \subset \mathscr{U}'$; this shows that $\mathscr{U}'$ is uniformly finer than $\mathscr{U}$ and establishes statement (c).

III. The proof of (d): if f is uniformly continuous, then each mapping in question is uniformly continuous as the composite of two uniformly continuous mappings (the projections are uniformly continuous by (b)). Conversely, suppose that all the mappings in question are uniformly continuous. Let $\mathscr{U}_1$ be the sub-base for $\mathscr{U}$ consisting of all the sets

$$U'_a = \mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in P \times P, \langle \mathrm{pr}_a x, \mathrm{pr}_a y\rangle \in U_a\}, a \in A, U_a \in \mathscr{U}_a .$$

By 23 C.2 it is sufficient to show that $(f \times f)^{-1}[U'_a] \in \mathscr{V}$ for each a in A and U_a in $\mathscr{U}_a$. But this is almost self-evident as $(f \times f)^{-1}[U'_a] = (\mathrm{pr}_a \circ f \times \mathrm{pr}_a \circ f)^{-1}[U_a]$ and $\mathrm{pr}_a \circ f$ is a uniformly continuous mapping of $\langle Q, \mathscr{V}\rangle$ into $\langle P_a, \mathscr{U}_a\rangle$.

IV. It remains to prove (e): If h is uniformly continuous then the mapping $h \circ \mathrm{pr}_\alpha$ of $\langle P, \mathscr{U}\rangle$ into $\langle R, \mathscr{W}\rangle$ is uniformly continuous as the composite of two uniformly continuous mappings (by 23 C.4), namely of the projection of $\langle P, \mathscr{U}\rangle$ into $\langle P_\alpha, \mathscr{U}_\alpha\rangle$ and h. Conversely, suppose that $k = h \circ \mathrm{pr}_\alpha : \langle P, \mathscr{U}\rangle \to \langle R, \mathscr{W}\rangle$ is uniformly continuous and the projection f_α into $\langle P_\alpha, \mathscr{U}_\alpha\rangle$ is surjective. Clearly $(k \times k)^{-1}[W] = (f_\alpha \times f_\alpha)^{-1}[(h \times h)^{-1}[W]]$ for each W in $\mathscr{W}$. Now the proof will be accomplished if we show that $U_\alpha \subset P_\alpha \times P_\alpha$, $(f_\alpha \times f_\alpha)^{-1}[U_\alpha] \in \mathscr{U}$ implies $U_\alpha \in \mathscr{U}_\alpha$ provided that f_α is surjective. But this is evident.

23 D.12. *If $\{P_a\}$ and $\{Q_a\}$ are families of semi-uniform spaces such that Q_a is a subspace of P_a for each a, then the product of $\{Q_a\}$ is a subspace of the product of $\{P_a\}$.* – Evident.

23 D.13. Definition. The *product of a family $\{f_a\}$ of mappings* for semi-uniform spaces, denoted by $\Pi\{f_a\}$, is defined to be the mapping of $\Pi\{\mathbf{D}^* f_a\}$ into $\Pi\{\mathbf{E}^* f_a\}$ which assigns to each point $\{x_a\}$ the point $\{f_a x_a\}$; thus

$$\Pi\{f_a\} = \langle \Pi_{\mathrm{rel}}\{\mathrm{gr}\, f_a\}, \Pi\{\mathbf{D}^* f_a\}, \Pi\{\mathbf{E}^* f_a\}\rangle .$$

The *reduced product of a family $\{f_a\}$ of mappings* for semi-uniform spaces is defined if and only if all f_a have a common domain carrier, say P, in which case the reduced product is that mapping of P into $\Pi\{\mathbf{E}^* f_a\}$ which assigns to each $x \in P$ the point $\{f_a x\}$; thus the graph of the reduced product is the relational reduced product of the graphs.

It is to be noted that if f is the product (reduced product) of a family $\{f_a\}$ of mappings for semi-uniform spaces and if g_a is f_a transposed to a mapping for closure spaces and g is f transposed to a mapping for closure spaces, then g is the product (reduced product) of $\{g_a\}$. By 17 C.13 the product (reduced product) of continuous mappings is a continuous mapping. The same is true for mappings for semi-uniform spaces.

23 D.14. Theorem. *The reduced product f of a family $\{f_a \mid a \in A\}$ of mappings for semi-uniform spaces is uniformly continuous if and only if all the mappings f_a are uniformly continuous.*

Proof. Since clearly $f_a = \langle \mathrm{pr}_a \circ \mathrm{gr}\, f, \mathbf{D}^* f_a, \mathbf{E}^* f_a \rangle$ for each a in A, the result follows form 23 D.11 (d).

23 D.15. Theorem. *Let f be the product of a family $\{f_a \mid a \in A\}$ of mappings for semi-uniform spaces. If all the f_a are uniformly continuous, then f is also uniformly continuous. Conversely, if $\mathbf{D}f \neq \emptyset$ and f is uniformly continuous, then all the f_a are uniformly continuous.*

Proof. I. For each a in A let g_a denote the mapping $f_a \circ (\mathrm{pr}_a : \mathbf{D}^* f \to \mathbf{D}^* f_a)$. Evidently f is the reduced product of the family $\{g_a \mid a \in A\}$. According to the preceding theorem the mapping f is uniformly continuous if and only if all the mappings g_a are uniformly continuous. – II. Now if all the f_a are uniformly continuous, then all the g_a are uniformly continuous as composites of uniformly continuous mappings, and finally f is continuous by I. – III. Now let f be uniformly continuous. By I all the g_a are uniformly continuous. If in addition $\mathbf{D}f \neq \emptyset$, then the mappings $\mathrm{pr}_a : \mathbf{D}^* f \to \mathbf{D}^* f_a$ are surjective and the uniform continuity of f_a follows from the uniform continuity of g_a by 23 D.11 (e).

23 D.16. Theorem. *Let f be the product of a family of mappings $\{f_a\}$. If each f_a is a uniform homeomorphism or a uniform embedding then f has the same property. Conversely, if $\mathbf{D}f \neq \emptyset$ and f is a uniform homeomorphism or a uniform embedding then each f_a has the same property.*

Proof. Applying 23 D.15 to both f and f^{-1} we obtain the statement concerning uniform homeomorphisms. Statements concerning uniform embeddings follow immediately from the corresponding statements for uniform homeomorphisms and 23 D.6.

Remark. In this connection, one may show that the operation of forming products is commutative in a certain sense, namely if $\{P_a \mid a \in A\}$ is a family of semi-uniform spaces and φ is a bijective mapping of A, then $\Pi\{P_a\}$ and $\Pi\{P_{\varphi a}\}$ are uniform homeomorphs of each other.

23 D.17. Definition. In agreement with the notation for the products of closure spaces, we shall denote the product $\Pi\{P \mid a \in A\}$ of semi-uniform spaces by P^A.

It is apparent from 23 D.16 that P^A and Q^B are uniformly homeomorphic provided that P and Q are uniformly homeomorphic and A and B are equipollent.

The next theorem implies that every semi-uniform space can be uniformly embedded into the product of semi-pseudometrizable semi-uniform spaces.

23 D.18. Theorem. *Suppose that* $\{\mathcal{U}_a \mid a \in A\}$ *is a non-void family of semi-uniformities for a set P and* $\mathcal{U}$ *is the smallest semi-uniformity containing all the* $\mathcal{U}_a$ *(thus* $\bigcup\{\mathcal{U}_a\}$ *is a sub-base for* $\mathcal{U}$ *by the corollary to* 23 A.4). *For each a in A let* f_a *be the identity mapping of* $\langle P, \mathcal{U}\rangle$ *onto* $\langle P, \mathcal{U}_a\rangle$. *The reduced product f of* $\{f_a \mid a \in A\}$ *is an embedding* $\big($*of* $\langle P, \mathcal{U}\rangle$ *into* $\Pi\{\langle P, \mathcal{U}_a\rangle \mid a \in A\}\big)$.

Corollary. *Suppose that a semi-uniformity* $\mathcal{U}$ *for a set P is generated by a collection* $\mathcal{M}$ *of semi-pseudometrics. For each d in* $\mathcal{M}$ *let* $\mathcal{U}_d$ *be the semi-uniformity induced by d and* f_d *be the identity mapping of* $\langle P, \mathcal{U}\rangle$ *onto* $\langle P, \mathcal{U}_d\rangle$. *Then the reduced product of the family* $\{f_d \mid d \in \mathcal{M}\}$ *is an embedding.*

In particular, every semi-uniform space admits an embedding into the product of a family of semi-pseudometrizable semi-uniform spaces (use 23 A.15).

Proof of 23 D.18. Obviously the mapping f is injective, and by 23 D.14 f is uniformly continuous because each f_a is uniformly continuous. It remains to find a sub-base $\mathcal{U}'$ for $\mathcal{U}$ so that each set $[f \times f][U]$, $U \in \mathcal{U}'$, contains a set of the form $V \cap (\mathbf{E}f \times \mathbf{E}f)$ for some element V of the semi-uniformity of $\mathbf{E}^*f$. Let $\mathcal{U}' = \bigcup\{\mathcal{U}_a\}$; if $U \in \mathcal{U}'$, then $U \in \mathcal{U}_a$ for some a and clearly we can take

$$V = \mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in P^A \times P^A, \langle \mathrm{pr}_a x, \mathrm{pr}_a y\rangle \in U\}$$

(indeed, $(f \times f)[U] = V \cap (\mathbf{E}f \times \mathbf{E}f)$).

Recall that, by 18 A.17, a pseudometric d for a closure space $\langle P, u\rangle$ is continuous (i.e. the closure induced by d is coarser than u) if and only if the function $d : \langle P, u\rangle \times \langle P, u\rangle \to \mathrm{R}$ is continuous. The final theorem asserts a similar result for uniform continuity.

23 D.19. Theorem. *In order that a pseudometric d for a semi-uniform space* $\langle P, \mathcal{U}\rangle$ *be uniformly continuous it is necessary and sufficient that the function* $d : \langle P, \mathcal{U}\rangle \times \langle P, \mathcal{U}\rangle \to \mathrm{R}$ *be uniformly continuous.*

Proof. If $d : \langle P, \mathcal{U}\rangle \times \langle P, \mathcal{U}\rangle \to \mathrm{R}$ is uniformly continuous, then for each $r > 0$ there exists a U in $\mathcal{U}$ such that $\langle x_1, x_2\rangle \in U$, $\langle y_1, y_2\rangle \in U$ implies $|d\langle x_1, x_2\rangle - d\langle y_1, y_2\rangle| < r$; in particular, if $y_1 = y_2$, then $\langle y_1, y_2\rangle \in U$ and $d\langle y_1, y_2\rangle = 0$, and hence $\langle x_1, x_2\rangle \in U$ implies $d\langle x_1, x_2\rangle < r$ which proves that d is a uniformly continuous semi-pseudometric for $\langle P, \mathcal{U}\rangle$. Notice that the triangle inequality was not used. Conversely, suppose that d is a uniformly continuous pseudometric. We must show that for each $r > 0$ there exist a U in $\mathcal{U}$ and a V in $\mathcal{U}$ so that $\langle x_1, y_1\rangle \in U$, $\langle x_2, y_2\rangle \in V$ implies $|d\langle x_1, x_2\rangle - d\langle y_1, y_2\rangle| < r$. Choose a positive s such that $2s \leqq r$ and a U in $\mathcal{U}$ such that $\langle z_1, z_2\rangle \in U$ implies $d\langle z_1, z_2\rangle < s$. Now, if $\langle x_1, y_1\rangle \in U$ and $\langle x_2, y_2\rangle \in U$, then (by 18 A.11)

$$|d\langle x_1, y_2\rangle - d\langle y_1, y_2\rangle| \leqq d\langle x_1, y_1\rangle + d\langle x_2, y_2\rangle < 2s \leqq r$$

which establishes the uniform continuity of the function d on $\langle P, \mathcal{U}\rangle \times \langle P, \mathcal{U}\rangle$.

24. UNIFORM SPACES

Each semi-pseudometric d for a set P induces a semi-uniformity $\mathcal{U}$; the collection of all $U_r = \mathsf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\}$, $r > 0$, is a base for $\mathcal{U}$. If d is a pseudometric, that is, if it fulfils the triangle inequality, then

(u 3) for each U in $\mathcal{U}$ there exists a V in $\mathcal{U}$ such that $V \circ V \subset U$.

Indeed, if $U \in \mathcal{U}$, then $U_r \subset U$ for some $r > 0$ and we may put $V = U_s$ where s is any positive real less than $2^{-1} . r$; in fact, by the triangle inequality, $U_s \circ U_s \subset U_{2s}$. In this section the semi-uniformities $\mathcal{U}$ satisfying (u 3), the so-called uniformities, will be studied. Roughly speaking, uniformities are related to semi-uniformities as pseudometrics to semi-pseudometrics. Pseudometrics play the same part in the theory of uniformities as semi-pseudometrics do in the theory of semi-uniformities, e.g. every semi-uniformity is generated by the collection of all uniformly continuous semi-pseudometrics and every uniformity is generated by the collection of all uniformly continuous pseudometrics. A semi-uniformity has a countable base if and only if it is semi-pseudometrizable, and a uniformity has a countable base if and only if it is pseudometrizable, i.e. induced by a pseudometric. We shall see that axiom (u 3) has topological consequences. Without doubt, uniformizable closures, i.e. the closures induced by a uniformity, form the most important class of closure operations. We know that semi-uniform spaces enable one to define uniformly continuous mappings in the most general situation; however, uniformly continuous mappings of uniform spaces have further important extension properties which are often included in the intuitive content of a uniformly continuous mapping.

In subsection A we shall prove some topological conclusions of axiom (u 3), we shall clarify the role of pseudometrics in the theory of uniform spaces (as indicated above) and prove that the class of all uniform spaces is completely productive, hereditary and closed under formation of sums.

If $\mathcal{U}$ is a semi-uniformity then there exists a (unique) largest uniformity contained in $\mathcal{U}$ which is said to be the uniform modification of $\mathcal{U}$. Uniform modifications are introduced and studied in subsection B.

If $\mathcal{G}$ is a topological group and $\mathcal{U}$ is a local base at the unit element, then the collection of all $U_L = \mathsf{E}\{\langle x, y\rangle \mid x^{-1} . y \in U\}$, $U \in \mathcal{U}$, is a base for a uniformity $\mathcal{L}$ called the left uniformity of $\mathcal{G}$, and the collection of all $U_R = \mathsf{E}\{\langle x, y\rangle \mid x . y^{-1} \in U\}$,

$U \in \mathscr{U}$, is a base for a uniformity $\mathscr{R}$ called the right uniformity of $\mathscr{G}$. The union $\mathscr{R} \cup \mathscr{L}$ is a sub-base for the so-called two-sided uniformity of $\mathscr{G}$. These uniformities are called the group uniformities of $\mathscr{G}$. It turns out that all the three group uniformities induce the closure structure of $\mathscr{G}$. Properties of group uniformities are examined in subsection C.

In 19 B.17 we introduced the closure of uniform convergence for the set $\mathbf{F}(\mathscr{S}, \mathscr{G})$ of all mappings of a struct $\mathscr{S}$ into a commutative topological group $\mathscr{G}$ and we proved that a uniform limit of continuous mappings is a continuous mapping. In subsection D we shall endow the set $\mathbf{F}(\mathscr{S}, \mathscr{P})$ of mappings of a struct $\mathscr{S}$ into a uniform space $\mathscr{P}$ with a uniformity such that, if $\mathscr{S}$ is a closure space, then $\mathbf{C}(\mathscr{S}, \mathscr{P})$ is closed in $\mathbf{F}(\mathscr{S}, \mathscr{P})$, i.e. the uniform limit of continuous mappings is continuous, and if $\mathscr{S}$ is a uniform space, then $\mathbf{U}(\mathscr{S}, \mathscr{P})$ is closed in $\mathbf{F}(\mathscr{S}, \mathscr{P})$, i.e. the uniform limit of uniformly continuous mappings is uniformly continuous. We shall show that the result of 19 B.17 mentioned above is a corollary of results of subsection D.

The concluding subsection E is concerned with the description of uniformities by means of uniform collections of covers. We shall introduce the important concepts of a uniform cover of a uniform space and a uniformizable cover of a closure space (this latter is often termed a normal cover).

A. UNIFORMITIES AND PSEUDOMETRICS

24 A.1. Definition. A *uniformity for a set P* is a semi-uniformity $\mathscr{U}$ for P satisfying condition (u 3) above. A *uniform space* is a semi-uniform space $\langle P, \mathscr{U} \rangle$ such that $\mathscr{U}$ is a uniformity. Recall that a closure operation induced by a semi-uniformity is said to be semi-uniformizable. Naturally, a closure operation induced by a uniformity will be called *uniformizable,* and a set endowed with a uniformizable closure operation will be called a *uniformizable space.*

Uniformizable spaces will be studied rather extensively in Section 28. Nevertheless, to clarify the force of condition (u 3), in the proposition which follows we shall prove several properties of uniformizable closures. It is to be noted that all of these properties also follow immediately from the description of a uniformity in terms of uniformly continuous pseudometrics (24 A.9), the proof of which depends essentially upon the pseudometrization lemma (18 B.10) where the proof is rather technical; as a result, the proofs of simple topological consequences of condition (u 3), stated in the next proposition, may not be clear. Therefore we prefer to give a direct proof. Let us recall that a semi-uniformizable space need not be topological, and the elements of a semi-uniformity are semi-neighborhoods of the diagonal (relative to the product of the induced closures) and need not be neighborhoods.

24 A.2. *Let $\mathscr{U}$ be a uniformity for a set P and let u be the closure induced by $\mathscr{U}$. Then*

(a) $\langle P, u \rangle$ *is a topological space (thus every uniformizable space is topological);*

(b) *The collection of all closed (in $\langle P, u\rangle \times \langle P, u\rangle$) elements of $\mathcal{U}$ is a base for $\mathcal{U}$;*

(c) *The collection of all open (in $\langle P, u\rangle \times \langle P, u\rangle$) elements of $\mathcal{U}$ is a base for $\mathcal{U}$, in particular every element of $\mathcal{U}$ is a neighborhood of the diagonal.*

Proof. (a) First let us recall that (by 23 B.5) we have $uX = \bigcap[\mathcal{U}][X]$ $(= \bigcap\{U[X] \mid U \in \mathcal{U}\})$ for each $X \subset P$. Fix a $Y \subset P$ and a $U \in \mathcal{U}$. It must be shown that $uuY \subset U[Y]$. Choose V in $\mathcal{U}$ so that $V \circ V \subset U$. We have $uY \subset V[Y]$ and hence $uuY \subset uV[Y]$; but $uV[Y] \subset V[V[Y]] = V \circ V[Y] \subset U[Y]$.

(b) Since $\langle P, u\rangle$ is a topological space, $\langle P, u\rangle \times \langle P, u\rangle$ is also topological, and consequently to prove (b) it is sufficient to show that each element U of $\mathcal{U}$ contains the closure of an element V of $\mathcal{U}$. But if $V \in \mathcal{U}$ is chosen symmetric and such that $V \circ V \circ V \subset U$, then the closure of V is contained in $V \circ V \circ V$ (by 23 B.6) and hence in U.

(c) Since $\langle P, u\rangle \times \langle P, u\rangle$ is topological, to prove (c) it is sufficient to show that the interior of any element U of $\mathcal{U}$ belongs to $\mathcal{U}$. Choose a symmetric element V of $\mathcal{U}$ such that $V \circ V \circ V \subset U$. It will be shown that $V \subset \operatorname{int} U$. By lemma 23 B.7 we have

$$V \circ V \circ V = \bigcup\{V[x] \times V[y] \mid \langle x, y\rangle \in V\}.$$

It follows that $V \circ V \circ V$, and hence U, is a neighborhood in $\langle P, u\rangle \times \langle P, u\rangle$ of each point $\langle x, y\rangle$ of V.

Corollary. *Every uniformizable space is locally closed, i.e., every neighborhood of any point x of a uniformizable space contains a closed neighborhood.*

The next proposition gives a necessary and sufficient condition for a semi-uniformity induced by a semi-pseudometric to be a uniformity. As a consequence, using the rather profound theorem (18 B.16), we obtain a pseudometrization theorem for semi-uniform spaces.

24 A.3. *A semi-pseudometric d induces a uniformity if and only if for each positive real r there exists a positive real s such that $d\langle x, y\rangle < s$, $d\langle y, z\rangle < s$ imply $d\langle x, z\rangle < r$. In particular, a semi-uniformity induced by a pseudometric is a uniformity.*

Proof. Let d be a semi-pseudometric for a set P, and let $\mathcal{U}$ be the semi-uniformity induced by d. — I. First suppose that $\mathcal{U}$ is a uniformity. Given an $r > 0$, we can choose a U in $\mathcal{U}$ such that $\langle x, y\rangle \in U \circ U$ implies $d\langle x, y\rangle < r$, and an $s > 0$ such that $d\langle x, y\rangle < s$ implies $\langle x, y\rangle \in U$. Now if $d\langle x, y\rangle < s$ and $d\langle y, z\rangle < s$, then $\langle x, z\rangle \in U$ and hence $d\langle x, z\rangle < r$, which shows that the condition is fulfilled. — II. Now suppose that the condition is fulfilled and U is any element of $\mathcal{U}$. Choose a positive r such that $d\langle x, y\rangle < r$ implies $\langle x, y\rangle \in U$. By the condition we can find a positive s such that $d\langle x, y\rangle < s$, $d\langle x, y\rangle < s$ imply $d\langle x, z\rangle < r$. Now if V is any element of $\mathcal{U}$ such that $\langle x, y\rangle \in V$ implies $d\langle x, y\rangle < s$, then $V \circ V \subset U$; indeed, if $\langle x, z\rangle \in V \circ V$, then $\langle x, y\rangle \in V$, $\langle y, z\rangle \in V$ for some y, and hence $d\langle x, y\rangle < s$, $d\langle y, z\rangle < s$; this implies $d\langle x, z\rangle < r$ which yields $\langle x, z\rangle \in U$.

24 A.4. Pseudometrization theorem. *A semi-uniformity $\mathscr{U}$ is induced by a pseudometric (i.e., is pseudometrizable) if and only if $\mathscr{U}$ is a uniformity with a countable base.*

Proof. It has already been shown (23 A.8) that a semi-uniformity $\mathscr{U}$ is induced by a semi-pseudometric if and only if $\mathscr{U}$ has a countable base. Combining this with 24 A.3 we find that $\mathscr{U}$ is a uniformity with a countable base if and only if $\mathscr{U}$ is induced by a semi-pseudometric d satisfying this condition: for each $r > 0$ there exists an $s > 0$ such that $d\langle x, y\rangle < s$, $d\langle y, z\rangle < s$ imply $d\langle x, z\rangle < r$. By 18 B.16, however, this condition is necessary and sufficient for d to be uniformly equivalent to a pseudometric.

The concepts of a base and a sub-base for a semi-uniformity are useful in situations such as the following. To define a semi-uniformity for a set P it is sufficient to declare an appropriate collection of subsets of $P \times P$ to be a sub-base or a base. To prove that a filter $\mathscr{U}$ on $P \times P$ is a semi-uniformity for P it is sufficient to show that a base or a sub-base for $\mathscr{U}$ is a base or a sub-base for a semi-uniformity for P. Finally, to prove that a mapping f of $\langle P, \mathscr{U}\rangle$ into $\langle Q, \mathscr{V}\rangle$ is uniformly continuous it is sufficient to show that $(f \times f)^{-1}[V] \in \mathscr{U}$ for each V from a sub-base for $\mathscr{V}$. In the first two cases it is necessary to use a sufficient condition for a collection of sets to be a base or a sub-base for a semi-uniformity.

24 A.5. It has been shown that a filter base $\mathscr{U}$ on $P \times P$ is a base for a semi-uniformity if and only if $\mathscr{U}$ fulfils conditions (u 1) and (u 2). It is easy to show that condition (u 3) is necessary and sufficient for a base $\mathscr{U}$ for a semi-uniformity to be a base for a uniformity. Thus conditions (u 1), (u 2) and (u 3) are necessary and sufficient for a filter base on $P \times P$ to be a base for a uniformity. Next, it has been shown (23 A.4) that conditions (u 1) and (u 2) are sufficient for a filter sub-base on $P \times P$ to be a sub-base for a semi-uniformity. It is clear that condition (u 3) is sufficient for a sub-base for a semi-uniformity to be a sub-base for a uniformity. Thus conditions (u 1), (u 2) and (u 3) are sufficient for a filter sub-base on $P \times P$ to be a sub-base for a uniformity. In particular, the union of a non-void family of uniformities for a set is a sub-base for a uniformity. We shall need a necessary and sufficient condition for a filter sub-base on $P \times P$ to be a sub-base for a uniformity.

24 A.6. *A collection $\mathscr{W}$ of sets is a sub-base for a uniformity if and only if $\mathscr{W}$ is a sub-base for a semi-uniformity and for each W in $\mathscr{W}$ there exists a finite family $\{V_a\}$ in $\mathscr{W}$ such that $V \circ V \subset W$ for $V = \bigcap\{V_a\}$.*

Proof. Let $\mathscr{V}$ be the set of all non-void finite intersections of sets from $\mathscr{W}$. Clearly $\mathscr{W}$ is a sub-base for a uniformity if and only if $\mathscr{V}$ is a base for a uniformity. If $\mathscr{W}$ is a sub-base for a uniformity, then $\mathscr{V}$ is a base for a uniformity and therefore $\mathscr{V}$ fulfils (u 3), in particular, if $W \in \mathscr{W}$ then $V \circ V \subset W$ for some $V \in \mathscr{V}$. Conversely suppose that $\mathscr{W}$ is a sub-base for a semi-uniformity and each $W \in \mathscr{W}$ contains a set $V \circ V$ for some V in $\mathscr{V}$. It is to be shown that $\mathscr{V}$ fulfils (u 3). If $U \in \mathscr{V}$, then $U = \bigcap\{W_a \mid a \in A\}$ for some finite family in $\mathscr{W}$; if $\{V_a\}$ is a family in $\mathscr{V}$ such that $V_a \circ$

$\circ V_a \subset W_a$ for each a, then $V = \bigcap\{V_a\}$ belongs to $\mathscr{V}$ and $V \circ V \subset V_a \circ V_a \subset W_a$ for each a and hence $V \circ V \subset \bigcap\{W_a\} = U$.

Combining 24 A.6 with 23 A.5 we obtain the following result:

24 A.7. Theorem. *A collection $\mathscr{W}$ of sets is a sub-base for a uniformity for a set P if and only if $\mathscr{W} \neq \emptyset$, each element of $\mathscr{W}$ is a vicinity of the diagonal of $P \times P$ and for each W in $\mathscr{W}$ there exist finite families $\{W_a\}$ and $\{W'_b\}$ in $\mathscr{W}$ such that $V \subset W^{-1}$ and $V' \circ V' \subset W$ where $V = \bigcap\{W_a\}$ and $V' = \bigcap\{W'_b\}$.*

24 A.8. Theorem. *The class of all uniform spaces is hereditary and closed under formation of products and sums.*

Proof. If $V \circ V \subset U$, $V' = (Q \times Q) \cup V$ and $U' = (Q \times Q) \cap U$ then $V' \circ V' \subset \subset U'$, which shows that the relativization of a uniformity is a uniformity. To prove that the sum of a family of uniform spaces is a uniform space notice that, in the notation of 23 D.8, if $U = \bigcup\{(\mathrm{inj}_a \times \mathrm{inj}_a)[U_a]$, $V = \bigcup\{(\mathrm{inj}_a \times \mathrm{inj}_a)[V_a]\}$ and $V_a \circ V_a \subset \subset U_a$ for each a, then $V \circ V \subset U$; now apply 24 A.5. Finally, to prove invariance under products, notice that, in the notation of 23 D.10, the collection $\mathscr{U}'_a$ consisting of all $U'_a = \{\langle x, y\rangle \mid \langle \mathrm{pr}_a x, \mathrm{pr}_a y\rangle \in U_a\}$ is a uniformity for ΠP_a whenever $\mathscr{U}_a$ is a uniformity for P_a, and $\bigcup\{\mathscr{U}'_a\}$ is a sub-base for the product semi-uniformity $\mathscr{U}$. By 24 A.5 $\mathscr{U}$ is a uniformity if each $\mathscr{U}'_a$ is a uniformity.

By 23 A.14 every semi-uniformity is generated (in the sense of 23 A.12) by the collection of all uniformly continuous semi-pseudometrics. Now we shall prove that a semi-uniformity is a uniformity if and only if it is generated by a collection of pseudometrics. Thus semi-uniformities are related to uniformities as semi-pseudometrics to pseudometrics.

24 A.9. Theorem. *Let $\mathscr{M}$ be the collection of all uniformly continuous pseudometrics for a semi-uniform space $\langle P, \mathscr{U}\rangle$. The following conditions are equivalent:*

(a) *$\mathscr{U}$ is a uniformity.*

(b) *For each $U \in \mathscr{U}$ there exists a d in $\mathscr{M}$ such that $d\langle x, y\rangle < 1$ implies $\langle x, y\rangle \in U$.*

(c) *$\mathscr{M}$ generates $\mathscr{U}$ (in the sense of* 23 A.12*).*

(d) *A subcollection of $\mathscr{M}$ generates $\mathscr{U}$.*

(e) *$\langle P, \mathscr{U}\rangle$ admits a uniform embedding into the product of pseudometrizable semi-uniform spaces.*

Proof. Obviously (b) ⇒ (c) ⇒ (d). The implication (d) ⇒ (e) follows from 23 D.18, and (e) ⇒ (a) follows from Theorem 24 A.3 asserting that a semi-uniformity induced by a pseudometric is a uniformity and Theorem 24 A.8 asserting that every subspace of a product of uniform spaces is a uniform space. It remains to show that (a) implies (b). Assuming that $\mathscr{U}$ is a uniformity and U is an element of $\mathscr{U}$, we can choose a sequence $\{U_n\}$ of symmetric elements of $\mathscr{U}$ such that $U_0 \subset U$ and $U_{n+1} \circ U_{n+1} \subset U_n$ for each n. By 24 A.5 the collection of all U_n is a base for a uniformity $\mathscr{V} \subset \mathscr{U}$ which is pseudometrizable by 24 A.4. Let d' be any pseudometric inducing $\mathscr{V}$. Since $U \in \mathscr{V}$, there exists a positive real r such that $d'\langle x, y\rangle < r$ implies $\langle x, y\rangle \in U$. Put $d = = r^{-1} . d'$.

Combining 24 A.9 and 23 C.10 we obtain at once:

24 A.10. Theorem. *A mapping f of a semi-uniform space $\langle P, \mathcal{U} \rangle$ into a uniform space $\langle Q, \mathcal{V} \rangle$ is uniformly continuous if and only if $d \circ (\operatorname{gr} f \times \operatorname{gr} f)$ $(= \{\langle x, y \rangle \to \to d\langle fx, fy \rangle\})$ is a uniformly continuous pseudometric for $\langle P, \mathcal{U} \rangle$ for each uniformly continuous pseudometric d for $\langle Q, \mathcal{V} \rangle$.*

Remark. Notice that Theorem 24 A.2 is a corollary of Theorem 24 A.9 (b).

24 A.11. *A semi-uniform space $\mathcal{P}$ is uniform if and only if $\mathcal{P}$ admits an embedding into a product of pseudometrizable uniform spaces.*

B. UNIFORM MODIFICATION

Let $\mathcal{U}$ be a semi-uniformity for a set P. Now consider the set $\mathfrak{U}$ consisting of all uniformities contained in $\mathcal{U}$. The set $\mathfrak{U}$ is non-void because $(P \times P)$ is a uniformity contained in $\mathcal{U}$. By 24 A.5 the union of $\mathfrak{U}$ is a sub-base for a uniformity $\mathcal{U}'$ which is necessarily contained in $\mathcal{U}$, and hence $\mathcal{U}' \in \mathfrak{U}$. Thus $\mathcal{U}'$ is the largest (i.e. uniformly finest) uniformity contained in $\mathcal{U}$ (i.e. uniformly coarser than $\mathcal{U}$).

24 B.1. Definition. The largest uniformity $\mathcal{U}'$ contained in a given semi-uniformity $\mathcal{U}$ will be called the *uniform modification* of $\mathcal{U}$, and the elements of $\mathcal{U}'$ will be called *uniform* elements of $\mathcal{U}$; the space $\langle P, \mathcal{U}' \rangle$ will be called the *uniform modification of* $\langle P, \mathcal{U} \rangle$.

24 B.2. Theorem. *The uniform modification of a semi-uniformity* of $\langle P, \mathcal{U} \rangle$ *always exists. An element U of a semi-uniformity $\mathcal{U}$ is uniform if and only if there exists a sequence $\{U_n\}$ in $\mathcal{U}$ such that $U_0 \subset U$ and $U_{n+1} \circ U_{n+1} \subset U_n$ for each n.*

Proof. The existence of uniform modifications has been already proved. If U is a uniform element of a semi-uniformity $\mathcal{U}$, then U belongs to the uniform modification $\mathcal{U}'$ of $\mathcal{U}$ and hence the required sequence $\{U_n\}$ can be found in $\mathcal{U}' \subset \mathcal{U}$. Conversely, if such a sequence $\{U_n\}$ exists and $V_n = U_n \cap U_n^{-1}$, then clearly $V_n = V_n^{-1}$ and $V_{n+1} \circ V_{n+1} \subset V_n$ (if we first show that $U_{n+1}^{-1} \circ U_{n+1}^{-1} \subset U_n^{-1}$ for each n) and $V_0 \subset U$. By virtue of 24 A.5 the collection of all V_n is a base for a uniformity $\mathcal{V}$. Clearly $\mathcal{V} \subset \mathcal{U}$ and hence $\mathcal{V} \in \mathcal{U}'$. Since $U \in \mathcal{V}$ we obtain $U \in \mathcal{U}'$, which shows that U is a uniform element of $\mathcal{U}$.

24 B.3. Lemma. *Let f be a uniformly continuous mapping of a semi-uniform space $\langle P, \mathcal{U} \rangle$ into another one $\langle Q, \mathcal{V} \rangle$. If V is a uniform element of $\mathcal{V}$, then $(f \times f)^{-1} [V]$*) is a uniform element of $\mathcal{U}$.*

Proof. If V is a uniform element of $\mathcal{V}$, then (by 24 B.2) there exists a sequence $\{V_n\}$ in $\mathcal{V}$ such that $V_0 \subset V$ and $V_{n+1} \circ V_{n+1} \subset V_n$ for each n. Put $U = (f \times f)^{-1} [V]$ and $U_n = (f \times f)^{-1}[V_n]$ for each n. Since f is uniformly continuous, the set U as well

*) Recall that we may write g instead of $\operatorname{gr} g$ (cf. 7 B.3)

as the sets U_n belong to $\mathcal{U}$. It is easily seen that $U_0 \subset U$ and $U_{n+1} \circ U_{n+1} \subset U_n$ for each n. By 24 B.2 U is a uniform element of $\mathcal{U}$.

24 B.4. Theorem. *Suppose that $\mathcal{U}'$ is the uniform modification of a semi-uniformity $\mathcal{U}$ for a set P. Then a mapping f of $\langle P, \mathcal{U} \rangle$ into a uniform space $\langle Q, \mathcal{V} \rangle$ is uniformly continuous if and only if the mapping $f : \langle P, \mathcal{U}' \rangle \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous.*

Proof. If $f : \langle P, \mathcal{U}' \rangle \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous, then $f : \langle P, \mathcal{U} \rangle \to \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous because $\mathcal{U}' \subset \mathcal{U}$. Conversely suppose that f : $\langle P, \mathcal{U} \rangle \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous, i.e. $(f \times f)^{-1} [V] \in \mathcal{U}$ for each V in $\mathcal{V}$. However, each element V of $\mathcal{V}$ is a uniform element of $\mathcal{V}$ and consequently each $(f \times f)^{-1} [V]$, $V \in \mathcal{V}$, is a uniform element of $\mathcal{U}$ (by 24 B.3) and hence belongs to $\mathcal{U}'$; this shows that the mapping $f : \langle P, \mathcal{U}' \rangle \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous.

Corollary. *If f is a uniformly continuous mapping of a semi-uniform space $\langle P, \mathcal{U}_1 \rangle$ into another one $\langle P_2, \mathcal{U}_2 \rangle$, and $\mathcal{U}'_i$ is the uniform modification of $\mathcal{U}_i$, then the mapping $f : \langle P_1, \mathcal{U}'_1 \rangle \to \langle P_2, \mathcal{U}'_2 \rangle$ is uniformly continuous.*

Proof. If f is uniformly continuous, then clearly $f : \langle P_1, \mathcal{U}_1 \rangle \to \langle P_2, \mathcal{U}'_2 \rangle$ is uniformly continuous and (by 24 B.4) the mapping $f : \langle P_1, \mathcal{U}' \rangle \to \langle P_2, \mathcal{U}'_2 \rangle$ is uniformly continuous.

It is useful to observe that the property of uniform modification stated in 24 B.4 is characteristic for uniform modifications; more precisely, the following theorem is true.

24 B.5. Theorem. *The uniform modification $\mathcal{W}$ of a semi-uniformity $\mathcal{U}$ for a set P is the unique uniformity for P such that a mapping f of $\langle P, \mathcal{U} \rangle$ into a uniform space $\langle Q, \mathcal{V} \rangle$ is uniformly continuous if and only if the mapping $f : \langle P, \mathcal{W} \rangle \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous.*

Proof. By 24 B.4 the uniform modification of $\mathcal{U}$ fulfils the condition. To prove uniqueness, suppose that uniformities $\mathcal{W}_1$ and $\mathcal{W}_2$ fulfil the condition (with $\mathcal{W}$ replaced by $\mathcal{W}_1$ and $\mathcal{W}_2$ respectively). Since $\mathsf{J} : \langle P, \mathcal{W}_1 \rangle \to \langle P, \mathcal{W}_1 \rangle$ is uniformly continuous, by the condition for $\mathcal{W}_1$ the mapping $\mathsf{J} : \langle P, \mathcal{U} \rangle \to \langle P, \mathcal{W}_1 \rangle$ must be uniformly continuous, and by the condition for $\mathcal{W}_2$ the mapping $\mathsf{J} : \langle P, \mathcal{W}_2 \rangle \to \to \langle P, \mathcal{W}_1 \rangle$ must be uniformly continuous, i.e., $\mathcal{W}_2 \supset \mathcal{W}_1$. The same is true with $\mathcal{W}_1$ and $\mathcal{W}_2$ interchanged, i.e., $\mathcal{W}_1 \subset \mathcal{W}_2$. Thus $\mathcal{W}_1 = \mathcal{W}_2$.

Remark. Recall that the topological modification τu of a closure operation u for a set P is the finest topological closure coarser than u. By 16 B.5 a subset U of P is a neighborhood of a point x in $\langle P, \tau u \rangle$ if and only if there exists a sequence $\{U_n\}$ of subsets of P such that $U_n \subset U$ for each n, $x \in U_0$ and U_{n+1} is a neighborhood of U_n for each n. Compare this result with 24 B.2. Theorem 24 B.5 corresponds to Theorem 16 B.4, which asserts that the topological modification of a closure operation u is the unique topological closure for the same set as u, say P, such that a mapping

f of $\langle P, u\rangle$ into a topological space $\mathscr{Q}$ is continuous if and only if the mapping $f : \langle P, \tau u\rangle \to \mathscr{Q}$ is continuous.

Now we shall describe the uniform modification of a semi-uniformity by means of uniformly continuous pseudometrics.

24 B.6. Theorem. *The uniform modification $\mathscr{U}'$ of a semi-uniformity $\mathscr{U}$ for a set P is generated by the collection $\mathscr{M}$ of all uniformly continuous pseudometrics for $\langle P, \mathscr{U}\rangle$.*

Proof. Let $\mathscr{M}'$ be the collection of all uniformly continuous pseudometrics for $\langle P, \mathscr{U}'\rangle$. Since $\mathscr{U}'$ is a uniformity, according to 24 A.9 it is generated by the collection $\mathscr{M}'$. Hence it is sufficient to show that $\mathscr{M}' = \mathscr{M}$. But this follows from 24 B.4 and the fact that a semi-uniformity induced by a pseudometric is a uniformity (24 A.3); in fact, if $\mathscr{U}_d$ is the semi-uniformity induced by a pseudometric d, then by 24 B.4 $\mathsf{J} : \langle P, \mathscr{U}\rangle \to \langle P, \mathscr{U}_d\rangle$ is uniformly continuous if and only if $\mathsf{J} : \langle P, \mathscr{U}'\rangle \to$ $\to \langle P, \mathscr{U}_d\rangle$ is uniformly continuous.

By 23 A.9 a uniform collection of semi-pseudometrics is the collection of all uniformly continuous semi-pseudometrics for a semi-uniform space. We have seen that a semi-uniform space is completely determined by its uniform collection of semi-pseudometrics and a uniform space is uniquely determined by the collection of all uniformly continuous pseudometrics (24 A.9). In 23 A.10 a uniform collection of semi-pseudometrics is described without any reference to semi-uniform spaces. Now we shall describe the collection of all uniformly continuous pseudometrics for a uniform space without any reference to uniform spaces. First we shall introduce some terminology.

24 B.7. Definition. A *uniform collection of pseudometrics* is the collection of all uniformly continuous pseudometrics for a uniform space.

24 B.8. Theorem. *A collection $\mathscr{M}$ is a uniform collection of pseudometrics if and only if $\mathscr{M}$ is non-void, all the elements of $\mathscr{M}$ are pseudometrics for the same set, say P, and the following two conditions are fulfilled:*

(a) *if $d_1 \in \mathscr{M}$ and $d_2 \in \mathscr{M}$, then $d_1 + d_2 \in \mathscr{M}$.*

(b) *if d is a pseudometric for P and if for each positive real r there exists a d' in $\mathscr{M}$ and an $s > 0$ such that $d'\langle x, y\rangle < s$ implies $d\langle x, y\rangle < r$, then $d \in \mathscr{M}$.*

Proof. First suppose that $\mathscr{M}$ is the collection of all uniformly continuous pseudometrics for a uniform space $\langle P, \mathscr{U}\rangle$. Clearly, $\{\langle x, y\rangle \to 0 \mid \langle x, y\rangle \in P \times P\} \in \mathscr{M}$ and hence $\mathscr{M} \neq \emptyset$. If $d_1, d_2, \in \mathscr{M}$, then by 23 A.10, $d_1 + d_2$ is a uniformly continuous semi-pseudometric for $\langle P, \mathscr{U}\rangle$; but $d_1 + d_2$ is a pseudometric and hence $d_1 + d_2$ belongs to $\mathscr{M}$. Finally, it follows from 23 A.10 that condition (b) is fulfilled. The second part of the proof is an immediate consequence of the proposition which follows.

24 B.9. *Let $\mathscr{M}$ be a non-void collection of pseudometrics for a set P and let $\mathscr{V}$ be the collection of all the sets of the form $\mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\}, d \in \mathscr{M}, r > 0$.*

Then $\mathscr{V}$ is a sub-base for a uniformity $\mathscr{U}$ for P and if $\mathscr{M}$ fulfils condition (a) *of* 24 B.8, *then $\mathscr{V}$ is a base for $\mathscr{U}$. If $\mathscr{V}$ is a base for $\mathscr{U}$ and $\mathscr{M}$ fulfils condition* (b) *of* 24 B.8, *then $\mathscr{M}$ is the set of all uniformly continuous pseudometrics for $\langle P, \mathscr{U} \rangle$.*

Proof. By 23 A.11 the set $\mathscr{V}$ is a sub-base for the semi-uniformity $\mathscr{U}$ generated by $\mathscr{M}$, and, each $d \in \mathscr{M}$ being a pseudometric, $\mathscr{U}$ is a uniformity by 24 A.9. The remaining statements follow from the corresponding statements of 23A.11.

24 B.10. It may be appropriate to point out that the relation $\{\mathscr{U} \to \mathscr{M}_{\mathscr{U}} \mid \mathscr{U}$ is a uniformity$\}$, where $\mathscr{M}_{\mathscr{U}}$ is the set of all uniformly continuous pseudometrics for the uniform space $\langle P, \mathscr{U} \rangle$ (P is uniquely determined by $\mathscr{U}$) is a one-to-one relation ranging on the class of all uniform collections of pseudometrics, and $\mathscr{U} \subset \mathscr{V}$ if and only if $\mathscr{M}_{\mathscr{U}} \subset \mathscr{M}_{\mathscr{V}}$.

Now we shall turn to the topological conclusions of the results of this subsection.

24 B.11. Theorem. *Suppose that $\langle P, u \rangle$ is a closure space. There exists a uniformly finest continuous uniformity $\mathscr{U}$ for the space $\langle P, u \rangle$. The uniformity $\mathscr{U}$ is the uniform modification of the uniformly finest semi-uniformity $\mathscr{V}$ for $\langle P, u \rangle$. The closure u is uniformizable if and only if u is induced by $\mathscr{U}$. The closure induced by $\mathscr{U}$ is the finest uniformizable closure coarser than u.*

Proof. By 23 B.2 there exists the uniformly finest continuous semi-uniformity $\mathscr{V}$ for $\langle P, u \rangle$. Obviously the uniform modification $\mathscr{U}$ of $\mathscr{V}$ is the uniformly finest continuous uniformity for $\langle P, u \rangle$, and the closure induced by $\mathscr{U}$ is the finest uniformizable closure coarser than u; in particular, u is uniformizable if and only if $\mathscr{U}$ induces u.

24 B.12. Theorem. *Suppose that $\mathscr{U}$ is the uniformly finest continuous uniformity for a closure space $\langle P, u \rangle$. The set of all uniformly continuous pseudometrics for $\langle P, \mathscr{U} \rangle$ coincides with the set of all continuous pseudometrics for $\langle P, u \rangle$. Stated in other words, the uniformly finest uniformity for a closure space $\langle P, u \rangle$ is generated by the collection of all continuous pseudometrics for $\langle P, u \rangle$.* – Obvious.

24 B.13. Definition. The *uniformizable modification of a closure operation u* is the finest uniformizable closure coarse than u. If $\langle P, u \rangle$ is a closure space, then a *uniformizable neighborhood of the diagonal of $\langle P, u \rangle \times \langle P, u \rangle$* is defined to be an element of the uniformly finest continuous uniformity for $\langle P, u \rangle$.

Thus the uniformly finest uniformity for a closure space $\langle P, u \rangle$ consists of all uniformizable neighborhoods of the diagonal of $\langle P, u \rangle \times \langle P, u \rangle$. Now we shall give two descriptions of uniformizable neighborhoods of the diagonal.

24 B.14. Theorem. *Let $\langle P, u \rangle$ be a closure space. Each of the following two conditions is necessary and sufficient for a subset U of $P \times P$ to be a uniformizable neighborhood of the diagonal of $\langle P, u \rangle \times \langle P, u \rangle$:*

(a) *there exists a continuous pseudometric d for $\langle P, u \rangle$ such that $d\langle x, y \rangle < 1$ implies $\langle x, y \rangle \in U$;*

(b) *there exists a sequence* $\{U_n\}$ *of semi-neighborhoods of the diagonal of* $\langle P, u\rangle \times \langle P, u\rangle$ *such that* $U_0 \subset U$ *and* $U_{n+1} \circ U_{n+1} \subset U_n$ *for each* n.

Proof. Condition (a) is necessary and sufficient by 24 B.12. The necessity of (b) follows from the fact that every element of a continuous semi-uniformity for $\langle P, u\rangle$ is a semi-neighborhood of the diagonal of $\langle P, u\rangle \times \langle P, u\rangle$ (23 B.2). Finally, assume (b), put $V_n = U_n \cap U_n^{-1}$ and consider the sequence $\{V_n\}$. Clearly, $V_0 \subset U$, $V_n = V_n^{-1}$ and $V_{n+1} \circ V_{n+1} \subset V_n$ for each n. Obviously the collection of all V_n is a base for a uniformity $\mathscr{V}$ for P and each element of $\mathscr{V}$ is a semi-neighborhood of the diagonal of $\langle P, u\rangle \times \langle P, u\rangle$, and consequently $\mathscr{V}$ is a continuous uniformity for $\langle P, u\rangle$ (by 23 B.2). Since $U \in \mathscr{V}$, U is a uniformizable neighborhood of the diagonal of $\langle P, u\rangle \times \times \langle P, u\rangle$.

24 B.15. Theorem. *Let u be a closure operation for a set P. The uniformizable modification of u is the unique uniformizable closure for P such that a mapping f of $\langle P, u\rangle$ into a uniformizable space $\langle Q, v\rangle$ is continuous if and only if the mapping $f: \langle P, w\rangle \to \langle Q, v\rangle$ is continuous.*

Proof. I. For uniqueness, assume that uniformizable closures w_1 and w_2 fulfil the condition. Since $\mathsf{J} = \langle P, w_1\rangle \to \langle P, w_1\rangle$ is continuous, with the condition applied to w_1, we find that $\mathsf{J} : \langle P, u\rangle \to \langle P, w_1\rangle$ is continuous; with the condition applied to w_2, we find that $\mathsf{J} : \langle P, w_2\rangle \to \langle P, w_1\rangle$ is continuous. Thus w_1 is coarser than w_2. The same argument may be applied with w_1 and w_2 interchanged, and hence w_2 is coarser than w_1; this proves $w_1 = w_2$. – II. Now let w be the uniformizable modification of u and let f be a mapping of $\langle P, u\rangle$ into a uniformizable space $\langle Q, v\rangle$. If $f : \langle P, w\rangle \to \langle Q, v\rangle$ is continuous then f is continuous because w is coarser than u. Next, suppose that f is continuous and let $\mathscr{U}$ be the largest continuous semi-uniformity for $\langle P, u\rangle$. Let $\mathscr{W}$ be the uniform modification of $\mathscr{U}$ and $\mathscr{V}$ be a uniformity inducing the closure operation v. Since f is continuous, by 23 C.8 the mapping $f: \langle P, \mathscr{U}\rangle \to \langle Q, \mathscr{V}\rangle$ is uniformly continuous, and by 24 B.4 the mapping f : $\langle P, \mathscr{W}\rangle \to \langle Q, \mathscr{V}\rangle$ is uniformly continuous; hence its transpose to $f: \langle P, w\rangle \to \to \langle Q, v\rangle$ is continuous (by 23 C.7).

It may be appropriate to point out the crucial step in the second part of the proof of the preceding theorem.

24 B.16. *Let f be a continuous mapping of a closure space $\langle P, u\rangle$ into a closure space $\langle Q, v\rangle$. If $\mathscr{V}$ is a continuous uniformity for $\langle Q, v\rangle$ and $\mathscr{W}$ is the uniformly finest (i.e. largest) continuous uniformity for $\langle P, u\rangle$, then the mapping $f: \langle P, \mathscr{W}\rangle \to \to \langle Q, \mathscr{V}\rangle$ is uniformly continuous.* – (23 C.8, 24 B.4).

Corollary a. *Let f be a mapping of a uniformizable closure space $\langle P_1, u_1\rangle$ into a uniformizable closure space $\langle P_2, u_2\rangle$ and $\mathscr{U}_i$ be the largest continuous uniformity for $\langle P_i, u_i\rangle$, $i = 1, 2$. Then f is continuous if and only if $f: \langle P_1, \mathscr{U}_1\rangle \to \langle P_2, \mathscr{U}_2\rangle$ is uniformly continuous.*

Corollary b. *If f is a continuous mapping of a closure space $\langle P, u\rangle$ into a closure space $\langle Q, v\rangle$ and V is a uniformizable neighborhood of the diagonal of $\langle Q, v\rangle \times$*

$\times \langle Q, v\rangle$, *then* $(f \times f)^{-1}[V]$ *is a uniformizable neighborhood of the diagonal of* $\langle P, u\rangle \times \langle P, u\rangle$.

It is to be noted that every uniformizable neighborhood of the diagonal is actually a neighborhood of the diagonal. The converse does not hold, see ex. 4.

C. GROUP UNIFORMITIES

Throughout this subsection, unless the contrary is explicitly stated, $\mathcal{G} = \langle G, \cdot, u\rangle$ will be a topological group and $\mathcal{O}$ will be the neighborhood system at the unit element denoted by 1, in $\mathcal{G}$. The definitions, conventions and results of Section 19, in particular those of 19 B, are assumed. For each U in $\mathcal{O}$ put

(1) $U_R = \mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in G \times G, x \,.\, y^{-1} \in U\}$,
$U_L = \mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in G \times G, x^{-1} \,.\, y \in U\}$

A simple calculation yields the following formulae (where V and U are elements of $\mathcal{O}$):

(2) $U_R \cap V_R = (U \cap V)_R$, $U_L \cap V_L = (U \cap V)_L$.
(3) $\Delta_G \subset U_R \cap U_L$ (where Δ_G is the diagonal of $G \times G$).
(4) $U = U^{-1} \Rightarrow U_R = U_R^{-1}$, $U_L = U_L^{-1}$.
(5) $U_R \circ V_R \subset ([U] \cdot [V])_R$, $U_L \circ V_L \subset ([U] \cdot [V])_L$.

24 C.1. Definition. It follows from 24 A.5 and formulae (2)–(5) that the collection $\mathcal{U}_R(\mathcal{U}_L)$ of all $U_R(U_L)$, $U \in \mathcal{O}$, is a base for a uniformity $\mathcal{R}(\mathcal{L})$; this uniformity will be called the *right* (*left*) *uniformity* of $\mathcal{G}$. Again by 24 A.5 the collection $\mathcal{R} \cup \mathcal{L}$ is a sub-base for a uniformity; this uniformity will be called the *two-sided uniformity* of $\mathcal{G}$.

24 C.2. *The collection* $[\mathcal{U}_R] \cap [\mathcal{U}_L]$ *(consisting of all* $U_R \cap V_L$, $U \in \mathcal{O}$, $V \in \mathcal{O}$*) is a base for the two-sided uniformity of* $\mathcal{G}$. *If* $\mathcal{G}$ *is commutative then all three uniformities of* $\mathcal{G}$ *coincide.* – Obvious.

Remark. If $\mathcal{R} = \mathcal{L}$, then $\mathcal{G}$ need not be commutative (see ex. 17).

24 C.3. Remark. Let $\mathcal{G}' = \langle G, *, u\rangle$ where $* = \{\langle x, y\rangle \to y \,.\, x \mid \langle x, y\rangle \in G \times G\}$. Then evidently $\mathcal{G}'$ is a topological group and the right (left) uniformity of $\mathcal{G}$ coincides with the left (right) uniformity of $\mathcal{G}'$. Next, a right translation $\{x \to x \,.\, a\}$ of $\mathcal{G}$ coincides with the left translation $\{x \to a * x\}$ of $\mathcal{G}'$. Using these facts we can derive from each proposition about left (right) uniformities a proposition about right (left) uniformities.

24 C.4. *All three uniformities of a topological group induce the closure structure of* $\mathcal{G}$.

Proof. First let us notice that

(6) $U_L[x] = x \,.\, [U]$, $U_R[x] = [U^{-1}] \,.\, x$

for each U in $\mathcal{O}$ and x in $\mathcal{G}$. According to 19 B.3 the collections $\mathbf{E}\{x \,.\, [U] \mid U \in \mathcal{O}\}$ and

$\mathsf{E}\{[U] . x \mid U \in \mathcal{O}\}$ form local bases at x in $\mathcal{G}$, and by 23 A.3 the collection $\mathsf{E}\{U_L[x] \mid U \in \mathcal{O}\}$ ($\mathsf{E}\{U_R[x] \mid U \in \mathcal{U}\}$) is a local base at x in the space $\langle G, \mathcal{L}\rangle$ ($\langle G, \mathcal{R}\rangle$). Now it follows from (6) that $\mathcal{L}$ as well as $\mathcal{R}$ induces the closure structure of $\mathcal{G}$. Since $\mathcal{L} \cup \mathcal{R}$ is a sub-base for the two-sided uniformity of $\mathcal{G}$ and both $\mathcal{L}$ and $\mathcal{R}$ induce the closure structure of $\mathcal{G}$, we find that the two-sided uniformity of $\mathcal{G}$ also induces the closure structure of $\mathcal{G}$.

24 C.5. Theorem. *The mapping* $\{x \to x^{-1}\} : \langle|\mathcal{G}|, \mathcal{R}\rangle \to \langle|\mathcal{G}|, \mathcal{L}\rangle$ *is a uniform homeomorphism.*

Proof. Let $\varrho = \{x \to x^{-1} \mid x \in G\}$. It is sufficient to prove that $(\varrho \times \varrho)[U_R] = U_L$ for each U in $\mathcal{O}$. If $\langle x, y\rangle \in U_R$, then $x . y^{-1} \in U$ and hence $x . y^{-1} = (\varrho x)^{-1} . \varrho y \in U$, which means that $\langle \varrho x, \varrho y\rangle \in U_L$ and consequently $(\varrho \times \varrho)[U_R] \subset U_L$. Similarly $(\varrho \times \varrho)[U_L] \subset U_R$. Since $\varrho \circ \varrho = J_G$ we obtain $(\varrho \times \varrho)[U_R] = U_L$.

Corollary. *If $\mathcal{U}$ is the two-sided uniformity for $\mathcal{G}$, then the mapping* $\{x \to x^{-1}\}$: $\langle|\mathcal{G}|, \mathcal{U}\rangle \to \langle|\mathcal{G}|, \mathcal{U}\rangle$ *is a uniform homeomorphism.* (Use 23 C.2).

24 C.6. Theorem. *Let f be a homomorphism of a topological group $\mathcal{G}_1$ into a topological group $\mathcal{G}_2$ and let $\mathcal{L}_i, \mathcal{R}_i$ and $\mathcal{U}_i$ be respectively the left uniformity, the right uniformity and the two-sided uniformity of $\mathcal{G}_i$. The following conditions are equivalent:*

(a) *f is continuous,*
(b) $f : \langle|\mathcal{G}_1|, \mathcal{L}_1\rangle \to \langle|\mathcal{G}_2|, \mathcal{L}_2\rangle$ *is uniformly continuous,*
(c) $f : \langle|\mathcal{G}_1|, \mathcal{R}_1\rangle \to \langle|\mathcal{G}_2|, \mathcal{R}_2\rangle$ *is uniformly continuous,*
(d) $f : \langle|\mathcal{G}_1|, \mathcal{U}_1\rangle \to \langle|\mathcal{G}_2|, \mathcal{U}_2\rangle$ *is uniformly continuous.*

Proof. We shall prove that (a) $\Rightarrow$ (b); (b) $\Leftrightarrow$ (c), (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a). The last implication is obvious (the two-sided uniformity of a group induces the closure structure of the group). Also the implication (b), (c) $\Rightarrow$ (d) is evident because $\mathcal{R}_i \cup \mathcal{L}_i$ is a sub-base for $\mathcal{U}_i$. The equivalence of (b) and (c) follows immediately from 24 C.5. It remains to show that (a) $\Rightarrow$ (b). Let $\mathcal{O}_i$ be the neighborhood system at the unit element of $\mathcal{G}_i$, $i = 1, 2$, and suppose that O_2 is an element of $\mathcal{O}_2$; it is required to find an element O_1 of $\mathcal{O}_1$ such that $(f \times f)[(O_1)_L] \subset (O_2)_L$. Choose O_1 so that $f[O_1] \subset O_2$. Now if $\langle x, y\rangle \in (O_1)_L$, then $x^{-1} . y \in O_1$ and hence $(fx)^{-1} . (fy) = f(x^{-1} . y) \in f[O_1] \subset O_2$ which implies that $\langle fx, fy\rangle \in (O_2)_L$.

Remark. In the proof of the implication (a) $\Rightarrow$ (b) we only needed the continuity of f at the unit element of $\mathcal{G}_1$. Thus we obtained a new proof of the fact that a homomorphism is continuous provided that it is continuous at the unit element.

Now we turn to an examination of translations.

24 C.7. Theorem. *If $\mathcal{V} = \mathcal{R}$, $\mathcal{V} = \mathcal{L}$ or $\mathcal{V} = \mathcal{U}$, then the mapping*

$$\{x \to a . x . b\} : \langle|\mathcal{G}|, \mathcal{V}\rangle \to \langle|\mathcal{G}|, \mathcal{V}\rangle$$

is a uniform homeomorphism for each $a, b \in \mathcal{G}$. Roughly speaking, the translations are uniform homeomorphisms with respect to each of the group uniformities.

Proof. The inverse of a translation is a translation and hence it will suffice to show that the translations are uniformly continuous. This will be proved for $\mathscr{V} = \mathscr{R}$. In the case $\mathscr{V} = \mathscr{L}$ the proof is similar. Finally, the case $\mathscr{V} = \mathscr{U}$ follows readily from the cases $\mathscr{V} = \mathscr{R}$ and $\mathscr{V} = \mathscr{L}$. Let U be any element of $\mathcal{O}$ and let us choose a V in $\mathcal{O}$ such that $a . V . a^{-1} \subset U$. If $\langle x, y \rangle \in V_R$, i.e. $x . y^{-1} \in V$, then $(a . x . b) . . (a . y . b)^{-1} = a . x . y^{-1} . a^{-1} \in a . V . a^{-1} \subset U$, and hence $\langle a . x . b, a . y . b \rangle \in \in U_R$.

Notice that $(f_b \times f_b)[U_R] = U_R$ for each right translation $f_b = \{x \to x . b\}$. We have proved that the inclusion $\subset$ is true for each right translation. Since the inverse of f_b is the right translation $f_{b^{-1}}$ we obtain the equality. Similarly $(f \times f)[U_L] = U_L$ for each left translation f. On the other hand the last equality need not be true if f is a right translation.

Recall that a semi-pseudometric d for a group $\langle G, . \rangle$ is called *right* (*left*) *invariant* if $d \circ (f \times f) = d$ for each right (left) translation f of $\langle G, . \rangle$.

24 C.8. Definition. A semi-uniformity $\mathscr{U}$ for a group $\langle G, . \rangle$ is said to be *right* (*left*) *invariant* if there exists a base $\mathscr{V}$ for $\mathscr{U}$ such that $(f \times f)[V] = V$ for each $V \in \mathscr{V}$ and each right (left) translation f of $\langle G, . \rangle$.

It is to be noted that in 24 C.8 it is sufficient to assume that $\mathscr{V}$ is a sub-base for $\mathscr{U}$. The main results are summarized in the following theorem.

24 C.9. Theorem. *The right (left) uniformity of a topological group $\mathscr{G}$ is right (left) invariant. The set of all right (left) invariant pseudometrics for $\mathscr{G}$ which are uniformly continuous with respect to the right (left) uniformity of $\mathscr{G}$ generates the right (left) uniformity of $\mathscr{G}$. If $\mathscr{G}$ is of a countable local character then the right (left) uniformity of $\mathscr{G}$ is induced by a right (left) invariant pseudometric.*

Corollary. *If $\mathscr{G}$ is of a countable local character then $\mathscr{G}$ permits a pseudometrization by a right invariant as well as a left invariant pseudometric.*

Theorem 24 C.9 is a particular case of more general results concerning F-invariant uniformities which will now be studied. (The first statement, which is elementary, has been already proved.)

24 C.10. Let P be a set and F be a collection of one-to-one relations such that $\mathbf{D}f = \mathbf{E}f = P$ for each f in F. A semi-pseudometric d for P is said to be *F-invariant* if $d \circ (f \times f) = d$ for each f in F, and a semi-uniformity $\mathscr{U}$ for P is said to be *F-invariant* if there exists a base $\mathscr{V}$ for $\mathscr{U}$ such that $(f \times f)[V] = V$ for each V in $\mathscr{V}$.

Remarks. (a) If F is the set of all right (left) translations of a group $\langle G, . \rangle$ then a semi-pseudometric d for G is F-invariant if and only if d is a right (left) invariant semi-pseudometric for $\langle G, . \rangle$. A similar result is true for semi-uniformities.

(b) A semi-pseudometric d is F-invariant if and only if the mapping $f : \langle P, d \rangle \to \to \langle P, d \rangle$ is distance-preserving for each f in F.

(c) In definition 24 C.10 it is sufficient to assume that $\mathscr{V}$ is a sub-base for $\mathscr{U}$. Indeed, if $(f \times f)[V_i] = V_i$, $i = 1, 2$, then $(f \times f)[V_1 \cap V_2] = V_1 \cap V_2$.

24 C.11. Theorem. *Let P be a set, F be a collection of one-to-one relations such that $\mathbf{D}f = \mathbf{E}f = P$ for each f in F, and let $\mathscr{U}$ be an F-invariant uniformity for P. The collection of all F-invariant uniformly continuous pseudometrics for $\langle P, \mathscr{U}\rangle$ generates the uniformity $\mathscr{U}$. If $\mathscr{U}$ has a countable base then $\mathscr{U}$ can be pseudometrized by an F-invariant pseudometric.*

Proof. I. First let $\mathscr{U}$ be an F-invariant semi-uniformity with a countable base. Clearly there exists a decreasing sequence $\{U_n\}$ of symmetric elements of $\mathscr{U}$ such that the set of all U_n is a base for $\mathscr{U}$, $U_0 = P \times P$ and $(f \times f)[U_n] = U_n$ for each $n \in \mathsf{N}$ and $f \in F$. Setting $d\langle x, y\rangle = 2^{-n}$ if $\langle x, y\rangle \in U_{n+1} - U_n$ and $d\langle x, y\rangle = 0$ if $\langle x, y\rangle \in \bigcap\{U_n\}$ we obtain a semi-pseudometric inducing $\mathscr{U}$ and it is easily seen that d is F-invariant. Now if, in addition, $\mathscr{U}$ is a uniformity, then d is uniformly equivalent with a pseudometric D (by 24 A.4) which can be chosen F-invariant according to 18 B.16; this proves the second statement. — II. The first statement is an immediate consequence of the second statement and the fact that the union of all F-invariant uniformities with a countable base contained in $\mathscr{U}$ is a sub-base (even a base) for $\mathscr{U}$.

24 C.12. *The closure structure of a topological group $\mathscr{G}$ is induced by a uniformity invariant under both right and left translations if and only if the collection of all $U \in \mathscr{O}$ fixed under the inner automorphisms of $\mathscr{G}$ is a local base at the unit element.*

Proof. I. First suppose that the closure structure of $\mathscr{G}$ is induced by a uniformity $\mathscr{V}$ invariant under both left and right translations. Thus the collection $\mathscr{W}$ of all the sets $W \in \mathscr{V}$ such that $\langle x, y\rangle \in W$ implies $\langle ax, ay\rangle \in W$, $\langle xb, yb\rangle \in W$ for each a and b in $\mathscr{G}$ (in particular, $\langle x, y\rangle \in W$ implies $\langle axa^{-1}, aya^{-1}\rangle \in W$ for each a in A) is a base for $\mathscr{V}$. Now if $O = W[1]$, $W \in \mathscr{W}$, then clearly $y \in O$ implies $aya^{-1} \in O$ (because $a \cdot 1 \cdot a^{-1} = 1$).

II. Now let $\mathscr{O}_1$ be the collection of all sets $O \in \mathscr{O}$ such that $f[O] = O$ for each inner automorphism $f = \{x \to axa^{-1}\}$ of $\mathscr{G}$, and suppose that $\mathscr{O}_1$ is a base for the filter $\mathscr{O}$. We shall assume that the right uniformity $\mathscr{R}$ of $\mathscr{G}$, which is always invariant under right translations, is also invariant under left translations. Clearly the collection $\mathscr{W}$ of all the sets U_R, $U \in \mathscr{O}_1$, is a base for $\mathscr{R}$. Fix U_R in $\mathscr{W}$. First we shall show that U_R is invariant under inner automorphisms. If $\langle x, y\rangle \in U_R$ and $a \in \mathscr{G}$, then $x \cdot y^{-1} \in U \in \mathscr{O}_1$ and hence $axy^{-1}a^{-1} \in U$; but $axa^{-1} \cdot (aya^{-1})^{-1} = axy^{-1}a^{-1} \in U$ and consequently $\langle axa^{-1}, aya^{-1}\rangle \in U_R$. Now let $b \in \mathscr{G}$; we shall show that $\langle x, y\rangle \in U_R \Rightarrow \langle bx, by\rangle \in U_R$. Suppose $\langle x, y\rangle \in U_R$. Since $(f \times f)[U_R] = U_R$ for each inner automorphism f, we obtain $\langle bxb^{-1}, byb^{-1}\rangle \in U_R$ and since $U_R = (f \times f)[U_R]$ for each right translation f, we obtain $\langle bxb^{-1}b, byb^{-1}b\rangle \in U_R$ and hence $\langle bx, by\rangle \in U_R$, which concludes the proof.

The multiplication $\{\langle x, y\rangle \to x \cdot y\}$ need not be a uniformly continuous mapping, more precisely, the mapping $\{\langle x, y\rangle \to x \cdot y\} : \langle|\mathscr{G}| \times |\mathscr{G}|, \mathscr{V} \times \mathscr{V}\rangle \to \langle|\mathscr{G}|, \mathscr{V}\rangle$ need not be uniformly continuous if $\mathscr{V}$ is a left uniformity, right uniformity or two-sided uniformity of $\mathscr{G}$. On the other hand, if $\mathscr{G}$ is commutative, then all three uniformities of $\mathscr{G}$ coincide and the multiplication is uniformly continuous. More precisely,

24 C.13. Theorem. *If $\mathscr{G}$ is a commutative topological group, then the mapping*

$$\{\langle x, y\rangle \to x \,.\, y\} : \langle |\mathscr{G}| \times |\mathscr{G}|, \mathscr{U} \times \mathscr{U}\rangle \to \langle |\mathscr{G}|, \mathscr{U}\rangle$$

where $\mathscr{U}$ is the uniformity of $\mathscr{G}$, is uniformly continuous.

Proof. Fix a neighborhood U of the unit element and choose a neighborhood V of the unit element such that $[V] \,.\, [V] \subset U$. Now if $x_1 \,.\, y_1^{-1} \in V$, $x_2 \,.\, y_2^{-1} \in V$ then $(x_1 x_2) \,.\, (y_1 y_2)^{-1} = (x_1 \,.\, y_1)^{-1} \,.\, (x_2 \,.\, y_2^{-1}) \in [V] \,.\, [V] \subset U$ which establishes the uniform continuity of the mapping in question.

Corollary. *The addition of* R *is uniformly continuous.*

D. UNIFORM PRODUCT

The purpose of this subsection is to prove that, roughly speaking, the limit of a uniformly convergent net of continuous (uniformly continuous) mappings into a uniform space is a continuous (uniformly continuous) mapping. This result was proved for mappings into a commutative topological group in 19 B.16a. We begin with a remark concerned with the box-product of semi-uniform spaces.

24 D.1. The box-product. Let $\langle P, \mathscr{U}\rangle$ be the product of a family $\{\langle P_a, \mathscr{U}_a\rangle \mid a \in A\}$ of semi-uniform spaces. By definition the collection of all the sets of the form

(*) $$\mathsf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in P \times P, a \in F \Rightarrow \langle \mathrm{pr}_a\, x, \mathrm{pr}_a\, y\rangle \in U_a\},$$

where F is a finite subset of A and $U_a \in \mathscr{U}_a$, is a base for $\mathscr{U}$. The elements of $\mathscr{U}$ are relations for P, i.e., subsets of $P \times P$, and the elements of each $\mathscr{U}_a$ are relations for P_a. Notice that the set (*) can be written in the form

(**) $$\Pi_{\mathrm{rel}}\{V_a \mid a \in A\}$$

where $V_a = U_a$ if $a \in F$ and $V_a = P_a \times P_a$ otherwise (thus always $V_a \in \mathscr{U}_a$).

Here the symbol Π_{rel} denotes the relational product introduced in 5 C.2. If $\langle P, u\rangle$ is the product of a family $\{\langle P_a, u_a\rangle \mid a \in A\}$ of closure spaces, then the canonical neighborhoods of a point $x \in P$ in $\langle P, u\rangle$ are sets of the form

(***) $$\Pi\{V_a \mid a \in A\}$$

where each V_a is a neighborhood of $\mathrm{pr}_a\, x$ in $\langle P_a, u_a\rangle$ and $V_a = P_a$ except for a finite number of a's. In 17 ex. 2 we introduced the box-product $\langle P, w\rangle$ of $\{\langle P_a, u_a\rangle\}$ by requiring that, for each x in P, the collection of all sets (***), where V_a is a neighborhood of $\mathrm{pr}_a\, x$ in $\langle P_a, u_a\rangle$ for each a, be a local base at x in $\langle P, w\rangle$. Similarly, we can define the box-product $\langle P, \mathscr{W}\rangle$ of a family $\{\langle P_a, \mathscr{U}_a\rangle\}$ of semi-uniform spaces by requiring that the collection of all the sets (**), where $V_a \in \mathscr{U}_a$ for each a, be a base for $\mathscr{W}$. It turns out that the box-product of semi-uniform spaces is not too important and therefore we leave the discussion of its properties to the exercises.

The box-product is mentioned here because the box-product semi-uniformity is the largest semi-uniformity having a base consisting of sets of the form (**); it is to be noted that the product semi-uniformity is the smallest semi-uniformity having a base consisting of sets of the form (**) and such that all the projections are uniformly continuous.

Now we turn to the subject proper of this subsection. Recall that, by **23 D.17**, the product of a family $\{\mathscr{P} \mid a \in A\}$ of semi-uniform spaces is denoted by $\mathscr{P}^A$. The product of a family $\{\mathscr{G} \mid a \in A\}$ of topological groups is denoted by $\mathscr{G}^A$, and the uniform product of $\{\mathscr{G} \mid a \in A\}$ is denoted by unif $\mathscr{G}^A$. Now we shall introduce the product unif $\mathscr{P}^A$ for semi-uniform spaces.

24 D.2. Definition. Suppose that A is a set and $\mathscr{P} = \langle P, \mathscr{U} \rangle$ is a semi-uniform space. The *uniform product of the family* $\{\mathscr{P} \mid a \in A\}$, denoted by unif $\mathscr{P}^A$, is the semi-uniform space $\langle P_A, \mathscr{V} \rangle$ where $\mathscr{V}$ is the semi-uniformity having as its base the collection of all sets of the form $\Pi_{\text{rel}}\{U \mid a \in A\} = \mathbf{E}\{\langle x, y\rangle \mid \langle x, y \rangle \in P^A \times P^A, a \in A \Rightarrow \langle \text{pr}_a x, \text{pr}_a y\rangle \in U\}$ where $U \in \mathscr{U}$.

Of course we must show that the collection in question is indeed a base for a semi-uniformity. Before doing this we shall introduce an abbreviated notation. The product $\Pi\{X \mid a \in A\}$ can be written as X^A, and similarly the relational product $\Pi_{\text{rel}}\{U \mid a \in A\}$ can be written as $U^{\text{rel} A}$, which will be abbreviated, if no confusion is likely to result, to U^A. Next, in accordance with the general rule, the collection consisting of all $U^{\text{rel} A}$, $U \in \mathscr{U}$, can be denoted by $[\mathscr{U}]^{\text{rel} A}$, or simply by $[\mathscr{U}]^A$. To prove that $[\mathscr{U}]^A$ is a base for a semi-uniformity it is sufficient to notice that

$$(U^A \cap V^A) = (U \cap V)^A \supset \mathsf{J}_{P^A} \quad \text{and} \quad (U^A)^{-1} = (U^{-1})^A$$

for each U and V in $\mathscr{U}$. Next, it is clear that $(U \circ V)^A = U^A \circ V^A$ for each U and V and consequently, if $\mathscr{P}$ is a uniform space then unif $\mathscr{P}^A$ is also a uniform space. Finally, if $\mathscr{U}'$ is a base for $\mathscr{U}$, then clearly $[\mathscr{U}']^A$ is a base for $\mathscr{V}$; in particular, if $\mathscr{U}$ has a countable base then $\mathscr{V}$ also has a countable base. As a consequence if $\mathscr{P}$ is pseudometrizable, then unif $\mathscr{P}^A$ is also pseudometrizable. Thus we have proved

24 D.3. *If $\mathscr{P}$ is a uniform space then* unif $\mathscr{P}^A$ *is a uniform space for each set A. If $\mathscr{P}$ is pseudometrizable, then $\mathscr{P}^A$ is also pseudometrizable.*

24 D.4. Example. Let $\langle P, d\rangle$ be a semi-pseudometric space and A a non-void set. Consider the relation

$$D = \{\langle x, y\rangle \to \sup\{d\langle \text{pr}_a x, \text{pr}_a y\rangle \mid a \in A\} \mid \langle x, y\rangle \in P^A \times P^A\}.$$

If d is not bounded, then $D\langle x, y\rangle$ may be ∞ for some $\langle x, y\rangle$, but if d is bounded, then

(a) D is a semi-pseudometric for P^A;

(b) the semi-uniformity induced by D is the uniform structure of the uniform product unif $\langle P, \mathscr{U}\rangle^A$, where $\mathscr{U}$ is the semi-uniformity induced by d;

(c) if d is a pseudometric, then D is a pseudometric; and

(d) if d is a semi-metric, then D is also a semi-metric.

In 19 B.14 we introduced the uniform product unif $\mathcal{G}^A$, where $\mathcal{G}$ is a commutative group and A is a set. If $\mathcal{G}$ is a commutative group, then the left, the right and the two-sided uniformities coincide and we usually speak about the uniformity of $\mathcal{G}$.

24 D.5. Theorem. *Suppose that $\mathcal{G}$ is a commutative topological group and A is a set. Then the uniformity of the group* unif $\mathcal{G}^A$ *coincides with the uniform structure of* unif $\langle|\mathcal{G}|, \mathcal{U}\rangle^A$ *where $\mathcal{U}$ is the uniformity of the group $\mathcal{G}$.*

Remark. If for each commutative group $\mathcal{H}$ the symbol $\sigma(\mathcal{H})$ denotes the uniform space of $\mathcal{H}$, that is, the uniform space $\langle|\mathcal{H}|, \mathcal{U}\rangle$ where $\mathcal{U}$ is the uniformity of $\mathcal{H}$, then Theorem 24 D.5 can be stated as follows: For each commutative topological group $\mathcal{G}$ and each set A,

$$\text{unif}\,(\sigma(\mathcal{G}))^A = \sigma(\text{unif}\ \mathcal{G}^A)\,.$$

Proof. Let $\mathcal{G} = \langle G, \cdot, u\rangle$ and let $\mathcal{O}$ be the neighborhood system at the unit in $\mathcal{G}$. By definition 19 B.14 the collection $[\mathcal{O}]^A$ (consisting of all O^A, $O \in \mathcal{O}$) is a local base at the unit in unif $\mathcal{G}^A$. Next, the collection of all the sets of the form

$$(*)\ \ \mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in G^A \times G^A, x\,.\,y^{-1} \in O^A\}\ \ (\text{in unif}\ \mathcal{G}^A)$$

where $O \in \mathcal{O}$, is a base for the uniform structure of unif $\mathcal{G}^A$, and the collection of all the sets of the form

$$(**)\ \ \mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in G^A \times G^A, (\text{pr}_a\,x)\,.\,(\text{pr}_a\,y)^{-1} \in O\ \text{ for each }\ a \in A\}\,,$$

where $O \in \mathcal{O}$, is a base for the uniform structure of unif $\langle G, \mathcal{U}\rangle^A$. Clearly, $x\,.\,y^{-1}$ (in unif $\mathcal{G}^A$) $= \{(\text{pr}_a\,x)\,.\,(\text{pr}_a\,y)^{-1}\ (\text{in}\ \mathcal{G}) \mid a \in A\}$, and consequently, the sets $(*)$ and $(**)$ coincide. The proof is complete.

In 19 B.17, given a commutative group $\mathcal{G}$ and a comprisable struct $\mathcal{S}$ we introduced the topological group $\mathbf{F}(\mathcal{S}, \mathcal{G})$ of mappings of $\mathcal{S}$ into $\mathcal{G}$ by requiring the mapping

$$\{f \to \text{gr}\,f\} : \text{unif}\ \mathbf{F}(\mathcal{S}, \mathcal{G}) \to \text{unif}\ \mathcal{G}^{|\mathcal{S}|}$$

to be a topological group-isomorphism.

24 D.6. Definition. If $\mathcal{S}$ is a comprisable struct and $\mathcal{P}$ is a semi-uniform space, then the symbol unif $\mathbf{F}(\mathcal{S}, \mathcal{P})$ will denote the set $\mathbf{F}(\mathcal{S}, \mathcal{P})$ endowed with a semi-uniformity such that the mapping

$$\{f \to \text{gr}\,f\} : \text{unif}\ \mathbf{F}(\mathcal{S}, \mathcal{P}) \to \text{unif}\ \mathcal{P}^{|\mathcal{S}|}$$

is a uniform homeomorphism.

Thus a symbol of the form unif $\mathbf{F}(\mathcal{S}, \mathcal{P})$ has two meanings: if $\mathcal{P}$ is a semi-uniform space, then unif $\mathbf{F}(\mathcal{S}, \mathcal{P})$ is a certain semi-uniform space and if $\mathcal{P}$ is a topological group, then unif $\mathbf{F}(\mathcal{S}, \mathcal{P})$ is a certain topological group; in addition, if $\mathcal{P}$ is a topological ring, then unif $\mathbf{F}(\mathcal{S}, \mathcal{P})$ denotes a certain topological ring. Nevertheless, in all cases a certain semi-uniformity of unif $\mathbf{F}(\mathcal{S}, \mathcal{P})$ is uniquely determined, in the former case the uniformity of (perhaps better the uniform structure of) the uniform space unif $\mathbf{F}(\mathcal{S}, \mathcal{P})$ and in the latter case the uniformity of the topological group unif $\mathbf{F}(\mathcal{S}, \mathcal{P})$.

Now we proceed to the formulation of the main results:

24 D.7. Theorem. *Let $\langle P, \mathcal{U}\rangle$ be a uniform space, and A be a set. Then*

(a) *If v is a closure operation for the set A and C is the set of all $f \in P^A$ such that the mapping $f : \langle A, v\rangle \to \langle P, \mathcal{U}\rangle$ is continuous, then C is a closed subset of* unif $\langle P, \mathcal{U}\rangle^A$.

(b) *If $\mathcal{V}$ is a semi-uniformity for A and C is the set of all $f \in P^A$ such that the mapping $f : \langle A, \mathcal{V}\rangle \to \langle P, \mathcal{U}\rangle$ is uniformly continuous, then the set C is closed in* unif $\langle P, \mathcal{U}\rangle^A$.

Obviously the theorem can be restated as follows:

24 D.7'. Theorem. *Let $\mathcal{P}$ be a uniform space. If $\mathcal{Q}$ is a closure space then the set $\mathbf{C}(\mathcal{Q}, \mathcal{P})$ of all continuous mappings of $\mathcal{Q}$ into $\mathcal{P}$ is closed in the space* unif $\mathbf{F}(\mathcal{Q}, \mathcal{P})$, *and if $\mathcal{Q}$ is a semi-uniform space then the set $\mathbf{U}(\mathcal{Q}, \mathcal{P})$ of all uniformly continuous mappings of $\mathcal{Q}$ into $\mathcal{P}$ is closed in the space* unif $\mathbf{F}(\mathcal{Q}, \mathcal{P})$.

Remark. According to 24 D.5, Theorem 24 D.7 is a generalization of a similar theorem (19 B.17) for topological groups, more precisely, combining 24 D.5 and 24 D.7 we obtain 19 B.17.

Proof of 24 D.7 (a). It is sufficient to show that the complement of C is open in the space unif $\langle P, \mathcal{U}\rangle^A$. Given an f in $P^A - C$ we must find a neighborhood G of f in the space $\langle P, \mathcal{U}\rangle^A$ such that no mapping $g : \langle A, u\rangle \to \langle P, \mathcal{U}\rangle$, $g \in G$, is continuous. There exists a point $\alpha \in A$ such that $f : \langle A, u\rangle \to \langle P, \mathcal{U}\rangle$ is not continuous at α. It follows that there exists a U in $\mathcal{U}$ such that $f^{-1}[U[f\alpha]]$ is not a neighborhood of α in $\langle A, u\rangle$. Choose a symmetric element W of $\mathcal{U}$ such that $W \circ W \circ W \subset U$. We shall prove that no mapping $g : \langle A, u\rangle \to \langle P, \mathcal{U}\rangle$ with g in $G = W^A[(f)]$ is continuous at the point α. It is sufficient to prove that

$$g^{-1}[W[g\alpha]] \subset f^{-1}[U[f\alpha]],$$

because the set on the right side is not a neighborhood of α in $\langle A, u\rangle$. Suppose that $a \in g^{-1}[W[g\alpha]]$, i.e., $\langle g\alpha, ga\rangle \in W$; we must show that $\langle f\alpha, fa\rangle \in U$. By our assumption $\langle f\alpha, g\alpha\rangle \in W$, $\langle ga, fa\rangle \in W^{-1} = W$, and consequently $\langle f\alpha, ga\rangle \in W \circ W$; finally $\langle f\alpha, fa\rangle \in W \circ W \circ W \subset U$, which concludes the proof.

Proof of 24 D.7 (b). We shall prove that the complement of C is open in unif $\langle P, \mathcal{U}\rangle^A$. Given an f in $P^A - C$, we must find a neighborhood G of f such that no mapping $g : \langle A, \mathcal{V}\rangle \to \langle P, \mathcal{U}\rangle$, $g \in G$, is uniformly continuous. Since $f : \langle A, \mathcal{V}\rangle \to \langle P, \mathcal{U}\rangle$ is not uniformly continuous, there exists a U in $\mathcal{U}$ so that $(f \times f)^{-1}[U]$ does not belong to $\mathcal{V}$. Choose a symmetric element W in $\mathcal{U}$ such that $W \circ W \circ W \subset U$. We shall prove that $(g \times g)^{-1}[W]$ belongs to $\mathcal{V}$ for no g from $G = W^A[(f)]$; this implies that no mapping $g : \langle A, \mathcal{V}\rangle \to \langle P, \mathcal{U}\rangle$, $g \in G$, is uniformly continuous. It is sufficient to show that

$$(g \times g)^{-1}[W] \subset (f \times f)^{-1}[U]$$

for each $g \in G$. The straightforward verification of this inclusion is left to the reader.

24 D.8. A net N of mappings of a given struct $\mathcal{S}$ into a given semi-uniform space $\mathcal{P}$ is said to be uniformly convergent to a mapping f of $\mathcal{S}$ into $\mathcal{P}$ if N converges to f in the space unif $\mathbf{F}(\mathcal{S}, \mathcal{P})$. Utilizing this terminology theorem 25 D.7 can be restated as follows: the limit of a uniformly convergent net of continuous (uniformly continuous) mappings into a uniform space is a continuous (uniformly continuous) mapping, or simply, the uniform limit of continuous (uniformly continuous) mappings into a uniform space is continuous (uniformly continuous); and this is the result announced at the beginning of the subsection.

Remark. It turns out that the assumption that $\mathcal{P} = \langle P, \mathcal{U} \rangle$ be a uniform space is essential in theorem 24 D.7 and 24 D.7′ (ex. 9).

The next theorem shows that the closure operation induced by the uniformity of unif $\langle P, \mathcal{U} \rangle^A$ depends essentially on $\mathcal{U}$.

24 D.9. Theorem. *Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are two semi-uniformities for a set P. Then the mapping*

$(*)$ $\mathsf{J} : \langle P, \mathcal{U}_1 \rangle \to \langle P, \mathcal{U}_2 \rangle$

is uniformly continuous (i.e., $\mathcal{U}_1 \supset \mathcal{U}_2$) if and only if the mapping

$(**)$ $\mathsf{J} : \text{unif} \langle P, \mathcal{U}_1 \rangle^P \to \text{unif} \langle P, \mathcal{U}_2 \rangle^P$

is continuous at the point J_P $(= \{\langle x, x \rangle \mid x \in P\})$.

Proof. The result follows immediately from the following formula which holds for each subset U of $P \times P$:

$$U^{\text{rel} P} [(\mathsf{J}_P)] = \mathbf{E}\{f \mid f \in P^P, f \subset U\}$$

Corollary. *Under the assumption of* 24 D.9, *the mapping $(*)$ is uniformly continuous if and only if the mapping $(**)$ is continuous. In particular $\mathcal{U}_1 = \mathcal{U}_2$ if and only if $(**)$ is a homeomorphism.*

Remark. Of course, the theorem remains true if the exponents P in $(**)$ are replaced by any set whose cardinal is at least the cardinal of P.

Recall that a net $\{f_a\}$ of mappings of a struct $\mathcal{S}$ into a closure space $\mathcal{P}$ is said to be pointwise convergent to a mapping f of $\mathcal{S}$ into $\mathcal{P}$ if for each x in $\mathcal{S}$ the net $\{f_a x\}$ converges to fx in $\mathcal{P}$. Thus $\{f_a\}$ converges to f pointwise if and only if $\{f_a\}$ converges to f in the set $\mathbf{F}(\mathcal{S}, \mathcal{P})$ endowed with a closure operation such that the bijective mapping

$$\{f \to \text{gr} f\}_L^V : \mathbf{F}(\mathcal{S}, \mathcal{P}) \to \mathcal{P}^{|\mathcal{S}|}$$

is a homeomorphism. We have seen that the pointwise limit of continuous mappings need not be continuous while the uniform limit of continuous mappings is always continuous. The theorem which follows is intended to clarify the difference between pointwise and uniform convergence.

24 D.10. Theorem. *Let $\mathcal{Q}$ be a closure space, $\mathcal{P}$ a uniform space and* unif $\mathbf{C}(\mathcal{Q}, \mathcal{P})$ *the subspace of* unif $\mathbf{F}(\mathcal{Q}, \mathcal{P})$ *with the underlying set* $\mathbf{C}(\mathcal{Q}, \mathcal{P})$ $(=$ *the set of all continuous mappings of $\mathcal{Q}$ into $\mathcal{P}$). Finally, let $\mathcal{C}$ be the closure space induced* *by* unif $\mathbf{C}(\mathscr{Q}, \mathscr{P})$. *Then the mapping*

$$\{\langle f, x\rangle \to fx\} : \mathscr{C} \times \mathscr{Q} \to \mathscr{P}$$

is continuous.

Corollary. *If a net* $\{f_a\}$ *of continuous mappings of a closure space* $\mathscr{Q}$ *into a uniform space* $\mathscr{P}$ *converges uniformly to* f, *then the net* $\{f_a x_a\}$ *converges to* fx *in* $\mathscr{P}$ *provided that* $\{x_a\}$ *converges to* x *in* $\mathscr{Q}$.

Proof. Write $\mathscr{P} = \langle P, \mathscr{U}\rangle$. Fix $\langle f, x\rangle \in \mathscr{C} \times \mathscr{Q}$ and let G be a neighborhood of fx in $\mathscr{P}$. We must find a neighborhood H_1 of f in $\mathscr{C}$ and a neighborhood H_2 of x in $\mathscr{Q}$ so that $g \in H_1$, $y \in H_2 \Rightarrow gy \in G$. Choose a U in $\mathscr{U}$ with $U[fx] \subset G$, and then a symmetric element $V \in \mathscr{U}$ such that $V \circ V \subset U$.

Since f is continuous we can choose a neighborhood H_2 of x in $\mathscr{Q}$ so that $f[H_2] \subset$ $\subset V[fx]$. Finally, put

$$H_1 = \mathbf{E}\{g \mid g \in \mathscr{C}, y \in |\mathscr{Q}| \Rightarrow \langle fy, gy\rangle \in V\}\,.$$

By definition, H_1 is a neighborhood of f in $\mathscr{C}$. Now, if $g \in H_1$, then clearly

$$g[H_2] \subset V[f[H_2]] \subset V[V[fx]] = (V \circ V)[fx] \subset U[fx] \subset G$$

which shows that H_1 and H_2 have the required properties.

E. UNIFORM COLLECTIONS OF COVERS

We shall introduce the concepts of a uniform cover and a semi-uniform cover of a semi-uniform space, and the concepts of a uniformizable cover and a semi-uniformizable cover of a closure space. In later developments only uniformizable covers will play an essential part (Sections 29 and 30). It is to be noted that semi-uniformizable covers (mainly of topological spaces) are often important in questions related to paracompactness (30 C) and metrization (30 B); however, particular attention will not be paid to these questions and semi-uniformizable covers will not be considered. For the sake of completeness we shall show that every uniform space is completely determined by the collection of all uniform covers. It is to be noted that a uniformity for a set P is often defined as a collection of covers of P subject to certain conditions. On the other hand, a semi-uniform space is not uniquely determined by the collection of all semi-uniform covers.

By 12 A.1 a cover of a set P is a collection or a family of sets the union of which is P. A cover of a struct $\mathscr{S}$ is defined as a cover of the underlying class of $\mathscr{S}$. We have considered, e.g., interior covers of a closure space; recall that an interior cover of a closure space is a cover $\mathscr{X}$ of P such that the interiors of elements of $\mathscr{X}$ or members of $\mathscr{X}$ (according as whether $\mathscr{X}$ is a family or a collection of sets) cover P.

24 E.1. Definition. A *semi-uniform cover* (*uniform cover*) of a semi-uniform space $\langle P, \mathscr{U}\rangle$ is a cover of P which is refined by some cover $\{U[x] \mid x \in P\}$ where U is an element (uniform element) of $\mathcal{U}$. *A semi-uniformizable (uniformizable) cover* of a closure space $\langle P, u\rangle$ is a cover of P which is a semi-uniform (uniform) cover for some semi-uniform space $\langle P, \mathcal{U}\rangle$ such that $\mathcal{U}$ is a continuous semi-uniformity for $\langle P, u\rangle$.

24 E.2. *Every uniform cover of a semi-uniform space $\mathcal{P}$ is a semi-uniform cover of $\mathcal{P}$ and every semi-uniform cover of a uniform space $\mathcal{P}$ is a uniform cover of $\mathcal{P}$. If $\mathcal{P}$ is a uniform modification of a semi-uniform space $\mathcal{Q}$, then uniform covers of $\mathcal{P}$ and $\mathcal{Q}$ coincide.*

Proof. The first statement is evident and the second follows from the fact that the uniform structure of $\mathcal{P}$ consists of all uniform elements of the uniform structure of $\mathcal{Q}$.

24 E.3. *Every uniformizable cover of a closure space $\mathcal{P}$ is a semi-uniformizable cover of $\mathcal{P}$ and every semi-uniformizable cover of $\mathcal{P}$ is an interior cover of $\mathcal{P}$. If $\mathcal{U}$ is the uniformly finest continuous semi-uniformity for a closure space $\langle P, u\rangle$, then $\mathcal{X}$ is a semi-uniformizable (uniformizable) cover of $\langle P, u\rangle$ if and only if $\mathcal{X}$ is a semi-uniform (uniform) cover of $\langle P, \mathcal{U}\rangle$.* – Evident.

24 E.4. *Let $\mathcal{P}$ be a closure space. If $\mathcal{X}$ is an interior cover of $\mathcal{P}$ then the closure of any subset X of $\mathcal{P}$ is contained in the star of X with respect to $\mathcal{X}$ (see Definition* 12 A.6). *A cover $\mathcal{X} = \{X_a \mid a \in A\}$ of $\mathcal{P}$ is semi-uniformizable if and only if there exists a mapping f of $|\mathcal{P}|$ into A such that X_{fx} is a neighborhood of x for each $x \in |\mathcal{P}|$ and*

$$\overline{X} \subset \bigcup\{X_{fx} \mid x \in X\}$$

for each $X \subset \mathcal{P}$.

Proof. The first statement is evident. We shall prove the second one. Let U be a semi-uniformizable vicinity of the diagonal of $\mathcal{P} \times \mathcal{P}$, i.e. U is a neighborhood of the diagonal of ind $(\mathcal{P} \times \mathcal{P})$ such that $\{U[x] \mid x \in |\mathcal{P}|\}$ refines $\mathcal{X}$, and let f assign to each $x \in |\mathcal{P}|$ an index a in A such that $U[x] \subset X_a$. By 23 B.4 we have $\overline{X} \subset U[X]$ for each $X \subset |\mathcal{P}|$ and hence $\overline{X} \subset U[X] \subset \bigcup\{X_{fx} \mid x \in X\}$. Conversely, given f, put $V = \Sigma\{X_{fx} \mid x \in \mathcal{P}\}$, $U = V \cap V^{-1}$. Since $\{U[x] \mid x \in \mathcal{P}\}$ refines X, it is enough to show that U is semi-uniformizable, or by 23 B.4, that $\overline{X} \subset U[X]$ for each $X \subset \mathcal{P}$. Clearly $X \subset \bigcup\{X_{fx} \mid x \in X\} = V[X]$, and as X_{fx} is a neighborhood of x, it results that $\overline{X} \subset V^{-1}[X]$.

24 E.5. Remarks. (a) An interior cover need not be semi-uniformizable. For example, let $\mathcal{P}$ be the ordered space of countable ordinals and let us consider $\mathcal{X} = \{X_x \mid x \in |\mathcal{P}|\}$, where X_x is the set of all ordinals less than x. Clearly $\mathcal{X}$ is an open cover of $\mathcal{P}$ and hence an interior cover of $\mathcal{P}$. We shall prove that $\mathcal{X}$ is not semi-uniformizable. Assuming that $\mathcal{X}$ is semi-uniformizable we can choose a mapping f of $|\mathcal{P}|$ into itself such that the formula of 24 E.4 holds for each $X \subset |\mathcal{P}|$; clearly there exists a sequence $\{x_n\}$ in $\mathcal{P}$ such that $x_{n+1} \notin X_{fx_n}$ for each n. Clearly $\{x_n\}$ is an increasing sequence and therefore $\{x_n\}$ converges to $x = \sup\{x_n\}$ (remember that $\mathcal{P}$ contains no countable cofinal sets). Thus x belongs to the closure of the set X of all

x_n. On the other hand, evidently $x \notin \bigcup\{X_{fx} \mid x \in X\}$, which contradicts our assumption and proves that $\mathscr{X}$ is not semi-uniformizable.

(b) In ex. 5 we shall show that every interior cover of a pseudometrizable space is uniformizable. It should be noted that regular topological spaces with this property are termed paracompact and will be studied in 30 C.

(c) A semi-uniformizable cover need not be uniformizable (see ex. 4).

24 E.6. *If $\mathscr{X}$ is an interior cover of a closure space $\mathscr{P}$ then the cover* st $\mathscr{X}$ *(see Definition* 12 A.6) *is semi-uniformizable.*

Proof. If U is the sum of st $\mathscr{X}$ (i.e. $U = \Sigma\{\operatorname{st}(\mathscr{X}, x) \mid x \in |\mathscr{P}|\}$) then $U[x] = \operatorname{st}(\mathscr{X}, x)$ is a neighborhood of x for each $x \in |\mathscr{P}|$ and clearly U is symmetric; thus U is a semi-neighborhood of the diagonal in $\mathscr{P} \times \mathscr{P}$. Since $\{U[x] \mid x \in |\mathscr{P}|\}$ refines st $\mathscr{X}$, the cover st $\mathscr{X}$ is semi-uniformizable.

Remark. It is to be noted that the set U of the proof is a neighborhood of the diagonal of $\mathscr{P} \times \mathscr{P}$. Indeed, $U = \bigcup\{X \times X \mid X \in \mathscr{X}\}$ if $\mathscr{X}$ is a collection and $U = \bigcup\{X_a \times X_a \mid a \in A\}$ if $\mathscr{X} = \{X_a \mid a \in A\}$.

Now we proceed to semi-uniform and uniform covers. We begin with a definition.

24 E.7. Definition. A semi-pseudometric d is said to be *subordinated to a cover* $\mathscr{X}$ if the collection of all open 1-spheres refines $\mathscr{X}$.

24 E.8. *In order that a cover $\mathscr{X}$ of a semi-uniform space $\mathscr{P}$ be semi-uniform it is necessary and sufficient that some uniformly continuous semi-pseudometric for $\mathscr{P}$ be subordinated to $\mathscr{X}$.* – Obvious, see 23 A.15.

Recall that, by Definition 12 A.2, for each cover $\mathscr{X}$ the symbol $\mathbf{V}\mathscr{X}$ denotes the vicinity associated with $\mathscr{X}$, i.e. the set $(\Sigma\mathscr{X}) \circ (\Sigma\mathscr{X})^{-1}$ which coincides with the set $\bigcup\{X \times X \mid X \in \mathscr{X}\}$ if $\mathscr{X}$ is a collection and $\bigcup\{X_a \times X_a\}$ if $\mathscr{X}$ is a family $\{X_a\}$; thus $\mathbf{V}\mathscr{X}$ is the vicinity considered in the remark following 24 E.6. We shall use the formula $\mathbf{V}\mathscr{X} \circ \mathbf{V}\mathscr{X} = \mathbf{V} \operatorname{st} \mathscr{X}$ of 12 A.7.

24 E.9. Theorem. *Each of the following two conditions is necessary and sufficient for a cover $\mathscr{X}$ of a semi-uniform space $\mathscr{P}$ to be a uniform cover:*

(a) *Some uniformly continuous pseudometric for $\mathscr{P}$ is subordinated to $\mathscr{X}$.*

(b) *There exists a sequence $\{\mathscr{X}_n\}$ of semi-uniform covers of $\mathscr{P}$ such that $\mathscr{X}_0$ refines $\mathscr{X}$ and each $\mathscr{X}_{n+1}$ is a star-refinement of $\mathscr{X}_n$ (i.e. each* st $\mathscr{X}_{n+1}$ *refines $\mathscr{X}_n$, see Definition* 12 A.12*).*

Proof. I. If $\mathscr{X}$ is a uniform cover then there exists a uniform element U such that $\{U[x] \mid x \in |\mathscr{P}|\}$ refines $\mathscr{X}$, and by 24 A.9 there exists a uniformly continuous pseudometric d for $\mathscr{P}$ such that $d\langle x, y\rangle < 1$ implies $\langle x, y\rangle \in U$; clearly d is subordinated to $\mathscr{X}$. – II. If a uniformly continuous pseudometric d for $\mathscr{P}$ is subordinated to $\mathscr{X}$ and $\mathscr{X}_n$ consists of all open 2^{-n}-spheres, then the sequence $\{\mathscr{X}_n\}$ has the properties of (b) and hence (a) implies (b). – III. It remains to show that (b) is sufficient. Assuming (b) let U_n be the vicinity associated with $\mathscr{X}_{n+1}$ for each n. It is easily seen that each U_n is an element of the semi-uniform structure of $\mathscr{P}$, $\{U_0[x] \mid x \in |\mathscr{P}|\}$ refines $\mathscr{X}$ and

$U_{n+1} \circ U_{n+1} \subset U_n$ for each n. Thus U_0 is a uniform element of the semi-uniform structure of $\mathscr{P}$ and $\mathscr{X}$ is a uniform cover.

Now we shall prove that a uniformity is uniquely determined by uniform covers.

24 E.10. Theorem. *Let $\mathscr{P} = \langle P, \mathscr{U} \rangle$ be a semi-uniform space. A vicinity U of the diagonal of $P \times P$ is a uniform element of $\mathscr{U}$ if and only if there exists a uniform cover $\mathscr{X}$ for $\mathscr{P}$ such that the vicinity associated with $\mathscr{X}$ is contained in U. A pseudometric d for $\mathscr{P}$ is uniformly continuous if and only if the cover of $\mathscr{P}$ consisting of all open r-spheres is a uniform cover of $\mathscr{P}$ for each positive real r.*

Proof. If U contains the vicinity $\mathbf{V}\mathscr{X}$ associated with a uniform cover $\mathscr{X}$ and V is a uniform element of $\mathscr{U}$ such that $\{V[x] \mid x \in P\}$ refines $\mathscr{X}$, then clearly $V[x] \subset \subset \operatorname{st}(\mathscr{X}, x) = (\mathbf{V}\mathscr{X})[x] \subset U[x]$ for each $x \in P$; hence $V \subset U$ so that U is a uniform element of $\mathscr{U}$. Conversely, let U be a uniform element of $\mathscr{U}$ and choose a symmetric uniform element V of $\mathscr{U}$ such that $V \circ V \subset U$. If $\mathscr{X} = \{V[x] \mid x \in P\}$ then $\mathscr{X}$ is a uniform cover of $\mathscr{P}$ (by definition) and evidently the vicinity associated with $\mathscr{X}$ is contained in U. The proof of the statement concerning pseudometrics is left to the reader.

Combining theorems 24 E.9 and 24 E.10 with 24 E.3 we obtain without difficulty the corresponding relations between uniformizable vicinities, uniformizable covers and continuous pseudometrics for a closure space.

24 E.11. Theorem. *Each of the following two conditions is necessary and sufficient for a cover $\mathscr{X}$ of a closure space $\mathscr{P}$ to be uniformizable:*

(a) *Some continuous pseudometric for $\mathscr{P}$ is subordinated to $\mathscr{X}$.*

(b) *There exists a sequence $\{\mathscr{X}_n\}$ of interior covers of $\mathscr{P}$ such that $\mathscr{X}_0$ refines $\mathscr{X}$ and each* $\operatorname{st} \mathscr{X}_{n+1}$ *refines $\mathscr{X}_n$.*

Proof. If $\{\mathscr{X}_n\}$ is the sequence of (b), then each $\mathscr{X}_n$ is a semi-uniformizable cover by 24 E.6. The result then follows from 24 E.3 and 24 E.9.

24 E.12. Theorem. *Let $\mathscr{P}$ be a closure space. A vicinity U of the diagonal of $\mathscr{P} \times \mathscr{P}$ is a uniformizable neighborhood of the diagonal of $\mathscr{P} \times \mathscr{P}$ if and only if U contains the vicinity associated with a uniformizable cover of $\mathscr{P}$. A pseudometric for $\mathscr{P}$ is continuous if and only if the cover consisting of all open r-spheres is uniformizable for each positive real r.* — 24 E.3, 24 E.10.

Remark. It is to be noted that a uniformizable cover of a closure space $\mathscr{P}$ is often said to be a normal cover of $\mathscr{P}$, and a sequence $\{\mathscr{X}_n\}$ of 24 E.11 (b) is said to be a normal sequence of covers of $\mathscr{P}$.

24 E.13. Definition. The *uniform collection of covers associated with a semi-uniform space* $\mathscr{P}$ is the set of all those uniform covers of $\mathscr{P}$ which are collections.

Notice that a uniform collection of covers consists of only those which are collections; families are excluded, as we want a uniform collection of covers to be a set.

From 24 E.9 and 24 E.10 we obtain immediately the following result.

24 E.14. Theorem. *The relation which assigns to each uniform space $\mathscr{P}$ the uniform collection of covers associated with $\mathscr{P}$ is one-to-one.*

Now we shall give a necessary and sufficient condition for a collection of covers to be a uniform collection of covers.

24 E.15. Theorem. *A class $\mathfrak{U}$ of covers of a set $P \neq \emptyset$ is a uniform collection of covers if and only if the following conditions are fulfilled:*

(a) *each element of $\mathfrak{U}$ is a collection;*

(b) *if a collection $\mathscr{X}$ of subsets of P is refined by a cover of $\mathfrak{U}$, then $\mathscr{X}$ belongs to $\mathfrak{U}$;*

(c) *if $\mathscr{X}_1$ and $\mathscr{X}_2$ belong to $\mathfrak{U}$, then some $\mathscr{X} \in \mathfrak{U}$ refines both $\mathscr{X}_1$ and $\mathscr{X}_2$;*

(d) *every $\mathscr{X} \in \mathfrak{U}$ has a star-refinement in $\mathfrak{U}$.*

Proof. Evidently every uniform collection of covers has properties (a)–(d). Conversely, assuming (a)–(d) let us consider the set $\mathscr{V}$ consisting of all the vicinities associated with covers of $\mathfrak{U}$. First we shall show that $\mathscr{V}$ is a base for a uniformity $\mathscr{U}$. Evidently each element of $\mathscr{V}$ is symmetric, and if $\mathscr{X}$ refines $\mathscr{X}_i$, $i = 1, 2$, then $\mathbf{V}\mathscr{X}$ is contained in both $\mathbf{V}\mathscr{X}_i$ and hence in $\mathbf{V}\mathscr{X}_1 \cap \mathbf{V}\mathscr{X}_2$. If st$\mathscr{Y}$ refines $\mathscr{X}$, then $(\mathbf{V}\mathscr{Y} \circ \mathbf{V}\mathscr{Y}) \subset \subset \mathbf{V}\mathscr{X}$ by 12 A.7.

It remains to show that $\mathfrak{U}$ is the uniform collection of covers of $\langle P, \mathscr{U} \rangle$. Let $\mathscr{X} \in \mathfrak{U}$. Choose a $\mathscr{Y}$ in $\mathfrak{U}$ such that the star of $\mathscr{Y}$ refines $\mathscr{X}$. Clearly, if V is the vicinity associated with $\mathscr{Y}$, then $\{V[x] \mid x \in P\}$ refines $\mathscr{X}$ and therefore $\mathscr{X}$ is a uniform cover of $\langle P, \mathscr{U} \rangle$. Conversely, let a collection $\mathscr{X}$ be a uniform cover of $\langle P, \mathscr{U} \rangle$. To prove $\mathscr{X} \in \mathfrak{U}$ it is sufficient to show that $\mathscr{X}$ is refined by some $\mathscr{Y} \in \mathfrak{U}$ (according to (b)). By our assumption there exists a U in $\mathscr{U}$ such that $\{U[x] \mid x \in P\}$ refines $\mathscr{X}$, and by the definition of $\mathscr{U}$ we can choose a $\mathscr{Y}$ in $\mathfrak{U}$ such that the vicinity associated with $\mathscr{Y}$ is contained in U. It is easily seen that $\mathscr{Y}$ refines $\mathscr{X}$.

Remark. Let $\mathfrak{W}$ be the set of all covers of a set P which are the collections ordered by the relation $\{\mathscr{X} \to \mathscr{Y} \mid \mathscr{X}$ is a refinement of $\mathscr{Y}\}$. Then a class $\mathfrak{U}$ of covers of P has the properties (a)–(c) if and only if $\mathfrak{U}$ is a right filter in $\mathfrak{W}$. Thus a class of covers of a set P is a uniform collection of covers if and only if $\mathfrak{U}$ is a right filter in $\mathfrak{W}$ satisfying condition (d).

In conclusion for the sake of completeness we shall state two theorems, leaving their simple proof to the reader.

24 E.16. *Let f be a uniformly continuous mapping of a semi-uniform space $\mathscr{P}$ into another one $\mathscr{Q}$. If $\mathscr{X}$ is a semi-uniform (uniform) cover of $\mathscr{Q}$, then the inverse image $\mathscr{Y}$ of $\mathscr{X}$ under f is a semi-uniform (uniform) cover of $\mathscr{P}$.*

24 E.17. *Let f be a continuous mapping of a closure space $\mathscr{P}$ into another one $\mathscr{Q}$. If $\mathscr{X}$ is a semi-uniformizable (uniformizable) cover of $\mathscr{Q}$, then the inverse image of $\mathscr{X}$ under f is a semi-uniformizable (uniformizable) cover of $\mathscr{P}$.*

25. PROXIMITY SPACES

Let $\mathscr{U}$ be a semi-uniformity for a set P and let us consider the relation p for $\exp P$ such that $\langle X, Y \rangle \in p$ if and only if $U[X] \cap Y \neq \emptyset$ for each U in $\mathscr{U}$. We shall write $X \, p \, Y$ instead of $\langle X, Y \rangle \in p$, and X non p Y instead of $\langle X, Y \rangle \in ((\exp P \times \exp P) - p)$. The following assertions will be proved:

(prox 1) $\emptyset$ non p P

(prox 2) p is symmetric, i.e. $X \, p \, Y \Rightarrow Y \, p \, X$

(prox 3) $X \subset P$, $Y \subset P$, $X \cap Y \neq \emptyset \Rightarrow X \, p \, Y$

(prox 4) If $X_1 \subset P$, $X_2 \subset P$ then $(X_1 \cup X_2) \, p \, Y$ if and only if $X_1 \, p \, Y$ or $X_2 \, p \, Y$.

Statement (prox 1) is obvious. If $U \subset P \times P$ is symmetric, then $U[X] \cap Y \neq \emptyset$ if and only if $U[Y] \cap X \neq \emptyset$; since the symmetric elements of $\mathscr{U}$ form a base for $\mathscr{U}$, the symmetry of p follows. Statement (prox 3) follows from the fact that each element of $\mathscr{U}$ contains the diagonal of $P \times P$, and hence $U[X] \supset X$ for each $X \subset P$ and $U \in \mathscr{U}$. If $P \supset Z \supset X$ and $X \, p \, Y$, then $Z \, p \, Y$ because $U[Z] \supset U[X]$ for each U; consequently, if $X_1 \, p \, Y$ or $X_2 \, p \, Y$, then also $(X_1 \cup X_2) \, p \, Y$. It remains to show that X_i non p Y, $i = 1, 2$, implies $(X_1 \cup X_2)$ non p Y. Now, if X_i non p Y, then we can choose U_i in $\mathscr{U}$ such that $U_i[X_i] \cap Y \neq \emptyset$; $U = U_1 \cap U_2$ belongs to $\mathscr{U}$ and $U[X_1 \cup X_2] \cap Y = \emptyset$ because $U[X_1 \cup X_2] = U[X_1] \cup U[X_2]$ and $U[X_i] \subset U_i[X_i]$, $i = 1, 2$, and hence, by definition of p, $(X_1 \cup X_2)$ non p Y.

Next, notice that if u is the closure operation induced by $\mathscr{U}$, then $uX = \mathbf{E}\{x \mid (x) \, p \, X\}$ for each $X \subset P$.

Given a set P, a relation for $\exp P$ satisfying conditions (prox 1)–(prox 4) will be called a proximity relation or a proximity for the set P.

In a closure space $\langle P, u \rangle$ we shall say that a point x is proximal to X in $\langle P, u \rangle$ if and only if $x \in uX$; the closure of a set X consists of all points proximal to X. One might say that, given a neighborhood U of x, the points of U are U-proximal to x; then x is proximal to X if and only if, for each U, X contains a point U-proximal to x. In a semi-uniform space $\langle P, \mathscr{U} \rangle$ we might define two points to be U-proximal, where $U \in \mathscr{U}$, if $\langle x, y \rangle \in U$, and two sets X and Y to be U-proximal if some $x \in X$ and $y \in Y$ are U-proximal. Finally, we might define two sets to be proximal if they are U-proximal for each U in $\mathscr{U}$. The resulting relation is just the proximity induced by $\mathscr{U}$.

In the rather elementary first subsection the basic concepts related to proximities will be introduced (e.g. a proximally continuous mapping, continuous proximity, proximally continuous semi-uniformity, relativization of a proximity) and the basic properties will be derived.

In the second subsection the relation between semi-uniformities and proximities will be studied. It will be shown that every proximity p is induced by a semi-uniformity $\mathcal{U}$ (in the sense described above, i.e., $X\,p\,Y \Leftrightarrow U[X] \cap Y \neq \emptyset$ for each U in $\mathcal{U}$), and that among all semi-uniformities inducing a given proximity p there exists a unique smallest one, which will be termed proximally coarse; if p is induced by a uniformity, i.e., if p is uniformizable, then the proximally coarse semi-uniformity inducing p is a uniformity. It turns out that the study of proximities is equivalent to the study of proximally coarse semi-uniformities. It is to be noted that proximally coarse uniformities coincide with totally bounded uniformities, i.e., uniformities such that for each element U of the uniformity there exists a finite set such that $U[A]$ is the whole underlying set of the corresponding space.

Subsection 25 C is concerned with developing the properties of uniformizable proximities. It is shown that a uniformizable proximity is uniquely determined by the set of all bounded proximally continuous functions. The important concept of the uniformizable modification q of a proximity p (for a set P) is introduced; it is shown that q is the unique uniformizable proximity for P such that a mapping f of $\langle P, p\rangle$ into a uniformizable proximity space Q is proximally continuous if and only if the mapping $f : \langle P, q\rangle \to Q$ is proximally continuous.

In the next two subsections, 25 D and 25 E, the set of all bounded proximally continuous functions is investigated. The main result of 25 D which asserts that the set of all bounded proximally continuous functions on a proximity space $\mathcal{P}$ is a closed sub-lattice-algebra of the topological lattice-algebra unif $\mathbf{F}^*(\mathcal{P},\mathrm{R})$ of all bounded mappings of $\mathcal{P}$ into R. The subject of 25 E is the famous Stone-Weierstrass theorem adapted for proximity spaces.

A. PROXIMITIES AND PROXIMALLY CONTINUOUS MAPPINGS

25 A.1. Definition. A *proximity for a set P* is a relation for exp P satisfying the conditions (prox 1)–(prox 4). *A proximity space* is a struct $\langle P, p\rangle$ such that P is a set and p is a proximity for P. If $\langle P, p\rangle$ is a proximity space and $X\,p\,Y$, then X and Y are said to be *proximal* in $\langle P, p\rangle$ or under p; the relation $(\exp P \times \exp P) - p$ is denoted by non p; if X non p Y, then X and Y are said to be *distant* or *non-proximal* in $\langle P, p\rangle$. If $\mathcal{U}$ is a semi-uniformity for a set P, then

$$p = \mathbf{E}\{\langle X, Y\rangle \mid X \subset P\,,\, Y \subset P\,,\, U \in \mathcal{U} \Rightarrow U[X] \cap Y \neq \emptyset\}$$

is a proximity for P which is said to be *induced by* $\mathcal{U}$. If p is a proximity for a set P, then the relation

$$\{X \to \mathbf{E}\{x \mid (x)\, p X\} \mid X \subset P\}$$

is a closure operation for P which is said to be *induced by p*.

25 A.2. *Let $\mathscr{U}$ be a semi-uniformity for a set P, p the proximity induced by $\mathscr{U}$ and u the closure induced by p. Then u is induced by $\mathscr{U}$.*

Proof. By definition $x \in uX$ if and only if the sets (x) and X are proximal, which means, by the definition of induced proximities, that $U[(x)] \cap X \neq \emptyset$ for each U in $\mathscr{U}$. It follows that, for each $x \in P$, the collection $[\mathscr{U}]\,[(x)]$ is a local base at x in $\langle P, u\rangle$. By the definition of semi-uniform closure the closure u is induced by $\mathscr{U}$.

Remarks. (a) A proximity for a set P is a subset of $\exp P \times \exp P$. – (b) Given a proximity p, there exists a unique set P such that p is a proximity for P, namely $P = \bigcup \mathbf{D} p$.

We shall often need the following simple proposition:

25 A.3. *Suppose that p is a proximity for a set P. Then*

(a) $X \subset Y \subset P,\ X p Z \Rightarrow Y p Z$

(b) *If $\{X_i\}$ and $\{Y_j\}$ are finite families in* $\exp P$ *such that* $(\bigcup\{X_i\})\ p\ (\bigcup\{Y_j\})$, *then $X_i\, p\, Y_j$ for some i and j.*

Proof. I. Statement (a) follows from (prox 4); $X\ p\ Z$, $Y \subset P$ imply $(X \cup Y)\ p\ Z$ by (prox 4), and $X \subset Y$ implies $X = X \cup Y$. – II. By induction it follows from (prox 4) that, for each finite family $\{X_i\}$, $(\bigcup\{X_i\})\ pY$ implies $X_i\, p\, Y$ for some i, and by (prox 2) (symmetry), $Y\, p\, (\bigcup\{X_i\})$ implies $X_i\, p\, Y$ for some i. Hence if $\{X_i\}$ and $\{Y_j\}$ are finite families such that $(\bigcup\{X_i\})\ p\ (\bigcup\{Y_j\})$, then $(\bigcup\{X_i\})\ p\ Y_j$ for some j and this implies $X_i\, p\, Y_j$ for some i.

25 A.4. Examples. (a) Let d be a semi-pseudometric for a set P, $\mathscr{U}$ the semi-uniformity induced by d and p the proximity induced by $\mathscr{U}$. It is almost self-evident that

$$p = \mathbf{E}\{\langle X, Y\rangle \mid \operatorname{dist}(X, Y) = 0\}\,.$$

This proximity will be said to be *induced* by d. – (b) Let P be the set consisting of all positive integers and let us consider the following two semi-pseudometrics d_1 and d_2 for P: if $x \neq y$ then $d_1\langle x, y\rangle = x^{-1} + y^{-1}$ and $d_2\langle x, y\rangle = 1$. Clearly both d_1 and d_2 induce the discrete closure for P. On the other hand d_1 and d_2 induce distinct proximities. If p_i is the proximity induced by d_i, then $X p_2 Y \Leftrightarrow X \subset P$, $Y \subset P$, $X \cap Y \neq \emptyset$, but $X\ p_1\ Y$ if and only if $X \subset P$, $Y \subset P$ and either $X \cap Y \neq \emptyset$ or both X and Y are infinite. Let $\mathscr{U}$ be the uniformity such that the sets of the form $\bigcup\{X_i \times X_i\}$, $\{X_i\}$ being a finite cover of P, form a base for $\mathscr{U}$. Clearly $\mathscr{U}$ induces p_2. On the other hand, p_2 is induced by the uniformity induced by d_2 which differs from $\mathscr{U}$. Thus distinct uniformities may induce the same proximity.

It turns out that a proximity may be described by means of proximal neighborhoods which will be introduced in the definition which follows. The concept of a proximal neighborhood is an adaptation of the concept of a neighborhood in a closure space to proximity spaces.

25 A.5. A *proximal neighborhood of a set* $X \subset P$ in a proximity space $\langle P, p \rangle$ is a set $Y \subset P$ such that X non $p(P - Y)$, that is, the complement of Y is distant to X in $\langle P, p \rangle$.

Let p be a proximity for a set P and let η be the relation consisting of all pairs $\langle X, Y \rangle$ such that Y is a p-proximal neighborhood of X, i.e., $\langle X, Y \rangle \in \eta$ if and only if $X \subset P$ and X non $p(P - Y)$. On the other hand, clearly X non $p\, Y$ if and only if $Y \subset P$ and $\langle X, P - Y \rangle \in \eta$. Thus a proximity is uniquely determined by the proximal neighborhoods. It is to be noted that some authors define a proximity as the relation η. One can easily formulate the corresponding conditions (see ex. 8).

25 A.6. Let $\langle P, p \rangle$ be a proximity space and let u be the closure induced by p. Every subset of P is a proximal neighborhood of the empty set. If $X \subset P$ is non-void, then the collection of all proximal neighborhoods of X is a proper filter on P the intersection of which contains X. If Y is a proximal neighborhood of X in $\langle P, p \rangle$, then Y is a neighborhood of X in $\langle P, u \rangle$, but the converse need not be true. On the other hand, every neighborhood of a singleton (x) is a proximal neighborhood of (x), more precisely, if $x \in P$ and Y is a neighborhood of (x) in $\langle P, u \rangle$, then Y is a proximal neighborhood of (x) in $\langle P, p \rangle$. The symmetry of a proximity implies that Y is a proximal neighborhood of X if and only if $P - X$ is a proximal neighborhood of $P - Y$. If p is induced by a semi-uniformity $\mathcal{U}$ and $X \neq \emptyset$, then $[\mathcal{U}][X]$ $(= \mathbf{E}\{U[X] \mid U \in \mathcal{U}\})$ is a base for the filter of all proximal neighborhoods of X in $\langle P, p \rangle$ (moreover, $[\mathcal{U}][X]$ coincides with this filter).

The proof is simple and therefore is left to the reader.

25. A.7. Definition. A mapping f of a proximity space $\langle P_1, p_1 \rangle$ into a proximity space $\langle P_2, p_2 \rangle$ is said to be *proximally continuous* if $X\, p_1\, Y$ implies $f[X]\, p_2 f[Y]$, i.e., if the relation $\{X \to f[X] \mid X \subset P_1\}$ is a "homomorphism relation under p_1 and p_2". A *proximal homeomorphism* is a one-to-one mapping of a proximity space $\langle P_1, p_1 \rangle$ onto a proximity space $\langle P_2, p_2 \rangle$ such that f as well as its inverse f^{-1} is proximally continuous. Finally, a proximity p_1 is said to be *proximally finer* than a proximity p_2, and p_2 is said to be *proximally coarser* than p_1, if p_1 as well as p_2 is for the same set, say P, and the identity mapping of $\langle P, p_1 \rangle$ onto $\langle P, p_2 \rangle$ is proximally continuous. A proximity space $\mathcal{P}$ is a *proximal homeomorph* of a proximity space $\mathcal{Q}$ if there exists a proximal homeomorphism of $\mathcal{Q}$ onto $\mathcal{P}$.

25 A.8. Theorem. *The composite of two proximally continuous mappings is a proximally continuous mapping; more precisely, if f and g are proximally continuous mappings and $\mathbf{E}^* f = \mathbf{D}^* g$, then $g \circ f$ is a proximally continuous mapping. The identity mapping of a proximity space onto itself is a proximal homeomorphism, if f is a proximal homeomorphism then so is f^{-1}, and finally, if f and g are proximal homeomorphisms and $\mathbf{E}^* f = \mathbf{D}^* g$, then $g \circ f$ is also a proximal homeomorphism.*

Corollary. *The relation* $\mathbf{E}\{\langle p, q\rangle \mid p$ *is proximally finer than* $q\}$ *is an order on the class of all proximities, and the relation* $\mathbf{E}\{\langle \mathscr{P}, \mathscr{Q}\rangle \mid \mathscr{Q}$ *is a proximal homeomorph of* $\mathscr{P}\}$ *is an equivalence on the class of all proximity spaces.*

Proof. Let f and g be proximally continuous and suppose that the composite $h = g \circ f$ exists; then $h[X_i] = g[f[X_i]]$, and if X_1 and X_2 are proximal in $\mathbf{D}^*f$, then $f[X_1]$ and $f[X_2]$ are proximal in $\mathbf{E}^*f (= \mathbf{D}^*g)$ by the proximal continuity of f, and finally $h[X_1]$ and $h[X_2]$ are proximal in $\mathbf{E}^*g = \mathbf{E}^*h$ by the proximal continuity of g, which establishes the proximal continuity of $g \circ f$. The proof of the other statements follows the proof of similar results for a closure space or a semi-uniform space.

The next theorem describes proximal continuity by means of proximal neighborhoods of sets in the same way as continuity is described by neighborhoods of points.

25 A.9. Theorem. *A mapping* f *of a proximity space* $\mathscr{P}$ *into a proximity space* $\mathscr{Q}$ *is proximally continuous if and only if the following condition is fulfilled: if* Y *is a proximal neighborhood of* X *in* $\mathscr{Q}$, *then* $f^{-1}[Y]$ *is a proximal neighborhood of* $f^{-1}[X]$ *in* $\mathscr{P}$.

Proof. Write $\mathscr{P} = \langle P, p\rangle$, $\mathscr{Q} = \langle Q, q\rangle$. I. Assuming f to be proximally continuous, given a proximal neighborhood Y of X in $\mathscr{Q}$ we must prove that $f^{-1}[Y]$ is a proximal neighborhood of $f^{-1}[X]$ in $\mathscr{P}$, i.e. $f^{-1}[X]$ non $p\,(P - f^{-1}[Y])$. Assuming the contrary, we obtain $(f[f^{-1}[X]])\, q\, (f[P - f^{-1}[Y]])$; but $f[f^{-1}[X]] = (\mathbf{E}f) \cap X$ and $f[P - f^{-1}[Y]] = (\mathbf{E}f) \cap (Q - Y)$, and consequently $X\, q\, (Q - Y)$, which contradicts our assumption that Y is a proximal neighborhood of X in $\mathscr{Q}$. — II. Now suppose that the condition is fullfilled and $X\, p\, Y$; we have to show that $f[X]\, q\, f[Y]$. Assuming the contrary we find that $Q - f[Y]$ is a proximal neighborhood of $f[X]$ in $\mathscr{Q}$, and by the condition, $f^{-1}[Q - f[Y]]\ (= P - f^{-1}[f[Y]])$ is a proximal neighborhood of $f^{-1}[f[X]]$, and hence $f^{-1}[f[X]]$ non $p\, f^{-1}[f[Y]]$, which contradicts our assumption $X\, p\, Y$ because $f^{-1}[f[X]] \supset X$ and $f^{-1}[f[Y]] \supset Y$.

Corollary. *A mapping* f *of a proximity space* $\mathscr{P}$ *into another one* $\mathscr{Q}$ *is proximally continuous if and only if for each subset* X *of* $\mathscr{P}$ *and each proximal neighborhood* U *of* $f[X]$ *in* $\mathscr{Q}$ *there exists a proximal neighborhood* V *of* X *in* $\mathscr{P}$ *such that* $f[V] \subset U$.

Remark. One might define a mapping f of a proximity space $\mathscr{P}$ into a proximity space $\mathscr{Q}$ to be proximally continuous about a subset $X \neq \emptyset$ of $|\mathscr{P}|$ if for each proximal neighborhood U of $f[X]$ in $\mathscr{Q}$ there exists a proximal neighborhood V of X in $\mathscr{P}$ such that $f[V] \subset U$. Then f is proximally continuous if and only if f is proximally continuous about each non-void subset of $|\mathscr{P}|$. Next, f is proximally continuous about a singleton (x) if and only if f is continuous at x with respect to the induced closure.

25 A.10. *Let* f *be a mapping of a proximity space* $\langle P_1, p_1\rangle$ *into a proximity space* $\langle P_2, p_2\rangle$. *If* p_i *is induced by a semi-uniformity* $\mathscr{U}_i$ *and the mapping* $f: \langle P_1, \mathscr{U}_1\rangle \to \to \langle P_2, \mathscr{U}_2\rangle$ *is uniformly continuous, then the mapping* $f: \langle P_1, p_1\rangle \to \langle P_2, p_2\rangle$ *is proximally continuous. If* u_i *is the closure induced by* p_i *and* $f: \langle P_1, p_1\rangle \to \to \langle P_2, p_2\rangle$ *is proximally continuous, then* $f: \langle P_1, u_1\rangle \to \langle P_2, u_2\rangle$ *is continuous.*

Proof. I. Suppose that $f : \langle P_1, \mathcal{U}_1 \rangle \to \langle P_2, \mathcal{U}_2 \rangle$ is uniformly continuous and $X\, p_1\, Y$. If $f[X]$ non $p_2 f[Y]$, then $U_2[f[X]] \cap f[Y] = \emptyset$ for some U_2 in $\mathcal{U}_2$, and consequently $U_1[X] \cap Y = \emptyset$, where $U_1 = (f \times f)^{-1}[U_2]$; but $U_1 \in \mathcal{U}$ by the uniform continuity of f, and hence X non $p_1 X$ which contradicts our assumption and establishes the proximal continuity of f. — II. Now let $f : \langle P_1, p_1 \rangle \to \langle P_2, p_2 \rangle$ be proximally continuous. If $x \in u_1 X$, then $(x)\, p_1\, X$ and hence $(fx)\, p_2 f[X]$ by the proximal continuity, which yields $fx \in u_2 f[X]$ and establishes the continuity of f.

Corollary. *If f is a Lipschitz continuous mapping of a semi-pseudometric space $\langle P_1, d_1 \rangle$ into another one $\langle P_2, d_2 \rangle$ and if p_i is the proximity induced by d_i, $i = 1, 2$, then the mapping $f : \langle P_1, p_1 \rangle \to \langle P_2, p_2 \rangle$ is proximally continuous.*

Propositions 25 A.2, 25 A.4 (a) and 25 A.10 enable us to extend our conventions. Let us recall that we have agreed to consider every semi-uniform space $\langle P, \mathcal{U} \rangle$ as a closure space $\langle P, u \rangle$ where u is the closure induced by $\mathcal{U}$; more precisely, if we say that $\langle P, \mathcal{U} \rangle$ has a property defined for closure spaces it is to be understood that $\langle P, u \rangle$ has this property. Similar conventions were made for mappings for semi-uniform spaces; roughly speaking, we agreed to speak about a mapping f for semi-uniform spaces as about its transpose to a mapping for closure spaces.

25 A.11. Definition and convention. The *transpose of a mapping f* for semi-uniform spaces to a mapping for proximity spaces is the mapping $f : \mathcal{P} \to \mathcal{Q}$ where $\mathcal{P}$ and $\mathcal{Q}$ are the proximity spaces induced by the semi-uniform spaces $\mathbf{D}^*f$ and $\mathbf{E}^*f$ respectively. The transpose of a mapping f for proximity spaces to a mapping for closure spaces is the mapping $f : \mathcal{P} \to \mathcal{Q}$ where $\mathcal{P}$ and $\mathcal{Q}$ are the closure spaces induced by $\mathbf{D}^*f$ and $\mathbf{E}^*f$ respectively.

If we say that a semi-uniform space (proximity space) has a property defined for proximity spaces (closure spaces) it is to be understood that the induced proximity space (closure space) has this property. The same conventions are made for mappings, i.e., if we say that a mapping f for semi-uniform spaces has a property defined for mappings for proximity spaces, e.g. that f is proximally continuous, it is to be understood that the transpose of f to a mapping for proximity spaces has this property, and if we say that a mapping f for proximity spaces has a property defined for closure spaces, e.g. f is continuous, it is to be understood that the transpose of f to a mapping for closure spaces has this property.

Now proposition 25 A.10 and its corollary can be restated as follows:

25 A.12. Theorem. *Every Lipschitz continuous mapping and every uniformly continuous mapping is proximally continuous. Every proximally continuous mapping is continuous.*

25 A.13. Corollary. *Every uniform homeomorphism (uniform embedding) is a proximal homeomorphism (proximal embedding). Every uniformly continuous pseudometric is a proximally continuous pseudometric.*

We recall that a uniformly continuous mapping for semi-pseudometric spaces need not be Lipschitz continuous, a proximally continuous mapping for semi-uniform spaces need not be uniformly continuous (25 A.4) and a continuous mapping for proximity spaces need not be proximally continuous (25 A.4). There are two important theorems asserting that, under certain assumptions, a proximally continuous mapping is uniformly continuous. One of these will be proved now while a second one, which requires the concept of proximally coarse semi-uniformity, will be given in subsection 25 B.

25 A.14. Theorem. *A proximally continuous mapping of a pseudometrizable uniform space into a pseudometrizable uniform space is uniformly continuous.*

Proof. Suppose that f is a proximally continuous but not uniformly continuous mapping of a pseudometric space $\langle P', d'\rangle$ into another one $\langle P, d\rangle$; we have to derive a contradiction. The mapping f is not uniformly continuous and therefore there exists a positive real r and sequences $\{\xi_n\}$ and $\{\eta_n\}$ in P' such that the sequence $\{d'\langle\xi_n, \eta_n\rangle\}$ converges to zero but $d\langle f\xi_n, f\eta_n\rangle \geqq r$ for each n. If n_i is an unbounded sequence in N, then the distance from $\mathbf{E}\{\xi_{n_i}\}$ to $\mathbf{E}\{\eta_{n_i}\}$ is zero in $\langle P', d'\rangle$ and consequently, f being proximally continuous, the distance from $f[\mathbf{E}\{\xi_{n_i}\}]$ $(= \mathbf{E}\{f\xi_{n_i}\})$ to $f[\mathbf{E}\{\eta_{n_i}\}]$ $(= \mathbf{E}\{f\eta_{n_i}\})$ in $\langle P, d\rangle$ is zero.

Write $x_n = f\xi_n$, $y_n = f\eta_n$ so that

(a) $d\langle x_n, y_n\rangle \geqq r > 0$ for each n in N, and

(b) the distance from $\mathbf{E}\{x_n \mid n \in M\}$ to $\mathbf{E}\{y_n \mid n \in M\}$ is zero for each infinite subset M of N.

We shall derive a contradiction.

I. If the net $\{d\langle x_n, x_m\rangle \mid \langle n, m\rangle \in \mathsf{N} \times \mathsf{N}\}$ converges to zero where $\mathsf{N} \times \mathsf{N}$ is endowed with the product order, then a contradiction is obtained as follows. Choose n_0 in N such that $n \geqq n_0$, $m \geqq n_0$ implies $d\langle x_n, x_m\rangle < \frac{1}{2}r$. The distance from $\mathbf{E}\{x_k \mid k \geqq n_0\}$ to the set $\mathbf{E}\{y_k \mid k \geqq n_0\}$ is zero and therefore, by (b), we can choose $m \geqq n_0$ and $n \geqq n_0$ such that $d\langle x_n, y_m\rangle < \frac{1}{2}r$. Now $d\langle x_m, y_m\rangle \leqq d\langle x_m, x_n\rangle + d\langle x_n, y_m\rangle < \frac{1}{2}r + \frac{1}{2}r = r$ which contradicts our assumption (a).

II. If there exists an infinite subset M of N such that the net $\{d\langle x_n, x_m\rangle \mid \langle n, m\rangle \in M \times M\}$ converges to zero, then a contradiction is obtained as in I.

III. If there exists an infinite subset M of N such that the net $\{d\langle y_n, y_m\rangle \mid \langle n, m\rangle \in M \times M\}$ converges to zero, then a contradiction is obtained by applying the argument of I with x_n and y_n interchanged.

IV. In the remaining case there exists no infinite subset M of N such that the net $\{d\langle x_n, x_m\rangle \mid \langle n, m\rangle \in M \times M\}$ or the net $\{d\langle y_n, y_m\rangle \mid \langle n, m\rangle \in M \times M\}$ converges to zero. Consequently, there exists a positive real s and an infinite subset M of N such that (see 18 ex. 11)

(c) $d\langle x_n, x_m\rangle \geqq s$, $d\langle y_n, y_m\rangle \geqq s$

for each $n \in M$, $m \in M$, $n \neq m$. Choose a positive real t such that $t \leqq \frac{1}{2}s$ and $t \leqq r$. It is easily seen that there exists an infinite subset L of M such that the distance from x_n to $\mathbf{E}\{y_k \mid k \in L\}$ as well as the distance from y_n to $\mathbf{E}\{x_k \mid k \in L\}$ is less than t for each

$n \in L$. Indeed, assuming the contrary we can construct an infinite subset K of M such that the distance from $\mathbf{E}\{x_n \mid n \in K\}$ to the set $\mathbf{E}\{y_n \mid n \in K\}$ is at least s, which contradicts our assumption (b). Let ϱ be the relation consisting of all $\langle n, m \rangle \in L \times L$ such that $d\langle x_n, y_m \rangle < t$. We have $\varrho[n] \neq \emptyset \neq \varrho^{-1}[n]$ for each n. It follows from (c) that the relations ϱ and ϱ^{-1} are single-valued. Indeed, if $d\langle x_n, y_k \rangle < t$, $d\langle x_m, y_k \rangle < t$, $n \in L$, $m \in L$, $k \in L$ then $d\langle x_n, x_m \rangle < d\langle x_n, y_k \rangle + d\langle x_m, y_k \rangle < 2t \leqq s$ which contradicts (c) and proves that ϱ^{-1} is single-valued. The same argument with x and y interchanged yields that ϱ is single-valued. Thus $\varrho : L \to L$ is a bijective mapping. If $n \in L$, then $n \in \mathsf{N}$ and hence $d\langle x_n, y_n \rangle \geqq r \geqq t$ (by (a)) which shows that $\varrho n \neq n$ for each n. Now it is easily seen that there exists an infinite subset K of L such that $\varrho[K] \cap K = \emptyset$. (Take a maximal element K of the ordered subset of $\langle \exp L, \subset \rangle$ consisting of all H such that $H \cap \varrho[H] = \emptyset$ and show that K is infinite.) Evidently the distance from $\mathbf{E}\{x_n \mid n \in K\}$ to the set $\mathbf{E}\{y_n \mid n \in K\}$ is at most s, which contradicts our assumption (b). The proof is complete.

25 A.15. Corollary. *Two pseudometrics are uniformly equivalent if and only if they are proximally equivalent; stated in other words, if d_1 and d_2 are pseudometrics for a set P, $\mathcal{U}_i$ is the uniformity induced by d_i and p_i is the proximity induced by d_i, $i = 1, 2$, then $\mathcal{U}_1 = \mathcal{U}_2$ if and only if $p_1 = p_2$.*

Proof. Any uniform homeomorphism is a proximal homeomorphism (by 25 A.13) and therefore $\mathcal{U}_1 = \mathcal{U}_2$ implies $p_1 = p_2$. It follows immediately from 25 A.14 that $p_1 = p_2$ implies $\mathcal{U}_1 = \mathcal{U}_2$.

25 A.16. Definition. The class of all proximities ordered by the relation $\mathbf{E}\{\langle p, q \rangle \mid p$ is proximally finer than $q\}$ will be denoted by $\mathbf{P}$, and, given a set P, the ordered subset of $\mathbf{P}$ consisting of all proximities for P will be denoted by $\mathbf{P}(P)$. The set of all proximally continuous mappings of a proximity space $\mathscr{P}$ into another one $\mathscr{Q}$ will be denoted by $\mathbf{P}(\mathscr{P}, \mathscr{Q})$.

If $\mathscr{P}$ and $\mathscr{Q}$ are semi-uniform spaces, then $\mathbf{U}(\mathscr{P}, \mathscr{Q})$ denotes the set of all uniformly continuous mappings of $\mathscr{P}$ into $\mathscr{Q}$; in accordance with 25 A.11, the symbol $\mathbf{P}(\mathscr{P}, \mathscr{Q})$ will denote the set of all proximally continuous mappings of $\mathscr{P}$ into $\mathscr{Q}$. Similarly, if $\mathscr{P}$ and $\mathscr{Q}$ are proximity spaces, then $\mathbf{C}(\mathscr{P}, \mathscr{Q})$ will denote the set of all continuous mappings of $\mathscr{P}$ into $\mathscr{Q}$. Our earlier results can now be restated as follows:

$$(*) \quad \mathbf{C}(\mathscr{P}, \mathscr{Q}) \supset \mathbf{P}(\mathscr{P}, \mathscr{Q}) \supset \mathbf{U}(\mathscr{P}, \mathscr{Q})$$

for all semi-uniform spaces $\mathscr{P}$ and $\mathscr{Q}$; the first inclusion holds for all proximity spaces $\mathscr{P}$ and $\mathscr{Q}$ whereas $\mathbf{U}(\mathscr{P}, \mathscr{Q})$ is not always defined. Roughly speaking, inclusions $(*)$ are true whenever the symbols are defined. Theorem 25 A.14 asserts that $\mathbf{P}(\mathscr{P}, \mathscr{Q}) \subset \mathbf{U}(\mathscr{P}, \mathscr{Q})$ for all pseudometric spaces $\mathscr{P}$ and $\mathscr{Q}$. Earlier, we have introduced the concepts of a continuous semi-uniformity and a continuous semi-pseudometric for a closure space, and of a uniformly continuous semi-pseudometric for a semi-uniform space. In a similar way we shall define a continuous proximity for a closure space, and a proximally continuous semi-uniformity and a proximally continuous semi-pseudometric for a proximity space. Although the definitions are evident we give the precise formulations.

25 A.17. Definition. A *continuous proximity* for a closure space $\langle P, u \rangle$ is a proximity for P such that the closure induced by p is coarser than u, i.e., the identity mapping of $\langle P, u \rangle$ onto $\langle P, p \rangle$ is continuous. A *proximally continuous semi-pseudometric* (a *proximally continuous semi-uniformity*) for a proximity space $\langle P, p \rangle$ is a pseudometric (semi-uniformity) ξ for P such that the proximity induced by ξ is proximally coarser than p, i.e., the identity mapping of $\langle P, p \rangle$ onto $\langle P, \xi \rangle$ is proximally continuous.

It is to be noted that, according to earlier results, if d is a proximally continuous semi-pseudometric for a proximity space $\mathscr{P}$ and if $\mathscr{U}$ is the semi-uniformity induced by d, then $\mathscr{U}$ is a proximally continuous semi-uniformity for $\mathscr{P}$, and similarly for continuous semi-pseudometrics, semi-uniformities and proximities for a closure space.

25 A.18. Examples. Suppose that P is a closure space.

(a) The relation

$$p_1 = \{\langle X, Y \rangle \mid X \subset P,\ Y \subset P,\ (\overline{X} \cap Y) \cup (X \cap \overline{Y}) \neq \emptyset\}$$

is the proximally finest continuous proximity for P. It is to be noted that two subsets X and Y of P are distant in $\langle |P|, p_1 \rangle$ if and only if they are semi-separated (20 A.1), i.e., X non p_1 Y if and only if X and Y are semi-separated in P. Verification of the conditions (prox i) is simple and therefore is omitted (one can use properties of non p_1 proved in 20 A). If p is any continuous proximity for P and $X\, p_1\, Y$, then $\overline{X} \cap Y \neq \emptyset$ or $X \cap \overline{Y} \neq \emptyset$; but $\overline{X} \cap Y \neq \emptyset$ implies $y \in \overline{X}$ for some $y \in Y$, and p being a continuous proximity, we obtain $(y)\, p\, X$ and hence $Y\, p\, X$ and thus also $X\, p\, Y$. Similarly $X \cap \overline{Y}$ yields $X\, p\, Y$. Thus always $X\, p\, Y$ whenever $X\, p_1\, Y$, which shows that p_1 is proximally finer than p. On the other hand, if $x \in \overline{X}$ then clearly $(x)\, p_1\, X$, which means that p_1 is a continuous proximity for the closure space P.

(b) The relation $p_2 = \{\langle X, Y \rangle \mid X\, p_1\, Y \text{ or both } X \text{ and } Y \text{ are infinite}\}$ is a continuous proximity for P, and if some proximity induces the closure structure of P, then p_2 is the proximally coarsest proximity inducing the closure structure of P.

(c) The relations $p_3 = \{\langle X, Y \rangle \mid X \subset P,\ Y \subset P,\ X \text{ and } Y \text{ are not separated in } P\}$ and $p_4 = \{\langle X, Y \rangle \mid \overline{X} \cap \overline{Y} \neq \emptyset\}$ are continuous proximities. It is to be noted that the relation non p_3 was studied in 20 A and the relation p_4, which will be called the Wallman proximity of P, will be studied in Section 29 devoted to normal spaces.

(d) The relation $p_5 = \mathbf{E}\{\langle X, Y \rangle \mid \text{if } f \text{ is a continuous function on } P, \text{ then } \overline{f[X]} \cap \cap \overline{f[Y]} \neq \emptyset\}$ is a continuous proximity for P, which is called the Čech proximity of P and will be studied in Section 28 devoted to uniformizable spaces.

Now we shall turn to the definition of a subspace of a proximity space and the sum of a family of proximity spaces. The product will be studied in Section 38.

25 A.19. Definition. If $\langle P, p\rangle$ is a proximity space and $Q \subset P$, then $q = p \cap (\exp Q \times \exp Q)$ is a proximity for Q which will be called the *relativization* of p to Q, and the space $\langle Q, q\rangle$ will be called a *subspace* of $\langle P, p\rangle$.

The verification of the fact that q is actually a proximity for Q is left to the reader. One can prove that q is the proximally coarsest proximity for Q such that the mapping $\mathsf{J} : \langle Q, q\rangle \to \langle P, p\rangle$ is proximally continuous. Now we have the following result:

25 A.20. *Let Q be a subset of a set P. If $\mathscr{V}$ is the relativization to Q of a semi-uniformity $\mathscr{U}$ for P, then the proximity induced by $\mathscr{V}$ is the relativization of the proximity induced by $\mathscr{U}$. If q is the relativization to Q of a proximity p for P, then the closure induced by q is the relativization of that induced by p.* – Evident.

25 A.21. Definition. A *restriction of a mapping f for proximity spaces* is a mapping $g = f : \mathscr{P} \to \mathscr{Q}$ where $\mathscr{P}$ is a subspace of $\mathbf{D}^*f$ and $\mathscr{Q}$ is a subspace of $\mathbf{E}^*f$; if $\mathscr{P} = \mathbf{D}^*f$, then g is a *range-restriction* of f, and if $\mathscr{Q} = \mathbf{E}^*f$, then g is a *domain-restriction*. A *proximal embedding* is a mapping f such that the range-restriction of f to the subspace $\mathbf{E}f$ of $\mathbf{E}^*f$ is a proximal homeomorphism.

Obviously, if $\mathscr{Q}$ is a subspace of a proximity space $\mathscr{P}$, then the mapping $\mathsf{J} : \mathscr{Q} \to \mathscr{P}$ is a proximal embedding, which is said to be the *identity embedding* of $\mathscr{Q}$ into $\mathscr{P}$.

25 A.22. Theorem. *The restriction of a proximally continuous mapping is proximally continuous. A mapping f for proximity spaces is proximally continuous if the range-restriction of f to the subspace $\mathbf{E}f$ of $\mathbf{E}^*f$ is proximally continuous.* – Evident.

25 A.23. Definition. The *sum of a family* $\{\langle P_a, p_a\rangle \mid a \in A\}$ *of proximity spaces*, denoted by $\Sigma\{\langle P_a, p_a\rangle \mid a \in A\}$, is the proximity space $\langle P, p\rangle$ where $P = \Sigma\{P_a\}$ and $X\, p\, Y$ if and only if $X \subset P$, $Y \subset P$ and $X_\alpha\, p_\alpha\, Y_\alpha$ for some α where $X = \Sigma\{X_a\}$, $Y = \Sigma\{Y_a\}$. The proximity p is termed the *sum of the family* $\{p_a\}$ and denoted by $\Sigma\{p_a\}$.

The straightforward verification of the fact that p is actually a proximity is left to the reader. Next, it is to be observed that the sets $\mathrm{inj}_a [P_a]$ and $\mathrm{inj}_b [P_b]$ are distant in $\langle P, p\rangle$ for each $a \neq b$. The basic properties are summarized in the theorem which follows. The simple proof is left to the reader as a convenient exercise.

25 A.24. Theorem. *Let $\langle P, p\rangle$ be the sum of a family $\{\langle P_a, p_a\rangle \mid a \in A\}$ of proximity spaces. Then*

(a) *If $\{\mathscr{U}_a\}$ is a family of semi-uniform spaces such that $\mathscr{U}_a$ induces p_a for each a, then the sum semi-uniformity $\Sigma\{\mathscr{U}_a\}$ induces p.*

(b) *If u_a is the closure induced by p_a, $a \in A$, then the sum closure $\Sigma\{u_a\}$ is induced by p.*

(c) *The mapping $\mathrm{inj}_a : \langle P_a, p_a\rangle \to \langle P, p\rangle$ is a proximal embedding for each a in A (which will be called the* canonical embedding).

(d) *The proximity p is the proximally finest proximity for P such that all mappings $\mathrm{inj}_a : \langle P_a, p_a\rangle \to \langle P, p\rangle$ are proximally continuous.*

(e) *A mapping f of $\langle P, p\rangle$ into a proximity space $\mathcal{Q}$ is proximally continuous if and only if the mapping $f \circ \mathrm{inj}_a : \langle P_a, p_a\rangle \to \mathcal{Q}$ is proximally continuous for each a in A.*

B. PROXIMALLY COARSE SEMI-UNIFORMITIES

We shall show that every proximity is induced by a semi-uniformity, and that among all uniformities inducing a given proximity there exists a uniformly coarsest (= smallest) one which will be called the proximally coarse semi-uniformity of $\langle P, p\rangle$. It turns out that this semi-uniformity is a uniformity if and only if p is induced by a uniformity.

25 B.1. Definition. A proximity induced by a uniformity will be called *uniformizable.*

Uniformizable proximities permit the following simple characterization:

25 B.2. Theorem. *The following condition is necessary and sufficient for a proximity p for a set P to be uniformizable*:

(prox 5) *If X non pY, then there exist $X_1 \subset P$ and $Y_1 \subset P$ such that $X_1 \cap Y_1 = \emptyset$, $(P - X_1)$ non pX and $(P - Y_1)$ non pY.*

Evidently, condition (prox 5) *can be restated as follows*:

(prox 5′) *If X non pY, then there exist proximal neighborhoods X_1 of X and Y_1 of Y such that $X_1 \cap Y_1 = \emptyset$.*

Proof. I. Necessity. Suppose that p is induced by a uniformity $\mathcal{U}$ and X non p Y. By the definition of induced proximities there exists a U in $\mathcal{U}$ such that $U[X] \cap Y = \emptyset$. Choose a symmetric element V of $\mathcal{U}$ so that $V \circ V \subset U$ and put $X_1 = V[X]$, $Y_1 = V[Y]$. By definition, X_1 and Y_1 are proximal neighborhoods of X and Y and it remains to show that $X_1 \cap Y_1 = \emptyset$. Assuming the contrary we obtain $V \circ V[X] \cap Y \neq \emptyset$ which implies $U[X] \cap Y \neq \emptyset$, and this contradicts our assumption $U[X] \cap Y = \emptyset$.

II. To prove the sufficiency we must construct a uniformity inducing p. The construction is not too simple. Three lemmas will be given, concerning the construction of the uniformly coarsest semi-uniformity inducing a given proximity p which will be proved (in 25 B.6) to be a uniformity if p fulfils the condition (prox 5). Thus sufficiency will follow from 25 B.6.

If a proximity p for a set P is induced by a semi-uniformity $\mathcal{U}$ and if $X\,p\,Y$, then $U[X] \cap Y \neq \emptyset$ for each U in $\mathcal{U}$. Therefore, given p, if we want to find a semi-uniformity inducing p it is natural to consider the collection $\mathcal{U}$ of all vicinities U of the diagonal of $P \times P$ such that $U[X] \cap Y \neq \emptyset$ whenever $X\,p\,Y$. It is easily seen that $U \in \mathcal{U}$, $U \subset V \subset P \times P$ implies $U^{-1} \in \mathcal{U}$ and $V \in \mathcal{U}$. On the other hand, the intersection of two elements of $\mathcal{U}$ need not belong to $\mathcal{U}$, as will be shown in example 25 B.10, and therefore $\mathcal{U}$ need not be a semi-uniformity. It turns out that the collection $\mathcal{U}'$ of all the elements V of $\mathcal{U}$ of the form $\bigcup\{X_i \times X_i\}$, where $\{X_i\}$ is a finite cover of P, possesses the following two properties: (1) if $U \in \mathscr{U}$ and $U' \in \mathscr{U}'$, then $U \cap U' \in \mathscr{U}$, and (2) if a semi-uniformity $\mathscr{V}$ induces p, then $\mathscr{U}' \subset \mathscr{V}$. It will follow from (1) that $\mathscr{U}'$ is a base for a proximally continuous semi-uniformity for $\langle P, p\rangle$; it turns out that this semi-uniformity induces p, and if p fulfils (prox 5), then this semi-uniformity is a uniformity. For convenience we shall introduce some terminology.

25 B.3. Definition. A *finite square vicinity* of the diagonal of $P \times P$ is a vicinity of the form $\bigcup\{X_i \times X_i\}$ where $\{X_i\}$ is a finite cover of P. If $\langle P, p\rangle$ is a proximity space then a *proximal vicinity of the diagonal* of $\langle P, p\rangle \times \langle P, p\rangle$, or a *p-proximal vicinity* of the diagonal of $P \times P$, is a subset U of $P \times P$ such that $X\, p\, Y$ implies $U[X] \cap Y \neq \emptyset$.

Remarks. (a) A subset U of $P \times P$ is a symmetric vicinity of the diagonal of $P \times P$ if and only if U is a union of squares $X \times X$. "If" is obvious and to prove "only if" notice that $V = \bigcup\{((x, y) \times (x, y)) \mid \langle x, y\rangle \in V\}$ provided that V is a symmetric vicinity of the diagonal. — (b) Every proximal vicinity U of the diagonal of $\langle P, p\rangle \times \langle P, p\rangle$ is a vicinity of the diagonal of $P \times P$; indeed, if $x \in P$, then $(x)\, p\, (x)$ (by (prox 3)) and hence $U[(x)] \cap (x) \neq \emptyset$, i.e. $\langle x, x\rangle \in U$.

For convenience, the main result, Lemma 25 B.6, will be preceded by two preparatory lemmas which are also important by themselves.

25 B.4. *Let P be a set. Every finite square vicinity of the diagonal of $P \times P$ is an intersection of a finite family of vicinities of the form $(X \times X) \cup (Y \times Y)$.*

Proof. Suppose that $U = \bigcup\{X_i \times X_i \mid i \leqq n\}$, $n \in \mathbb{N}$, is a vicinity of the diagonal of $P \times P$, i.e., $\{X_i\}$ is a cover of P. Assuming that $\langle x, y\rangle \in (P \times P) - U$ let us consider the union X of all X_i such that $x \in X_i$, and the union Y of all the remaining sets X_i. Since $y \notin X$ we have $\langle x, y\rangle \notin X \times X$ and since $x \notin Y$ we have $\langle x, y\rangle \notin$ $\notin Y \times Y$. Thus $U \subset ((X \times X) \cup (Y \times Y)) \subset (P \times P) - (\langle x, y\rangle)$. This concludes the proof.

25 B.5. *Suppose that $\langle P, p\rangle$ is a proximity space. Each of the following two conditions is necessary and sufficient for a set $V = ((X_1 \times X_1) \cup (X_2 \times X_2)) \subset$ $\subset P \times P$ to be a proximal vicinity of the diagonal:*

(a) $(P - X_1)$ non $p\,(P - X_2)$ *(and hence $X_1 \cup X_2 = P$)*;

(b) *if $X\, p\, Y$, then* $(X_1 \cap X)\, p\, (X_1 \cap Y)$ or $(X_2 \cap X)\, p\, (X_2 \cap Y)$.

Proof. First notice that $P - X_1 = X_2 - X_1$ and $P - X_2 = X_1 - X_2$ if $X_1 \cup X_2 = P$.

I. Condition (a) is necessary because $V[X_2 - X_1] = X_2$ and $X_2 \cap (X_1 - X_2) = \emptyset$.

II. Condition (b) is sufficient, for $(X_i \cap X)\, p\, (X_i \cap Y)$ implies $X_i \cap X \neq \emptyset$, $X_i \cap Y \neq \emptyset$, and hence $V[X] \cap Y \supset V[X_i \cap X] \cap (X_i \cap Y) = X_i \cap (X_i \cap Y) =$ $= X_i \cap Y \neq \emptyset$.

III. It remains to show that (a) implies (b). Assuming (a) suppose $X\, p\, Y$ and consider the following decompositions of X and Y:

$$X = ((X_1 - X_2) \cap X) \cup ((X_1 \cap X_2) \cap X) \cup ((X_2 - X_1) \cap X)$$
$$Y = ((X_1 - X_2) \cap Y) \cup ((X_1 \cap X_2) \cap Y) \cup ((X_2 - X_1) \cap Y) .$$

The sets X and Y being proximal, by 25 A.3 at least one of the sets of the decomposition of X must be proximal to a set from the decomposition of Y. But $(X_2 - X_1)$ non $p(X_1 - X_2)$ and hence also $(X \cap (X_2 - X_1))$ non $p(Y \cap (X_1 - X_2))$, $(Y \cap (X_2 - X_1))$ non $p(X \cap (X_1 - X_2))$. It follows that both of the proximal sets in question must be contained in X_1 or in X_2; this concludes the proof.

25 B.6. Lemma. *Suppose that $\langle P, p\rangle$ is a proximity space, $\mathscr{V}$ is the set of all finite square proximal vicinities (of the diagonal of $\langle P, p\rangle \times \langle P, p\rangle$) and $\mathscr{W}$ is the set of all elements of $\mathscr{V}$ of the form $(X \times X) \cup (Y \times Y)$. Obviously $\mathscr{V}$ is a sub-base for a semi-uniformity $\mathscr{U}$ for P. The following assertions hold:*

(a) *$\mathscr{V}$ consists of finite intersections of elements of $\mathscr{W}$ and hence $\mathscr{W}$ is a sub-base for $\mathscr{U}$.*

(b) *If $W \in \mathscr{W}$ and U is any proximal vicinity, then $W \cap U$ is also a proximal vicinity.*

(c) *$\mathscr{V}$ is multiplicative, hence a base for $\mathscr{U}$. Thus every element of $\mathscr{U}$ is a proximal vicinity and hence $\mathscr{U}$ is a proximally continuous semi-uniformity for $\langle P, p\rangle$.*

(d) *$\mathscr{U}$ induces p.*

(e) *If a semi-uniformity $\mathscr{U}_1$ induces p, then $\mathscr{U} \subset \mathscr{U}_1$.*

(f) *If p fulfils the condition* (prox 5), *then $\mathscr{U}$ is a uniformity.*

Proof. I. Statement (a) follows from 25 B.4 and the definition of a sub-base for a semi-uniformity.

II. To prove (b) let $W = (X_1 \times X_1) \cup (X_2 \times X_2) \in \mathscr{W}$, and let U be any proximal vicinity. Assuming $X\, p\, Y$ we must show that $(U \cap W)[X] \cap Y \neq \emptyset$. By 25 B.5 we obtain that $(X_i \cap X)\, p\, (X_i \cap Y)$ for some $i = 1, 2$. Since U is a proximal vicinity we have $U[X_i \cap X] \cap (X_i \cap Y) \neq 0$. However $(U \cap W)[X_i \cap X] = X_i \cap U[X_i \cap X]$, and consequently $(U \cap W)[X] \cap Y \supset (U \cap W)[X_i \cap X] \cap Y \supset X_i \cap U[X_i \cap X] \cap Y \neq \emptyset$.

III. Statement (c) follows immediately from (a) and (b) (by induction).

IV. For (d) it remains to show that if X non p Y then $U[X] \cap Y = \emptyset$ for some U in $\mathscr{U}$. Put $X_1 = P - X$ and $Y_1 = P - Y$. It follows from 25 B.5 (a) that $U = (X_1 \times X_1) \cup (Y_1 \times Y_1)$ is a proximal vicinity and hence $U \in \mathscr{W} \subset \mathscr{U}$. But clearly $U[X] = Y_1 = P - Y$.

V. To prove (e), suppose that a semi-uniformity $\mathscr{U}_1$ induces p. To prove that $\mathscr{U}$ is contained in $\mathscr{U}_1$ it is sufficient to show that the sub-base $\mathscr{W}$ of $\mathscr{U}$ is contained in $\mathscr{U}_1$. Let $W = (X \times X) \cup (Y \times Y)$ be any element of $\mathscr{W}$. By 25 B.5 we obtain $(P - X)$ non $p(P - Y)$. By our assumption there exists a U in $\mathscr{U}_1$ such that

$$(*) \qquad U[P - X] \cap (P - Y) = \emptyset.$$

Without any loss of generality we may and shall assume that U is symmetric, i.e. $U = U^{-1}$. Now the proof will be accomplished if we show that $U \subset W$; and this inclusion will be derived from (*) as follows.

It is sufficient to show that $U[x] \subset W[x]$ for each x in P. It follows from (*) that $U[P - X] \subset Y$; but clearly $W[P - X] = Y$ and hence $U[x] \subset W[x]$ for each x in $P - X$. Since U is symmetric, we obtain from (*) that $U[P - Y] \cap (P - X) = \emptyset$ and the same argument as above gives $U[x] \subset W[x]$ for each x in $P - Y$. It remains to consider the case when $x \in P - ((P - X) \cup (P - Y)) = X \cap Y$. However if $x \in (X \cap Y)$, then $W[x] = (X \cup Y) = P$ and therefore $U[x] \subset P = W[x]$.

VI. We must now prove assertion (f). Suppose that p fulfils condition (prox 5) of 25 B.2. According to 24 A.6, to prove that $\mathcal{U}$ is a uniformity it is sufficient to show that for each element W of the sub-base $\mathcal{W}$ for $\mathcal{U}$ there exists an element V of $\mathcal{V}$ such that $V \circ V \subset W$. Suppose that $W = ((X \times X) \cup (Y \times Y)) \in \mathcal{W}$ (see fig. 1).

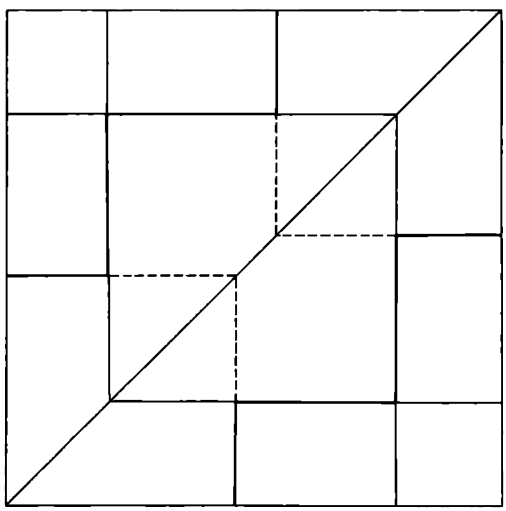

Fig. 1.

Since $(P - X)$ non $p\,(P - Y)$ there exists a proximal neighborhood Y_1 of $P - X$ and X_1 of $P - Y$ such that $X_1 \cap Y_1 = \emptyset$. Put $V = (X_1 \times X_1) \cup ((X \cap Y) \times (X \cap Y)) \cup (Y_1 \times Y_1)$. Now $V \in \mathcal{V}$ because V is the intersection of two elements of $\mathcal{W}$, namely $(X_1 \times X_1) \cup (Y \times Y)$ and $(Y_1 \times Y_1) \cup (X \times X)$, use 25 B.5 (a). It will be shown that $V \circ V \subset W$. By 23 B.7 we have $V \circ V = \bigcup\{V[x] \times V[x] \mid x \in P\}$. If $x \in X_1$, then $V[x] \subset X$ and hence $(V[x] \times V[x]) \subset X \times X \subset W$. If $x \in P - X_1$, then $V[x] \subset Y$ and hence $(V[x] \times V[x]) \subset Y \times Y \subset W$.

It is to be pointed out that Lemma 25 B.6 accomplishes the proof of Theorem 25 B.2. If p is a proximity for a set P then by 25 B.6 the set of all finite square p-proximal vicinities of the diagonal of $P \times P$ is a base for a semi-uniformity $\mathcal{U}$ for P, which is the smallest semi-uniformity inducing the proximity p. If $\mathcal{U}'$ is any semi-uniformity inducing p such that the set $\mathscr{V}'$ of all finite square vicinities from $\mathscr{U}'$ is a base for $\mathscr{U}'$, then necessarily $\mathscr{U} \subset \mathscr{U}'$; but $\mathscr{V}' \subset \mathscr{U}$, and $\mathscr{V}'$ being a base for $\mathscr{U}'$, we obtain $\mathscr{U}' \subset \mathscr{U}$ and hence $\mathscr{U}' = \mathscr{U}$. Thus we have proved

25 B.7. *Suppose that a semi-uniformity $\mathscr{U}$ induces a proximity p. Then $\mathscr{U}$ is the uniformly coarsest (i.e. smallest) semi-uniformity inducing p if and only if the finite square elements of $\mathscr{U}$ form a base for $\mathscr{U}$.*

25 B.8. Definition. A semi-uniformity $\mathscr{U}$ will be called *proximally coarse* if finite square elements of $\mathscr{U}$ form a base for $\mathscr{U}$, i.e. by 25 B.7, if a semi-uniformity $\mathscr{U}'$ induces the same proximity as $\mathscr{U}$, then $\mathscr{U} \subset \mathscr{U}'$ (i.e. $\mathscr{U}$ is uniformly coarser than $\mathscr{U}'$).

25 B.9. Theorem. *Every proximity is induced by a semi-uniformity. Among all the semi-uniformities inducing a given proximity p there exists a unique proximally coarse semi-uniformity $\mathscr{U}$; the set of all finite square p-proximal vicinities is a base for $\mathscr{U}$ and $\mathscr{U}$ is a uniformity if and only if p is uniformizable.* – (25 B.2, 25 B.6 and 25 B.7).

25 B.10. We shall construct some discrete proximities for a given infinite set P and we shall describe the corresponding proximal vicinities and proximal coarse semi-uniformities. We shall show that the intersection of two proximal vicinities need not be a proximal vicinity and the sum of two proximally continuous pseudometrics need not be proximally continuous.

According to Convention 25 A.11 a proximity p is said to be discrete if the closure induced by p is discrete. A proximity p will be called proximally discrete if every proximity proximally finer than p coincides with p, i.e. $p = \mathbf{E}\{\langle X, Y\rangle \mid X \subset P, Y \subset P, X \cap Y \neq \emptyset\}$ for some set P. A proximity space $\langle P, p\rangle$ will be called discrete or proximally discrete if p is discrete or proximally discrete. In what follows let P be an infinite set.

(a) *Let p be the proximally discrete proximity for P. Let d be the pseudometric for P which is 1 outside the diagonal of $P \times P$ (thus d is a metric). Evidently d induces p and the uniformity $\mathscr{U}$ induced by d is the uniformly finest uniformity for P. Thus $\mathscr{U}$ is the uniformly finest semi-uniformity inducing p and $\mathscr{U}$ consists of all proximal vicinities, i.e. of all vicinities of the diagonal. By* 25 B.9 *the finite square elements of $\mathscr{U}$ form a base for the proximally coarse semi-uniformity $\mathscr{V}$ of $\langle P, p\rangle$ which is a uniformity (by* 25 B.9*) because p is uniformizable. On the other hand, $\langle P, \mathscr{U}\rangle$ is metrizable and hence $\langle P, p\rangle$ is metrizable, but $\langle P, \mathscr{V}\rangle$ is not pseudometrizable.* Indeed, assuming that $\mathscr{V}$ is pseudometrizable we obtain that $\mathscr{U} = \mathscr{V}$ because any two proximally equivalent pseudometrizable uniformities coincide (by 25 A.15); but clearly $\mathscr{U} \neq \mathscr{V}$.

(b) *Suppose that ζ is a free proper filter of sets on P, i.e. each element of ζ is non-void and $\bigcap \zeta = \emptyset$. It is easily seen that the relation*

$$p = \mathbf{E}\{\langle X, Y\rangle \mid X \subset P,\ Y \subset P,\ X \cap Y \neq \emptyset \text{ or } (Z \in \zeta \Rightarrow Z \cap X \neq \emptyset \neq Z \cap Y)\}$$

is a discrete proximity for the set P. Next, p is proximally discrete if and only if

$X \cap Y = \emptyset$ implies that there exists a Z in ζ such that either $X \cap Z = \emptyset$ or $Y \cap Z = \emptyset$. It is almost self-evident that this condition is equivalent with the statement that ζ is an ultrafilter. Thus p is proximally discrete if and only if ζ is an ultrafilter. Clearly each set $J_P \cup (Z \times Z)$, $Z \in \zeta$, is a proximal vicinity. If $U = (X \times X) \cup (Y \times Y)$ is a proximal vicinity, then the sets $P - X$ and $P - Y$ are proximally distant and hence, being disjoint, there exists a Z in ζ such that $Z \cap (P - X) = \emptyset = Z \cap (P - Y)$ and hence $Z \times Z \subset U$. As a consequence, each proximal finite square vicinity contains a set of the form $Z \times Z$, $Z \in \zeta$. We have proved that a finite square vicinity U of the diagonal of $P \times P$ is a proximal vicinity if and only if U contains a set of the form $Z \times Z$, $Z \in \zeta$. Stated in other words, a finite square vicinity $\bigcup\{X_i \times X_i\}$ is proximal if and only if $\zeta \cap \mathsf{E}\{X_i\} \neq \emptyset$. It is interesting to notice that the proximity p is a relativization of the Wallman proximity q of a closure space $\langle Q, v\rangle$ (that is, $q = \mathsf{E}\{\langle X, Y\rangle \mid vX \cap vY \neq \emptyset\}$). Let Q consist of all points of P and a single further point x, and let v be the closure for Q such that P is an open isolated subset of $\langle Q, v\rangle$ and $[\zeta] \cup (x) = \mathsf{E}\{Z \cup (x) \mid Z \in \zeta\}$ is the neighborhood system at x. It is easily seen that p is the relativization of the Wallman proximity of $\langle Q, v\rangle$.

(c) *Let ζ_i, $i = 1, 2$, be proper filters on P and let p_i be proximities for P defined as in* (b), *i.e. $X p_i Y$ if and only if $X \subset P$, $Y \subset P$ and either $X \cap Y \neq \emptyset$ or $X \cap Z_i \neq \emptyset \neq Y \cap Z_i$* for each $Z_i \in \zeta_i$. *Let us consider the filter ζ on $P \times P$ having the collection $[\zeta_1] \times [\zeta_2] = \mathsf{E}\{Z_1 \times Z_2 \mid Z_i \in \zeta_i\}$ for a base, and the proximity p defined as in* (b), *i.e. $X p Y$ if and only if $X \subset P \times P$, $Y \subset P \times P$ and either $X \cap Y \neq \emptyset$ or $X \cap Z \neq \emptyset \neq Y \cap Z$ for each Z in ζ. Let π_1 and π_2 denote the projections $\{\langle x, y\rangle \to x \mid \langle x, y\rangle \in P \times P\}$ and $\{\langle x, y\rangle \to y \mid \langle x, y\rangle \in P \times P\}$. We shall prove that the projections π_i: $\langle P \times P, p\rangle \to \langle P, p_i\rangle$, $i = 1, 2$, are proximally continuous.* Suppose $X p Y$. If $X \cap Y \neq \emptyset$, then $\pi_i[X] \cap \pi_i[Y] \neq \emptyset$ and hence $\pi_i[X]\, p_i\, \pi_i[Y]$. Let $X \cap Y = \emptyset$ and $Z_i \in \zeta_i$. Since $Z = Z_1 \times Z_2$ belongs to ζ, we have $Z \cap X \neq \emptyset \neq Z \cap Y$ and hence $\pi_i[Z] \cap \pi_i[X] \neq \emptyset \neq \pi_i[Z] \cap \pi_i[Y]$, $i = 1, 2$. However, $\pi_i[Z] = Z_i$ and hence $\pi_i[X]\, p_i\, \pi_i[Y]$.

(d) *Under the assumptions of* (c) *let ζ_i be ultrafilters. Then, by* (b), *p_i are proximally discrete, i.e. $X p_i Y$ implies that $X \cap Y \neq \emptyset$, in particular $p_1 = p_2$. On the other hand, ζ need not be an ultrafilter and therefore the proximity p need not be proximally discrete. E.g., if $\zeta_1 \neq \zeta_2$ and ζ_i contains a countable set then ζ is not an ultrafilter by* 12 C.13.

(e) Let p be a proximity for $P \times P$ which is not proximally discrete and assume that the mappings $\pi_i : \langle P \times P, p\rangle \to \langle P, q\rangle$ are proximally continuous, where q is the proximally discrete proximity for P (such a p exists by (c) and (d)). If d is any proximally continuous pseudometric for $\langle P, q\rangle$ (and hence, by (a), if d is any pseudometric for P), then $d_i = d \circ (\pi_i \times \pi_i)$, $i = 1, 2$, are proximally continuous pseudometrics because the mapping $J : \langle P \times P, p\rangle \to \langle P \times P, d_i\rangle$ is proximally continuous as the composite of two proximally continuous mappings, namely π_i: $\langle P \times P, p\rangle \to \langle P, q\rangle$ and $J : \langle P, q\rangle \to \langle P, d\rangle$. In particular, if d is the pseudometric for P which is 1 outside the diagonal of $P \times P$, then d_1 and d_2 are proximally continuous pseudometrics for $\langle P \times P, p\rangle$; however their sum $D = d_1 + d_2$ is not proximally continuous because $D\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle = d\langle x_1, x_2\rangle + d\langle y_1, y_2\rangle \geqq 1$ whenever $\langle x_1, y_1\rangle \neq \langle x_2, y_2\rangle$, and hence D induces the proximally finest proximity for $P \times P$, which is, by our assumption, strictly proximally finer than p. *Thus the sum of two proximally continuous pseudometrics need not be proximally continuous. From this fact it follows at once that the intersection of two proximal vicinities need not be a proximal vicinity.*

25 B.11. Definition. A semi-uniformity $\mathscr{U}$ for a set P is said to be *totally bounded* if for each U in $\mathscr{U}$ there exists a finite subset X of P such that $U[X] = P$.

25 B.12. *Every proximally coarse semi-uniformity is totally bounded and every totally bounded uniformity is proximally coarse.*

Proof. I. Let $\mathscr{U}$ be a proximally coarse semi-uniformity for a set P and let $\mathscr{V}$ be the collection of all finite square elements of $\mathscr{U}$; thus $\mathscr{V}$ is a base for $\mathscr{U}$. If $U \in \mathscr{U}$, then $V \subset U$ for some $V = \bigcup\{X_i \times X_i\} \in \mathscr{V}$, where $\{X_i\}$ is a finite cover of P; now if X is a finite set intersecting each X_i, then clearly $V[X] = \bigcup\{X_i\} = P$ and hence $U[X] = P$.

II. Suppose that $\mathscr{U}$ is a totally bounded uniformity for a set P and U is any element of $\mathscr{U}$. We must find a finite square element W in $\mathscr{U}$ contained in U. Choose a symmetric element V in $\mathscr{U}$ such that $V \circ V \circ V \circ V \subset U$ and a finite subset X of P with $V[X] = P$, and put

$$W = \bigcup\{(V \circ V)[x] \times (V \circ V)[x] \mid x \in X\}.$$

Since $(V \circ V) \circ (V \circ V) \subset U$, the set W is contained in U by Lemma 23 B.7. To prove that $W \in \mathscr{U}$, we shall show that $W \supset V$. Given any y in P choose an x in X with $y \in V[x]$. We have

$$V[y] \subset V[V[x]] = (V \circ V)[x] \subset W[y].$$

Corollary. *A uniformity is proximally coarse if and only if it is totally bounded.*

Remark. A totally bounded semi-uniformity need not be proximally coarse. For example, consider an infinite set P, choose a point x in P and let us consider the proximity p for P such that $X \, p \, Y$ if and only if $X \cap Y \neq \emptyset$ or $X \neq \emptyset \neq Y$ and $x \in X \cup Y$. If u is the closure induced by p, then $u(y) = (x, y)$ if $y \in (P - (x))$ and $u(x) = P$. Thus P is the only neighborhood of x in $\langle P, u\rangle$ and consequently, if $\mathscr{U}$ is a continuous semi-uniformity for P, then $U[x] = P$ for each U in $\mathscr{U}$; this shows that every continuous semi-uniformity for $\langle P, u\rangle$ is totally bounded. Let $\mathscr{U}$ be the largest continuous semi-uniformity for $\langle P, u\rangle$. Clearly the set $U = \Delta_P \cup ((x) \times P) \cup \cup (P \times (x))$ forms a base for $\mathscr{U}$ and $\mathscr{U}$ induces p. On the other hand $\mathscr{U}$ is not proximally coarse because the set U contains no finite square element of $\mathscr{U}$ (P is infinite).

25 B.13. By our convention that every uniform concept applies to semi-pseudometrics, a semi-pseudometric is said to be totally bounded if the induced semi-uniformity is totally bounded. It is evident that a semi-pseudometric d for a set P is totally bounded if and only if, for each positive real r, there exists a finite subset X of P such that the distance from each $y \in P$ to X is less than r.

25 B.14. Theorem. *The class of all proximally coarse semi-uniformities is hereditary and closed under finite sums and arbitrary products.*

Proof. If $Q \subset P$ and U is a finite square vicinity of the diagonal of $P \times P$, then $(Q \times Q) \cap U$ is a finite square vicinity of the diagonal of $Q \times Q$ and therefore every relativization of a proximally coarse semi-uniformity is proximally coarse. If $\langle P, \mathscr{U} \rangle$ is the sum of a finite family $\{\langle P_a, \mathscr{U}_a \rangle \mid a \in A\}$ of proximally coarse semi-uniformities, and $U_a \in \mathscr{U}_a$ is finite square, then $\bigcup \{(\text{inj}_a \times \text{inj}_a) [U_a]\}$ is also finite square; this shows that $\mathscr{U}$ is proximally coarse. Finally, if $\langle P, \mathscr{U} \rangle$ is the product of a family $\{\langle P_a, \mathscr{U}_a \rangle\}$ and $U_a \in \mathscr{U}_a$ is finite square, then $\{\langle x, y \rangle \mid \langle \text{pr}_a x, \text{pr}_a y \rangle \in U_a\}$ is finite square and hence finite square elements form a sub-base for $\mathscr{U}$; this shows that $\mathscr{U}$ is proximally coarse.

25 B.15. Theorem. *Suppose that there exists a uniformly continuous mapping of a semi-uniform space $\mathscr{P}$ onto another one $\mathscr{Q}$. If $\mathscr{P}$ is totally bounded, then $\mathscr{Q}$ is also totally bounded. If $\mathscr{P}$ is totally bounded, in particular if $\mathscr{P}$ is proximally coarse, and if $\mathscr{Q}$ is a uniform space, then $\mathscr{Q}$ is proximally coarse.*

Proof. The first statement is an immediate consequence of the corresponding definition and the second one follows from the first and 25 B.12.

Remark. It is to be noted that there exists a uniformly continuous mapping of a proximally coarse uniform space onto a semi-uniform space which is not proximally coarse. For example, take an infinite set P, fix a point x in P and consider the proximally coarse uniformity $\mathscr{U}$ inducing the proximity $p = \mathbf{E}\{\langle X, Y \rangle \mid X \subset P, Y \subset P, X \cap Y \neq \emptyset\}$ (see 25 B.10). Next, fix a point x in P and consider the proximity $q = \mathbf{E}\{\langle X, Y \rangle \mid X \subset P,\ Y \subset P,\ X \cap Y \neq \emptyset \text{ or } X \neq \emptyset \neq Y \text{ and } x \in (X \cup Y)\}$. The proximity q is induced by the semi-uniformity $\mathscr{V}$ which consists of all $V \subset P \times P$ containing a set of the form $U \cup ((x) \times P) \cup (P \times (x))$, $U \in \mathscr{U}$. Clearly, $\mathrm{J} : \langle P, \mathscr{U} \rangle \to \langle P, \mathscr{V} \rangle$ is uniformly continuous but $\mathscr{V}$ is not proximally coarse because the proximally coarse semi-uniformity of $\langle P, q \rangle$ has for a base the set of all finite square vicinities $\bigcup \{X_i \times X_i\}$ such that $x \in \bigcap \{X_i\}$.

25 B.16. *A subspace Q of the uniform space of reals is proximally coarse if and only if Q is contained in a bounded interval in* $\mathbf{R}$.

Proof. I. If Q is contained in no bounded interval, then one can easily construct a sequence $\{x_n\}$ in Q such that $|x_n - x_m| \geqq 1$ for $n \neq m$. Now if $\{y_n\}$ is a sequence in Q such that $|x_n - y_n| < 2^{-1}$, then $1 \leqq |x_n - x_m| \leqq |x_n - y_n| + |y_n - y_m| + |x_m - y_m| < 1 + |y_n - y_m|$ whenever $n \neq m$, and hence $|y_n - y_m| > 0$ for $n \neq m$. But this implies that $\{y_n\}$ is a one-to-one sequence. Consequently (see 25 B.13), Q is not totally bounded and hence Q is not proximally coarse (by 25 B.12).

II. Now let Q be contained in a bounded interval $J = [\![-r, r]\!]$. But, according to

Theorem 25 B.14 it is sufficient to show that J is proximally coarse; by 25 B.12 this will follow if the interval J is totally bounded. Given a positive s, let S be the set of all the points $s \cdot n$, $n \in \mathrm{N}$ or $-n \in \mathrm{N}$. Clearly $S \cap [\![-r, r]\!]$ is finite and if $x \in J$, then $|x - y| < s$ for some y in S. The proof is complete.

By 25 A.10 every uniformly continuous mapping is proximally continuous but a proximally continuous mapping for semi-uniform spaces need not be uniformly continuous (it is sufficient to take two different proximally equivalent semi-uniformities). On the other hand one has the following

25 B.17. Theorem. *If $\mathscr{P}$ is a proximally coarse semi-uniform space, then every proximally continuous mapping of a semi-uniform space into $\mathscr{P}$ is uniformly continuous.*

Proof. Suppose that f is a proximally continuous mapping of a semi-uniform space $\langle P_1, \mathscr{U}_1 \rangle$ into a proximally coarse semi-uniform space $\langle P, \mathscr{U} \rangle$. To prove that f is uniformly continuous it is merely necessary to find a sub-base $\mathscr{W}$ for $\mathscr{U}$ such that $(f \times f)^{-1} [W] \in \mathscr{U}_1$ for each W in $\mathscr{W}$. Of course for $\mathscr{W}$ we take the sub-base for $\mathscr{U}$ described in lemma 25 B.6, i.e. the collection of all sets W of the form $W = (X \times X) \cup (Y \times Y)$ such that $(P - X)$ non p $(P - Y)$, where p is the proximity induced by $\mathscr{U}$. Since f is proximally continuous we obtain $(P_1 - X_1)$ non p_1 $(P_1 - Y_1)$ where $X_1 = f^{-1}[X]$, $Y_1 = f^{-1}[Y]$ and p_1 is the proximity induced by $\mathscr{U}_1$. Thus, from 25 B.5, $W_1 = (X_1 \times X_1) \cup (Y_1 \times Y_1)$ is a p_1-proximal vicinity of the diagonal of $P_1 \times P_1$, and consequently, by 25 B.6, $W_1 \in \mathscr{U}_1$. But clearly $W_1 = (f \times f)^{-1} [W]$.

Remark. It is to be noted that the property of proximally coarse semi-uniformities stated in 25 B.17 is characteristic for proximally coarse semi-uniformities, more precisely, a semi-uniform space $\mathscr{P}$ is proximally coarse if and only if every proximally continuous mapping of a semi-uniform space into $\mathscr{P}$ is uniformly continuous. "Only if" is proved in 25 B.17, and to prove "if" we need only take, for a given semi-uniformity $\mathscr{U}$ for a set P, the proximally coarse semi-uniformity $\mathscr{V}$ which is proximally equivalent to $\mathscr{U}$ and to consider the identity mapping of $\langle P, \mathscr{U} \rangle$ onto $\langle P, \mathscr{V} \rangle$ which is a proximal homeomorphism but which is not uniformly continuous if $\mathscr{U} \neq \mathscr{V}$.

25 B.18. Theorem. *Suppose that $\langle P, \mathscr{U} \rangle$ is the product of a non-void family $\{\langle P_a, \mathscr{U}_a \rangle \mid a \in A\}$ of proximally coarse semi-uniform spaces. The proximity p induced by $\mathscr{U}$ is the proximally coarsest proximity for P such that all mappings $\mathrm{pr}_a : \langle P, p \rangle \to \langle P_a, \mathscr{U}_a \rangle$ are proximally continuous.*

Proof. All the mappings in question are proximally continuous because all the mappings $\mathrm{pr}_a : \langle P, \mathscr{U} \rangle \to \langle P_a, \mathscr{U}_a \rangle$ are uniformly continuous and every uniformly continuous mapping is proximally continuous. Let q be any proximity for P such that all mappings $\mathrm{pr}_a : \langle P, q \rangle \to \langle P_a, \mathscr{U}_a \rangle$ are proximally continuous and let $\mathscr{V}$ be a semi-uniformity inducing q. Since $\mathscr{U}_a$ are proximally coarse, by 25 B.17 all mappings

$\mathrm{pr}_a : \langle P, \mathscr{V} \rangle \to \langle P_a, \mathscr{U}_a \rangle$ are uniformly continuous and consequently, by the definition of the product semi-uniformity, $\mathscr{V} \supset \mathscr{U}$; but this implies that q is proximally finer than p, which completes the proof.

Remark. One can prove that the last theorem remains true if all the $\mathscr{U}_a$ except one are proximally coarse (ex. 18). If two semi-uniformities are not proximally coarse, then the conclusion need not be true as will be shown by the example which follows.

25 B.19. Example. Let q be the proximally discrete proximity (see 25 B.10) for an infinite set P, let $\mathscr{U}$ be the proximally coarse semi-uniformity of $\langle P, q \rangle$ and let $\mathscr{V}$ be the largest semi-uniformity for P. Clearly $\mathscr{V}$ induces q. We shall prove that

(a) the product semi-uniformities (indeed, uniformities) $\mathscr{U} \times \mathscr{U}$, $\mathscr{V} \times \mathscr{U}$ and $\mathscr{U} \times \mathscr{V}$ induce the same proximity p for $P \times P$;

(b) p is not proximally discrete;

(c) $\mathscr{V} \times \mathscr{V}$ induces the proximally discrete proximity for P.

By 25 B.18 the proximity p, induced by $\mathscr{U} \times \mathscr{U}$, is the proximally coarsest proximity for $P \times P$ such that the projections $\{\langle x, y \rangle \to x\} : \langle P \times P, p \rangle \to \langle P, \mathscr{U} \rangle$ and $\{\langle x, y \rangle \to y\} : \langle P \times P, p \rangle \to \langle P, \mathscr{U} \rangle$ are proximally continuous. By example 25 B.10 (d) there exists a proximity for $P \times P$ which is not proximally discrete and such that the projections onto $\langle P, \mathscr{U} \rangle$ are proximally continuous. Thus (b) is true. Statement (c) is almost evident, since the product of two largest semi-uniformities is a largest semi-uniformity. The proof of (a) follows readily from lemma 25 B.6 (b).

25 B.20. Theorem. *The proximally coarse semi-uniformity $\mathscr{V}$ proximally equivalent with a given semi-uniformity $\mathscr{U}$ for a set P is the unique semi-uniformity for P with the following property:*

A mapping f of $\langle P, \mathscr{U} \rangle$ into a proximally coarse semi-uniform space $\mathscr{Q}$ is uniformly continuous if and only if the mapping $f : \langle P, \mathscr{V} \rangle \to \mathscr{Q}$ is uniformly continuous.

Proof. I. Let f be a uniformly continuous mapping of $\langle P, \mathscr{U} \rangle$ into a proximally coarse semi-uniform space $\langle Q, \mathscr{W} \rangle$ and let $\mathscr{V}$ be the proximally coarse semi-uniformity which is proximally equivalent to $\mathscr{U}$. The collection $\mathscr{W}'$ of all finite square elements of $\mathscr{W}$ is a base for $\mathscr{W}$, and the set $\mathscr{V}'$ of all $(f \times f)^{-1} [W]$, $W \in \mathscr{W}'$ consists of finite square elements of $\mathscr{U}$. On the other hand, the finite square elements of $\mathscr{U}$ form a base for $\mathscr{V}$ and therefore $\mathscr{V}' \subset \mathscr{V}$. Since $\mathscr{W}'$ is a base for $\mathscr{W}$, the mapping $f : \langle P, \mathscr{V} \rangle \to \langle Q, \mathscr{W} \rangle$ is uniformly continuous. Conversely, if $f : \langle P, \mathscr{V} \rangle \to$ $\to \langle Q, \mathscr{W} \rangle$ is uniformly continuous, then $f : \langle P, \mathscr{U} \rangle \to \langle Q, \mathscr{W} \rangle$ is uniformly continuous because $\mathscr{U}$ is uniformly finer than $\mathscr{V}$. Thus $\mathscr{V}$ fulfils the condition. – II. The uniqueness of $\mathscr{V}$ is evident.

In concluding we shall collect and complete some results concerning functions and pseudometrics. The set of all bounded functions of $\mathsf{P}(\mathscr{P}, \mathsf{R})$ will be denoted by $\mathsf{P}^*(\mathscr{P}, \mathsf{R})$.

25 B.21. Theorem. *Let $\mathscr{U}$ be a semi-uniformity for a set P, p the proximity induced by $\mathscr{U}$, and $\mathscr{V}$ the proximally coarse semi-uniformity inducing p (that is, $\mathscr{V}$ is the unique proximally coarse semi-uniformity which is proximally equivalent to $\mathscr{U}$). Then*

(a) *A pseudometric d for P is a uniformly continuous pseudometric for $\langle P, \mathscr{V} \rangle$ if and only if d is a totally bounded uniformly continuous pseudometric for $\langle P, \mathscr{U} \rangle$.*

(b) *A function f on $\langle P, \mathscr{V} \rangle$ is uniformly continuous if and only if the function $f : \langle P, \mathscr{U} \rangle \to \mathsf{R}$ is bounded and uniformly continuous.*

(c) *If a function f on $\langle P, \mathscr{U} \rangle$ is uniformly continuous, then f is proximally continuous, in symbols, $\mathbf{U}(\langle P, \mathscr{U} \rangle, \mathsf{R}) \subset \mathbf{P}(\langle P, \mathscr{U} \rangle, \mathsf{R})$.*

(d) *A function f on $\langle P, \mathscr{V} \rangle$ is uniformly continuous if and only if f is a bounded proximally continuous function, in symbols,*

$$\mathbf{U}(\langle P, \mathscr{V} \rangle, \mathsf{R}) = \mathbf{P}^*(\langle P, \mathscr{V} \rangle, \mathsf{R}) .$$

Proof. I. A totally bounded pseudometric is proximally coarse by 25 B.12, and therefore, by 25 B.20, a totally bounded pseudometric for P is uniformly continuous for $\langle P, \mathscr{V} \rangle$ if and only if it is uniformly continuous for $\langle P, \mathscr{U} \rangle$. Thus to prove statement (a) it remains to show that every uniformly continuous pseudometric for a proximally coarse semi-uniform space is totally bounded, and this follows from 25 B.15. — II. If f is a bounded function on P, then the subspace $\mathbf{E}f$ of R is proximally coarse (by 25 B.16) and therefore, by 25 B.20, the function $f : \langle P, \mathscr{U} \rangle \to \mathsf{R}$ is uniformly continuous if and only if the function $f : \langle P, \mathscr{V} \rangle \to \mathsf{R}$ is uniformly continuous. It remains to show that every uniformly continuous function f on a proximally coarse semi-uniform space is bounded. By 25 B.15 the subspace $\mathbf{E}f$ of $\mathbf{E}^*f$ is proximally coarse and therefore, by 25 B.16, $\mathbf{E}f$ is a bounded subset of R. — III. Statement (c) is a particular case of the fact that every uniformly continuous mapping is proximally continuous. — IV. Statements (b) and (c) imply the inclusion $\subset$ in (d). Conversely, if $f : \langle P, \mathscr{V} \rangle \to \mathsf{R}$ is a bounded proximally continuous function, then f is uniformly continuous by 25 B.17 because $\mathbf{E}f$ is a proximally coarse subspace of R.

25 B.22. Theorem. *Let $\mathscr{U}$ be a uniformity and let p be the proximity induced by $\mathscr{U}$. Every uniformly continuous pseudometric for $\langle P, \mathscr{U} \rangle$ is a proximally continuous pseudometric for $\langle P, p \rangle$. If every proximally continuous pseudometric for $\langle P, p \rangle$ is a uniformly continuous pseudometric for $\langle P, \mathscr{U} \rangle$, then $\mathscr{U}$ is the uniformly finest (i.e. largest) uniformity inducing p. Finally, if $\mathscr{U}$ is the uniformly finest uniformity inducing p, then every proximally continuous pseudometric for $\langle P, p \rangle$ is uniformly continuous for $\langle P, \mathscr{U} \rangle$.*

Proof. The first statement is a particular case of the fact that every uniformly continuous mapping is proximally continuous. If every proximally continuous pseudometric for $\langle P, p \rangle$ is uniformly continuous for $\langle P, \mathscr{U} \rangle$ and $\mathscr{W}$ is any proximally continuous uniformity for $\langle P, p \rangle$, then every uniformly continuous pseudometric for $\langle P, \mathscr{W} \rangle$ is proximally continuous for $\langle P, p \rangle$, and hence uniformly continuous for $\langle P, \mathscr{U}\rangle$; this implies that $\mathscr{U}$ is uniformly finer than $\mathscr{W}$ and establishes the second statement. The last statement follows from the following result:

25 B.23. *If d_1 and d_2 are proximally continuous pseudometrics for a proximity space $\langle P, p\rangle$ and d_1 is totally bounded, then $d_1 + d_2$ is proximally continuous.*

Indeed, if d is any proximally continuous pseudometric for $\langle P, p\rangle$, then all totally bounded proximally continuous pseudometrics for $\langle P, p\rangle$ together with d generate a proximally continuous uniformity $\mathscr{W}$ for $\langle P, p\rangle$ (by 25 B.23) which evidently induces p, and hence $\mathscr{W} \subset \mathscr{U}$. Thus d is a uniformly continuous pseudometric for $\langle P, \mathscr{U}\rangle$.

Proof of 25 B.23. Let $\mathscr{V}$ be the proximally coarse semi-uniformity which induces p and let $\mathscr{U}_i$, $i = 1, 2$, be the uniformity induced by d_i. Since $\mathscr{U}_1$ is proximally coarse and proximally continuous, by 25 B.15 the identity mapping of $\langle P, \mathscr{V}\rangle$ into $\langle P, \mathscr{U}\rangle$ is uniformly continuous and hence $\mathscr{U}_1 \subset \mathscr{V}$. By lemma 25 B.6, $[\mathscr{V}] \cap [\mathscr{U}_2]$ consists of p-proximal vicinities and hence $[\mathscr{U}_1] \cap [\mathscr{U}_2]$ consists of p-proximal vicinities. Since $[\mathscr{U}_1] \cap [\mathscr{U}_2]$ is a base for the uniformity induced by $d_1 + d_2$, the pseudometric $d_1 + d_2$ is proximally continuous for $\langle P, p\rangle$.

Remark. Remember that the sum of two proximally continuous pseudometrics need not be proximally continuous (25 B.10) and hence a uniformly finest proximally continuous uniformity for a given proximity space need not exist.

25 B.24. *Let p be a proximity for a set P induced by a pseudometric d and let $\mathscr{U}$ be the uniformity induced by d. Then $\mathscr{U}$ is the uniformly finest proximity which induces p.*

Proof. If D is a proximally continuous pseudometric for $\langle P, p\rangle$, then the mapping $\mathrm{J} : \langle P, d\rangle \to \langle P, D\rangle$ is proximally continuous and hence, by 25 A.14, uniformly continuous. Thus every proximally continuous pseudometric is a uniformly continuous pseudometric for $\langle P, \mathscr{U}\rangle$. By the preceding theorem $\mathscr{U}$ has the property in question.

25 B.25. Corollary. *If d is a totally bounded pseudometric, then the uniformity $\mathscr{U}$ induced by d is the unique uniformity inducing the same proximity p as d.*

Proof. Since $\mathscr{U}$ is proximally coarse, $\mathscr{U}$ is the smallest uniformity among all the uniformities inducing p. By 25 B.24, $\mathscr{U}$ is the largest among these uniformities.

C. UNIFORMIZABLE PROXIMITIES

By Definition 25 B.1 a proximity is uniformizable if it is induced by a uniformity, and by Theorem 25 B.2 a proximity is uniformizable if and only if it fulfils condition (prox 5). Here we shall describe uniformizable proximities by means of proximally continuous pseudometrics and functions and we shall introduce the concept of the uniformizable modification of a proximity.

25 C.1. Definition. The *uniformizable modification of a proximity p* is the proximally finest uniformizable proximity coarser than p.

25 C.2. Theorem. *The uniformizable modification q of a proximity p is induced by the uniform modification of any semi-uniformity inducing p. The proximally coarse semi-uniformity of the uniformizable modification $\mathcal{Q}$ of a proximity space $\mathcal{P}$ is the uniform modification of the proximally coarse semi-uniformity of $\mathcal{P}$.*

Proof. Let $\mathcal{V}$ be the uniform modification of the proximally coarse semi-uniformity $\mathcal{U}$ of $\langle P, p\rangle$. I. First we shall show that the proximity q induced by $\mathcal{V}$ is the uniformizable modification of p. It is clear that q is a uniformizable proximity proximally coarser than p. If q' is a uniformizable proximity proximally coarser than p, then the proximally coarse semi-uniformity $\mathcal{V}'$ of $\langle P, q'\rangle$ is a uniformity contained in $\mathcal{U}$ and hence in $\mathcal{V}$ because $\mathcal{V}$ is the largest uniformity contained in $\mathcal{U}$. As a consequence, q' is proximally coarser than q. – II. Now let $\mathcal{U}_1$ be any semi-uniformity inducing p, and $\mathcal{V}_1$ the uniform modification of $\mathcal{U}_1$. The proximity q_1 induced by $\mathcal{V}_1$ is proximally coarser than p, and q_1 being uniformizable, by I it is also proximally coarser than q. On the other hand, since $\mathcal{U} \subset \mathcal{U}_1$, we have $\mathcal{V} \subset \mathcal{V}_1$ and consequently q is proximally coarser than q_1. Thus $q = q_1$.

25 C.3. Theorem. *The uniformizable modification q of a proximity p for a set P is the unique uniformizable proximity for P satisfying the following condition:*
A mapping f of $\langle P, p\rangle$ into a uniformizable proximity space $\mathcal{R}$ is proximally continuous if and only if the mapping $f: \langle P, q\rangle \to \mathcal{R}$ is proximally continuous.

Proof. I. Let $\mathcal{U}$ be the proximally coarse semi-uniformity of $\langle P, p\rangle$ and let $\mathcal{V}$ be the uniform modification of $\mathcal{U}$. By 25 C.2 the uniformizable modification q of p is induced by $\mathcal{V}$. To prove that q fulfils the condition, suppose that f is any mapping of $\langle P, p\rangle$ into a uniformizable proximity space $\mathcal{R}$ and let us consider the proximally coarse semi-uniformity $\mathcal{W}$ of $\mathcal{R}$. By 25 B.6, $\mathcal{W}$ is a uniformity. Thus by 24 B.4 the mapping $f: \langle P, \mathcal{U}\rangle \to \langle|\mathcal{R}|, \mathcal{W}\rangle$ is uniformly continuous if and only if the mapping $f: \langle P, \mathcal{V}\rangle \to \langle|\mathcal{R}|, \mathcal{W}\rangle$ is uniformly continuous. Since $\mathcal{W}$ is proximally coarse, proximal continuity is equivalent to uniform continuity (by 25 B.17) which shows that the condition indeed obtains.

II. Uniqueness can be derived from 24 B.4 but a direct proof is simpler. Assuming the condition for uniformizable proximities q_1 and q_2, we find that $\mathsf{J}: \langle P, p\rangle \to \to \langle P, q_2\rangle$ is proximally continuous, because $\mathsf{J}: \langle P, q_2\rangle \to \langle P, q_2\rangle$ has this property, and therefore $\mathsf{J}: \langle P, q_1\rangle \to \langle P, q_2\rangle$ is proximally continuous; the same is true if q_1 and q_2 are interchanged, and consequently $q_1 = q_2$.

III. It might be appropriate to give a more direct proof than I of the fact that the uniformizable modification q of p fulfils the indicated condition. Since q is proximally coarser than p, if $f: \langle P, q\rangle \to \mathcal{R}$ is proximally continuous then necessarily $f: \langle P, p\rangle \to \mathcal{R}$ is proximally continuous. Conversely, let $f: \langle P, p\rangle \to \mathcal{R}$ be proximally continuous. Clearly, it is sufficient to find a uniformizable proximity q_1 proximally coarser than p such that $f: \langle P, q_1\rangle \to \mathcal{R}$ is proximally continuous (because then q_1 is proximally coarser than q). Let r be the proximity structure of $\mathcal{R}$ and put $q_1 = = \mathsf{E}\{\langle X, Y\rangle \mid X \subset P, Y \subset P, f[X]\, r\, f[Y]\}$. It is easily seen that q_1 is a proximity satisfying (prox 5) (because r fulfils (prox 5)) and hence q_1 is uniformizable. The remaining properties are almost self-evident.

25 C.4. Corollary. *A pseudometric d is a proximally continuous pseudometric for a proximity space $\mathscr{P}$ if and only if it is a proximally continuous pseudometric for the uniformizable modification of $\mathscr{P}$.*

Proof. The proximity induced by a pseudometric is uniformizable. – It is to be noted that 25 C.4 is also a corollary of 25 C.2 because the semi-uniformity induced by a pseudometric is a uniformity.

25 C.5. Theorem. *Each of the following three conditions is necessary and sufficient for a proximity space $\langle P, p\rangle$ to be uniformizable:*

(a) *$X\ p\ Y$ provided that $X \subset P$, $Y \subset P$ and the distance from X to Y is zero for each totally bounded proximally continuous pseudometric for $\langle P, p\rangle$;*

(b) *$X\ p\ Y$ provided that $X \subset P$, $Y \subset P$ and the distance from X to Y is zero for each proximally continuous pseudometric for $\langle P, p\rangle$;*

(c) *if X non $p\ Y$ then there exists a bounded proximally continuous function f on $\langle P, p\rangle$ which is 0 on X and 1 on Y.*

Proof. I. First we shall show that conditions (a), (b) and (c) are equivalent to each other. It is sufficient to prove (a) ⇒ (b) ⇒ (c) ⇒ (a). Clearly (a) ⇒ (b), and to prove (b) ⇒ (c) assume (b) and let X non $p\ Y$; by (b) we can take a proximally continuous pseudometric d for $\langle P, p\rangle$ such that the distance from X to Y in $\langle P, d\rangle$ is positive, say r; now consider the function $g = \{x \to \operatorname{dist}(X, (x))\}$ on $\langle P, p\rangle$ and put $f = \{x \to \min(1, r^{-1} . gx)\} : \langle P, p\rangle \to \mathsf{R}$. Clearly $0 \leqq f \leqq 1$ and f is 0 on X and 1 on Y. Next, $g : \langle P, d\rangle \to \mathsf{R}$ is a Lipschitz mapping, hence uniformly continuous and thus proximally continuous. Since g is proximally continuous, f is also proximally continuous. It is to be noted that it is easy to prove directly, without reference to semi-uniformities, that f is proximally continuous. It remains to show that (c) ⇒ (a). Assuming (c), let X non $p\ Y$; we must find a proximally continuous totally bounded pseudometric d for $\langle P, p\rangle$ such that the distance from X to Y in $\langle P, d\rangle$ is positive. Take a bounded proximally continuous function f on $\langle P, p\rangle$ which is 0 on X and 1 on Y, and consider the pseudometric $d = \{\langle x, y\rangle \to |fx - fy| \mid \langle x, y\rangle \in P \times P\}$. Evidently d is totally bounded and the distance from X to Y in $\langle P, d\rangle$ is 1. It remains to show that d is a proximally continuous pseudometric for $\langle P, p\rangle$. This follows immediately from the fact that, denoting by $\mathscr{U}$ the proximally coarse semi-uniformity of $\langle P, p\rangle$, the function $d_1 = d : \langle P, \mathscr{U}\rangle \times \langle P, \mathscr{U}\rangle \to \mathsf{R}$ is uniformly continuous since it is the composite of two uniformly continuous mappings; namely $d_1 = (\{\langle r, s\rangle \to |r - s|\} : \mathsf{R} \times \mathsf{R} \to \mathsf{R}) \circ (f \times f : \langle P, \mathscr{U}\rangle \times \langle P, \mathscr{U}\rangle \to \mathsf{R} \times \mathsf{R})$; this shows that d is a uniformly continuous pseudometric for $\langle P, \mathscr{U}\rangle$ and hence a proximally continuous pseudometric for $\langle P, p\rangle$.

It is to be noted that the proximal continuity of d can be proved directly: if $X\ p\ Y$, then the distance from $f[X]$ to $f[Y]$ is zero in R and therefore, clearly, the distance from X to Y in $\langle P, d\rangle$ is zero; this establishes the proximal continuity of d.

II. Condition (c) is sufficient. Assuming (c) we shall prove that condition (prox 5) is fulfilled (remember that, by 25 B.2, condition (prox 5) implies that p is uniformizable). If X non p Y and f is a proximally continuous function on $\langle P, p \rangle$ which is 0 on X and 1 on Y, then the sets $U = \mathbf{E}\{x \mid fx < \frac{1}{2}\} = f^{-1}[\,]\!\!] \rightarrow, \frac{1}{2} [\![\,]$ and $V = \mathbf{E}\{x \mid fx > \frac{1}{2}\} = f^{-1}[\,]\!\!] \frac{1}{2}, \rightarrow]\!\!]\,]$ are disjoint proximal neighborhoods of X and Y in $\langle P, p \rangle$.

III. Condition (b) is necessary. Let $\langle P, p \rangle$ be uniformizable and let $\mathscr{U}$ be a uniformity which induces p. If X non p Y, then $U[X] \cap Y = \emptyset$ for some U in $\mathscr{U}$, and $\mathscr{U}$ being a uniformity, we can choose a uniformly continuous pseudometric d for $\langle P, \mathscr{U} \rangle$ such that $d\langle x, y \rangle < 1$ implies $\langle x, y \rangle \in U$; clearly the distance from X to Y in $\langle P, d \rangle$ is at least 1. Since d is a uniformly continuous pseudometric for $\langle P, \mathscr{U} \rangle$, d is a proximally continuous pseudometric for $\langle P, p \rangle$.

Remark. The equivalence of conditions (a)–(c) was proved partly by means of semi-uniformities and partly without any reference to semi-uniform spaces. The proof was accomplished by showing that condition (c) implies (prox 5) and that, if a uniformity induces p, then condition (b) is fulfilled. We want to point out that, without any reference to the theory of semi-uniform spaces, one can prove that the condition (c) is equivalent to (prox 5). The proof of the implication (c) $\Rightarrow$ (prox 5) was given in II. The proof of the implication (prox 5) $\Rightarrow$ (c) is rather difficult; it parallels Urysohn's construction of continuous functions on normal spaces. It is to be noted that in our exposition the proofs of all results asserting the existence of continuous, proximally continuous or uniformly continuous functions were, in fact, based on lemma 18 B.10. The Urysohn procedure, just mentioned, gives another method of construction of continuous and proximally continuous functions (see ex. 14).

25 C.7. *If p and q are uniformizable proximities for a set P, then p is proximally coarser than q if and only if, for each bounded proximally continuous function f on $\langle P, p \rangle$, the function $f: \langle P, q \rangle \rightarrow \mathsf{R}$ is proximally continuous* (25 C.6).

Roughly speaking, a uniformizable proximity space is uniquely determined by the collection of all bounded proximally continuous functions.

D. PROXIMALLY CONTINUOUS FUNCTIONS

The purpose of this subsection is to prove that, for each proximity space $\mathscr{P}$, the set of all bounded proximally continuous functions on $\mathscr{P}$, denoted by $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$, is a closed sub-lattice-algebra of the topological lattice-algebra unif $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ of all bounded mappings of $\mathscr{P}$ into R.

25 D.1. Conventions. If $\mathscr{S}$ is a struct, then the symbol $\mathbf{F}^*(\mathscr{S}, \mathsf{R})$ will denote the normed lattice-algebra of all bounded mappings of $\mathscr{S}$ into R (see 19 D.14); we shall utilize the usual notation, i.e. $\| \;\|$ denotes the norm, $+$ and $.$ the addition and the multiplication both in $\mathbf{F}^*$ and in R; moreover, $.$ also denotes the external multiplication.

Next, $|f|$ denotes the function $\{x \to |fx|\} : \mathscr{P} \to \mathsf{R}$, and the symbols sup and inf stand for lattice operations, i.e., $\sup(f, g) = \{x \to \max(fx, gx)\}$, and similarly for $\inf(f, g)$. We shall say that $\mathscr{K}$ is a sub-lattice-algebra of $\mathscr{H}$ if the underlying lattice of $\mathscr{K}$ is a sublattice of the underlying lattice of $\mathscr{H}$ and if the underlying algebra of $\mathscr{K}$ is a subalgebra of that of $\mathscr{H}$. Similarly we use the terms sub-lattice-module, etc.

The main result is the following:

25 D.2. Theorem. *The set* $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$ *of all bounded proximally continuous functions on a proximity space* $\mathscr{P}$ *is a closed sub-lattice-algebra of the normed lattice-algebra* $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ *of all bounded mappings of* $\mathscr{P}$ *into* R.

Proof. Clearly every constant function on $\mathscr{P}$ is proximally continuous. Next, if f is a proximally continuous function, then $|f|$ is also proximally continuous because

$$\operatorname{dist}(|f|[X], |f|[Y]) \leqq \operatorname{dist}(f[X], f[Y])$$

for each $X \subset \mathscr{P}$ and $Y \subset \mathscr{P}$; this inequality follows from the inequality $||x| - |y|| \leqq$ $\leqq |x - y|$ which holds for all real numbers x and y. Clearly, if f is proximally continuous and r is a real number, then $r \cdot f$ is also proximally continuous. It remains to show that $f + g$ and $f \cdot g$ are proximally continuous functions whenever f and g are bounded proximally continuous functions, and that if a net $\{f_a\}$ of proximally continuous functions converges to f in unif $\mathbf{F}(\mathscr{P}, \mathsf{R})$, then f is proximally continuous. Indeed, the proximal continuity of the functions $\sup(f, g)$ and $\inf(f, g)$, where f and g are bounded proximally continuous functions, follows from the following obvious equalities:

$$\sup(f, g) = f + \sup(0, g - f) = f + \tfrac{1}{2}(|g - f| + (g - f)) =$$
$$= \tfrac{1}{2}(|g - f| + (f - g)),$$
$$\inf(f, g) = -\sup(-f, -g).$$

The remaining statements are particular cases of propositions 25 D.3 and 25 D.5 to follow.

25 D.3. *Let* $\mathscr{P} = \langle P, p\rangle$ *be a proximity space. The sum of two proximally continuous functions on* $\mathscr{P}$ *of which one is bounded, is a proximally continuous function. The product of two bounded proximally continuous functions on* $\mathscr{P}$ *is a proximally continuous function.*

Proof. I. We shall need the following property of bounded proximally continuous functions: if f is a bounded proximally continuous function on $\langle P, p\rangle$, r is a positive real and $X \, p \, Y$, then there exist $X' \subset X$ and $Y' \subset Y$ such that $X' \, p \, Y'$ and the diameters of the sets $f[X']$ and $f[Y']$ are at most r. As the set $\mathbf{E}f$ is contained in a bounded interval, we can choose a finite family $\{I_i\}$ of intervals which covers $\mathbf{E}f$ and such that the length of each I_i is r. Thus $\{f^{-1}[I_i]\}$ is a finite cover of $\mathscr{P}$ and the diameter of each set $f[f^{-1}[I_i]] \subset I_i$ is at most r. Now if $X \, p \, Y$, then, for some i and j, $(X \cap f^{-1}[I_i]) \; p \; (Y \cap f^{-1}[I_j])$ (by 25 A.2) and the diameters of the sets $f[X \cap$ $\cap f^{-1}[I_i]]$ and $f[Y \cap f^{-1}[I_j]]$ are at most r.

II. Now let f and g be two proximally continuous functions, f bounded and $h = f + g$. Suppose $X\,p\,Y$. To prove that the distance from $h[X]$ to $h[Y]$ is zero it is sufficient to show that the distance from $h[X]$ to $h[Y]$ is at most $3r$ for each positive real r. Let $r > 0$. Choose $X' \subset X$ and $Y' \subset Y$ such that $X'\,p\,Y'$ and the diameters of the sets $f[X']$ and $f[Y']$ are at most r (this is possible by I). Now if $x \in X'$ and $y \in Y'$, then the distance from fx to fy is at most $2r$ because the distance of the set $f[X']$ from $f[Y']$ is zero (f is proximally continuous) and their diameters are at most r. Since g is proximally continuous, the distance from $g[X']$ to $g[Y']$ is zero and therefore we can choose x in X' and y in Y' so that $|gx - gy| < r$. Now $|hx - hy| \leqq |fx - fy| + |gx - gy| \leqq 2r + r = 3r$, which shows that the distance from $h[X]$ to $h[Y]$ is at most $3r$.

III. Suppose that f and g are bounded proximally continuous functions, $|fx| \leqq K$ and $|gx| \leqq K$ for each x, where $K > 0$, $h = f \cdot g$, and $X\,p\,Y$. To prove that the distance from $h[X]$ to $h[Y]$ is zero it is sufficient to show that, for each $r > 0$, the distance from $h[X]$ to $h[Y]$ is at most $3\,K \,.\, r$. Let $r > 0$. By I we can choose $X' \subset X$ and $Y' \subset Y$ so that $X'\,p\,Y'$ and the diameters of the sets $f[X']$ and $f[Y']$ are at most r. Since the distance from $g[X']$ to $g[Y']$ is zero, we can choose x in X' and y in Y' such that $|gx - gy| < r$; since the distance from $f[X']$ to $f[Y']$ is zero and the diameters of these sets are at most r, we obtain $|fx \,.\, gx - fy \,.\, gy| \leqq |fx|\,|gx - gy| + |gy| \,.\, |fx - fy| \leqq K \,.\, 3r$, and consequently the distance from $h[X]$ to $h[Y]$ is at most $3rK$; this concludes the proof.

25 D.4. Examples. (a) The sum of two unbounded proximally continuous functions need not be proximally continuous. For example, let $P = \mathsf{N} \times \mathsf{N}$, and p be a proximity for P such that $X\,p\,Y$ implies $\pi_i[X] \cap \pi_i[Y] \neq \emptyset$, $i = 1, 2$, where $\pi_1 = \{\langle x, y\rangle \to x \mid \langle x, y\rangle \in P\}$ and $\pi_2 = \{\langle x, y\rangle \to y \mid \langle x, y\rangle \in P\}$ and p is not proximally discrete, i.e. $X\,p\,Y$ for some disjoint X and Y. Such a proximity p exists by 25 B.10. Let us take single-valued relations $\varrho_i \subset \mathsf{N} \times \mathsf{N}$, $i = 1, 2$, such that $\mathbf{D}\varrho_i = \mathsf{N}$ and the equality $\varrho_1 x + \varrho_2 y = \varrho_1 x' + \varrho_2 y'$ implies $x = x'$, $y = y'$, e.g., $\varrho_1 n = 2^{2n+1}$, $\varrho_2 n = 2^{2n}$. Consider the functions $f = \varrho_1 \circ \pi_1 : \langle P, p\rangle \to \mathsf{R}$, $g = \varrho_2 \circ \pi_2 : \langle P, p\rangle \to \mathsf{R}$. By the choice of ϱ_i we obtain that the values of $h = f + g$ are integers and $h\langle x, y\rangle = h\langle x', y'\rangle$ implies $\langle x, y\rangle = \langle x', y'\rangle$. It follows that if X and Y are disjoint subsets of P, then $h[X]$ and $h[Y]$ are disjoint subsets of N and hence the distance from $h[X]$ to $h[Y]$ is at least 1. As a consequence, h is not proximally continuous. On the other hand, both f and g are proximally continuous because if $X\,p\,Y$, then $\pi_i[X] \cap \pi_i[Y] \neq \emptyset$, $i = 1, 2$, and hence $f[X] \cap f[Y] \neq \emptyset$ and $g[X] \cap g[Y] \neq \emptyset$.

(b) The product of two proximally continuous functions need not be proximally continuous, and the following example is based on the fact that the multiplication in R is not uniformly continuous. Let f be the identity mapping of R onto itself. Thus f is a proximal homeomorphism. We proceed to prove that the function $h = f \,.\, f = \{x \to x^2\} : \mathsf{R} \to \mathsf{R}$ is not proximally continuous. Let X be the set of all integers $n \geqq 2$ and let Y be the set of all $n + n^{-1}$, $n \in X$. The distance from X to Y is zero because $(n + n^{-1}) - n = n^{-1}$, but the distance from $h[X]$ to $h[Y]$ is at least 2 because $((n + n^{-1})^2 - n^2) = 2 + n^{-2} \geqq 2$ and also $|(n + m)^2 - (n + n^{-1})^2| \geqq \geqq |n^2 - (n + n^{-1})^2|$ for each $n \in X$ and $|m| \in \mathsf{N}$. Thus X and Y are proximal but $h[X]$ and $h[Y]$ are not, which shows that h is not proximally continuous.

(c) The product of two proximally continuous functions need not be proximally continuous even if one of the functions is bounded. For example the functions $f = \{x \to x\} : \mathsf{R} \to \mathsf{R}$ and $g = \{x \to \sin x\} : \mathsf{R} \to \mathsf{R}$ are proximally continuous, g is bounded but $h = f \,.\, g$ is not proximally continuous. To prove that g is proximally continuous it is sufficient to show that g is Lipschitz continuous; indeed $|\sin x - - \sin y| \leqq |x - y|$. To show that h is not proximally continuous consider the set X of all $2k\pi$, $k \in \mathsf{N}$, and the set Y of all $2K\pi + \delta_K$, $K \in \mathsf{N}$, where the sequence $\{\delta_K\}$ is so chosen that $K \sin (2K\pi + \delta_K) = K \sin \delta_K \geqq 2^{-1}$ and 0 is a limit point of $\{\delta_K\}$. Thus the distance from X to Y is zero but the distance from $h[X]$ to $h[Y]$ is at least 2^{-1}. It may be noted that, on taking for g the mapping $\{x \to \to \operatorname{dist}(x, \mathsf{Z})\}$, the verification becomes considerably simpler.

Remark. A mapping of a pseudometric space into another one is proximally continuous if and only if it is uniformly continuous (25 A.14), and therefore examples (b) and (c) can be formulated for uniform continuity: the product of two uniformly continuous functions need not be uniformly continuous (even if one of the functions is bounded). It should also be noted that the proofs could be given, probably with some advantage, by means of more uniform-theoretical tools.

25 D.5. *The uniform limit of proximally continuous functions is a proximally continuous function. Stated in other words,* $\mathbf{P}(\mathscr{P}, \mathsf{R})$ *is closed in* unif $\mathbf{F}(\mathscr{P}, \mathsf{R})$ *for each proximity space* $\mathscr{P}$*.*

Proof. Suppose that a net $\{f_a\}$ of proximally continuous functions on a proximity space $\mathscr{P}$ converges uniformly to f, i.e., $\{f_a\}$ converges to f in unif $\mathbf{F}(\mathscr{P}, \mathsf{R})$. Let $X \, p \, Y$ and r be a positive real. We shall prove that the distance from $f[X]$ to $f[Y]$ is at most $3r$. Since $\{f_a\}$ converges to f uniformly, there exists an index a so that $|f_a x - fx| \leqq r$ for each $x \in \mathscr{P}$. Since f_a is proximally continuous, the distance from $f_a[X]$ to $f_a[Y]$ is zero and therefore we can choose an x in X and a y in Y so that $|f_a x - f_a y| < r$. Now

$$|fx - fy| \leqq |fx - f_a x| + |f_a x - f_a y| + |f_a y - fy| < 3r\,.$$

An alternate proof of 25 D.2 can be based on the theory of semi-uniform spaces and the fact that if $\mathscr{P}$ is a proximally coarse semi-uniform space, then $\mathbf{U}(\mathscr{P}, \mathsf{R}) = = \mathbf{P}^*(\mathscr{P}, \mathsf{R})$, i.e., a function f on a proximally coarse semi-uniform space is uniformly continuous if and only if f is a bounded proximally continuous function (25 B.21).

25 D.6. Theorem. (a) *If* $\mathscr{P}$ *is a semi-uniform space and* $\mathscr{G}$ *is a commutative topological group, then* $\mathbf{U}(\mathscr{P}, \mathscr{G})$ *is a closed subgroup of the group* unif $\mathbf{F}(\mathscr{P}, \mathscr{G})$ *and contains all constant mappings.*

(b) *If $\mathscr{P}$ is a semi-uniform space and $\mathscr{R}$ is a normed ring, then the set* $\mathbf{U}^*(\mathscr{P}, \mathscr{R})$ *of all bounded uniformly continuous mappings of $\mathscr{P}$ into $\mathscr{R}$ is a ring; if $\mathscr{R} = \mathsf{R}$, then* $\mathbf{U}^*(\mathscr{P}, \mathscr{R})$ *contains, with each f, the function $|f|$.*

Proof. I. Proof of (a). Suppose that $\mathscr{G} = \langle G, +, u\rangle$. The set $\mathbf{U}(\mathscr{P}, \mathscr{G})$ is closed in unif $\mathbf{F}(\mathscr{P}, \mathscr{G})$ by 24 D.7. Since $\mathscr{G}$ is commutative, the mapping $h = \{\langle x, y\rangle \to (x - y)\} : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ is uniformly continuous. Now, if f and g are uniformly continuous mappings, then $f - g = h \circ (f \times g)$, and consequently $f - g$ is uniformly continuous as the composite of two uniformly continuous mappings; hence $\mathbf{U}(\mathscr{P}, \mathscr{G})$ is a subgroup. — II. Proof of (b). Suppose that d is the pseudometric corresponding to the norm of $\mathscr{R}$, i.e. $d = \{\langle x, y\rangle \to \|x - y\|\}$ and let d_1 be the pseudometric for $|\mathscr{R}| \times |\mathscr{R}|$ such that $d_1\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle = d\langle x_1, x_2\rangle + d\langle y_1, y_2\rangle$. It is easily seen that the mapping $\{\langle x, y\rangle \to x \,.\, y\} = \langle|\mathscr{R}| \times |\mathscr{R}|, d_1\rangle \to \langle|\mathscr{R}|, d\rangle$ is Lipschitz continuous and hence uniformly continuous on each set $X \times X$, where X is a bounded subset of $\mathscr{R}$. Now, as in I, we find that $f \,.\, g$ is uniformly continuous whenever f and g are bounded uniformly continuous mappings. Finally, if $\mathscr{R} = \mathsf{R}$, then evidently $h = \{x \to |x|\} : \mathsf{R} \to \mathsf{R}$ is uniformly continuous and hence, if f is a uniformly continuous mapping into R, then $|f|$ is uniformly continuous as the composite of f and h.

E. STONE-WEIERSTRASS THEOREM

By the so-called Weierstrass theorem, for each bounded continuous function f on a bounded closed interval I of reals and for each positive real r there exists a polynomial function $g = \{x \to \Sigma\{a_i x^i \mid i \leqq n\}\}$ such that $|fx - gx| < r$ for each x in I; stated in other words, if $\mathscr{F}$ is the set of all polynomial functions on I, then $\mathscr{F}$ is dense in the normed algebra $\mathbf{C}^*(I, \mathsf{R})$ of all bounded continuous functions on I. Notice that $\mathscr{F}$ is the smallest subalgebra of $\mathbf{C}^*(I, \mathsf{R})$ containing the functions $\{x \to 1\} : I \to \mathsf{R}$ and $\mathsf{J} : I \to \mathsf{R}$. Thus the Weierstrass theorem can be stated as follows: the smallest subalgebra of $\mathbf{F}^*(I, \mathsf{R})$ containing the constant function $\{x \to 1\}$ and the function $\mathsf{J} : I \to \mathsf{R}$ is dense in $\mathbf{C}^*(I, \mathsf{R})$. Next, clearly the proximity of I is the proximally coarsest proximity for I such that $\mathsf{J} : I \to \mathsf{R}$ is a proximally continuous function, and it turns out that $\mathbf{C}^*(I, \mathsf{R}) = \mathbf{P}^*(I, \mathsf{R})$. (This follows from compactness of I.) Thus $\mathsf{J} : I \to \mathsf{R}$ "entirely determines" the proximity of I, and the smallest subalgebra of $\mathbf{P}^*(I, \mathsf{R})$ containing $\mathsf{J} : I \to \mathsf{R}$ and the constant function $\{x \to 1\}$ is dense in the normed algebra $\mathbf{P}^*(I, \mathsf{R})$. It turns out that this is true in general, for an appropriate definition of "entirely determines".

25 E.1 Definition. We shall say that a collection $\mathscr{M}$ of mappings of a proximity space $\mathscr{P} = \langle P, p\rangle$ into a proximity space $\mathscr{Q}$ *projectively generates the proximity of* $\mathscr{P}$ (or *projectively generates* $\mathscr{P}$) if p is the proximally coarsest proximity for P such that all mappings $f \in \mathscr{M}$ are proximally continuous.

The desired result can be stated as follows:

25 E.2. Stone-Weierstrass Theorem (*for proximity spaces*). *Let $\mathscr{P}$ be a proximity space projectively generated by a collection $\mathscr{M}$ of bounded functions, and let $\mathscr{F}$ be the smallest subalgebra of* $\mathbf{F}^*(\mathscr{P}, \mathrm{R})$ *containing $\mathscr{M}$ and the constant function* $\{x \to 1\} : \mathscr{P} \to \mathrm{R}$. *Then the closure of $\mathscr{F}$ in* $\mathbf{F}^*(\mathscr{P}, \mathrm{R})$ *is* $\mathbf{P}^*(\mathscr{P}, \mathrm{R})$. *Stated in other words, a bounded function f on $\mathscr{P}$ is proximally continuous if and only if the following condition is fulfilled:*

For each positive real r there exists a polynomial function $P = \{\langle z_0, \ldots, z_n\rangle \to \to \Sigma\{a_{i_0 \ldots i_n} \cdot z_0^{i_0} \cdot \ldots \cdot z_n^{i_n} \mid i_j \leqq k\} \mid z_j \in \mathrm{R}\} : \mathrm{R}^n \to \mathrm{R}$ *and functions* $f_0, \ldots, f_n$ *in $\mathscr{M}$ such that* $|fx - P(f_0x, \ldots, f_nx)| \leqq r$ *for each* $x \in |\mathscr{P}|$ *i.e.* $(\|f - P \circ (f_0 \times_{\mathrm{red}} \ldots \times_{\mathrm{red}} f_n)\| \leqq r)$.

25 E.3. Remarks. (a) If $\mathscr{M} = \emptyset$, then clearly the proximity structure of $\mathscr{P}$ is the proximally coarsest proximity for $|\mathscr{P}|$, and the Stone-Weierstrass Theorem states that precisely the constant functions are bounded proximally continuous functions; this is, of course, trivial.

(b) The Stone-Weierstrass Theorem states that if $\mathscr{M}$ projectively generates $\mathscr{P}$, then exactly the bounded proximally continuous functions can be obtained from $\mathscr{M}$ and the constant function $\{x \to 1\} : \mathscr{P} \to \mathrm{R}$ by the following operations:

(1) addition, multiplication and external multiplication (algebraic operations);

(2) taking uniform limits (a topological operation).

In other words, f is a bounded proximally continuous function if and only if, for each positive real r, there exists a linear combination g of finite products of functions of $\mathscr{M}$ and the function $\{x \to 1\} : \mathscr{P} \to \mathrm{R}$ such that $\|f - g\| \leqq r$.

(c) Instead of the assumption that $\mathscr{F}$ is the smallest subalgebra containing $\mathscr{M}$ and the function $\{x \to 1\}$ we can assume that $\mathscr{F}$ is the smallest ring containing $\mathscr{M}$ and all constant functions.

(d) By Theorem 25 D.2, $\mathbf{P}^*(\mathscr{P}, \mathrm{R})$ is a closed subalgebra of $\mathbf{F}^*(\mathscr{P}, \mathrm{R})$ and therefore it is sufficient to prove that the closure in $\mathbf{F}^*(\mathscr{P}, \mathrm{R})$ of $\mathscr{F}$ contains $\mathbf{P}^*(\mathscr{P}, \mathrm{R})$, i.e. every bounded proximally continuous function on $\mathscr{P}$ is a uniform limit of functions of $\mathscr{F}$.

The proof of 25 E.2 will be given in 25 E.10. We begin with a discussion of the proximity space projectively generated by a family of mappings into proximity spaces. It is to be noted that a more advanced theory will be given in Section 39.

25 E.4. *Let $\mathscr{F}$ be a collection of bounded functions on a set P. There exists a unique proximity p for P such that $\langle P, p\rangle$ is projectively generated by the collection of all functions* $f : \langle P, p\rangle \to \mathrm{R}$, $f \in \mathscr{F}$. *The set $\mathscr{D}$ of all pseudometrics* $d_f = \{\langle x, y\rangle \to \to |fx - fy| \mid \langle x, y\rangle \in P \times P\}$, $f \in \mathscr{F}$, *generates the proximally coarse semi-uniformity of $\langle P, p\rangle$. If $\mathscr{D}'$ is the smallest set containing $\mathscr{D}$ and such that* $d_1, d_2 \in \in \mathscr{D}' \Rightarrow (d_1 + d_2) \in \mathscr{D}'$, *then $X\,p\,Y$ if and only if the distance in $\langle P, d\rangle$ from X to Y is zero for each d in $\mathscr{D}'$.*

Proof. I. Let $\mathscr{U}$ be the semi-uniformity generated by the collection $\mathscr{D}$ of pseudometrics; by 23 A.12 the sets of the form $\mathbf{E}\{\langle x, y\rangle \mid d\langle x, y\rangle < r\}$, $d \in \mathscr{D}$, $r > 0$, form a sub-base for $\mathscr{U}$, and $\mathscr{U}$ is a uniformity by 24 A.9. Clearly each $d \in \mathscr{D}$ is totally bounded and hence $\mathscr{U}$ is totally bounded; $\mathscr{U}$ being a uniformity, it is proximally coarse (by 25 B.12). — II. Let p be the proximity induced by $\mathscr{U}$. Clearly the last statement of 25 E.4 holds. Hence every $f : \langle P, p \rangle \to \mathrm{R}, f \in \mathscr{F}$ is proximally continuous. — III. It remains to prove that p is the proximally coarsest proximity for P such that all the functions $f : \langle P, p \rangle \to \mathrm{R}$, $f \in \mathscr{F}$, are proximally continuous. Let q be any proximity for P such that all the functions $f : \langle P, p \rangle \to \mathrm{R}$, $f \in \mathscr{F}$, are proximally continuous; we shall show that q is proximally finer than p. Since each $f : \langle P, q \rangle \to \to \mathrm{R}$, $f \in \mathscr{F}$, is proximally continuous, each d_f, $f \in \mathscr{F}$, is a proximally continuous pseudometric for $\langle P, p \rangle$; each d_f being totally bounded, all the elements of $\mathscr{D}'$ are proximally continuous pseudometrics for $\langle P, p \rangle$ (by 25 B.23), and hence $\mathscr{U}$ is a proximally continuous uniformity for $\langle P, q \rangle$. Thus p is proximally coarser than q.

Assume that a proximity space $\langle P, p \rangle$ is projectively generated by a collection $\mathscr{F}$ of bounded functions, and for each f in $\mathscr{F}$ let d_f be the pseudometric defined in 25 E.4. If $X \, p \, Y$, then the distance from X to Y is zero in each $\langle P, d_f \rangle$. It is easy to find an example such that X and Y are distant in $\langle P, p \rangle$ but proximal in each $\langle P, d_f \rangle$. If the set of all d_f is addition-stable, then (by 25 E.4) X non $p \, Y$ implies that X and Y are distant in some $\langle P, d_f \rangle$. Similarly, if $X \, p \, Y$ then $f[X]$ is proximal to $f[Y]$ in R for each $f \in \mathscr{F}$, but the converse is not true; this follows from the similar result for d_f. It is interesting to show that the converse is not true even if $\mathscr{F}$ is a linear space. We shall only construct such an $\mathscr{F}$ with the following algebraic property: $f_1, f_2 \in \mathscr{F} \Rightarrow \Rightarrow f_1 + f_2 \in \mathscr{F}$. Using this example the reader may construct without difficulty such a linear space $\mathscr{F}$.

25 E.5. Example. Let $\langle P, p \rangle$ be a subspace of R, $P = I_1 \cup I_2 \cup I_3$, $I_1 = = [\![0, 1]\!]$, $I_2 = [\![2, 3]\!]$, $I_3 = [\![4, 5]\!]$, and let us consider the following two functions f and g on $\langle P, p \rangle$: $fx = gx = x$ for $x \in I_1$, $fx = x - 2$ and $gx = x$ for $x \in I_2$ and finally, $fx = x - 2$ and $gx = x - 4$ for $x \in I_3$.

It is easily seen that the collection (f, g) projectively generates $\langle P, p \rangle$. Let $\mathscr{F}$ be the set of all linear combinations $rf + sg$ with non-negative r and s. We shall show that $h[I_1] \cap h[I_2 \cup I_3] \neq \emptyset$ for each h in $\mathscr{F}$ (on the other hand, I_1 and $I_2 \cup I_3$ are distant in $\langle P, p \rangle$). Let $h = rf + sg$, $r \geqq 0$, $s \geqq 0$. It is easily seen that $h[I_1] = = [\![0, r + s]\!]$, $h[I_2] = [\![2s, 3s + r]\!]$, $h[I_3] = [\![2r, 3r + s]\!]$. It is clear that $t = = \min (2r + 2s) \leqq r + s$ and hence $t \in h\,[I_1] \cap h\,[I_2 \cap I_3]$.

Suppose that a proximity space $\mathscr{P}$ is generated by a collection $\mathscr{F}$ of bounded proximally continuous functions. By the preceding example it is not true that if X and Y are distant in $\mathscr{P}$ then $f[X]$ and $f[Y]$ are distant in R for some f in $\mathscr{F}$. On the other hand one has the following essentially weaker result:

25 E.6. *Suppose that a proximity space $\langle P, p \rangle$ is projectively generated by a collection $\mathscr{F}$ of bounded functions. Then $X \, p \, Y$ if and only if the following condition is fulfilled: If X is the union of a finite family $\{X_i\}$ and Y is the union of a finite family $\{Y_j\}$, then there exist indices i and j such that $f[X_i]$ is proximal to $f[Y_j]$ for each f in $\mathscr{F}$.*

Proof. For each f in $\mathscr{F}$ put $p_f = \mathsf{E}\{\langle X, Y\rangle \mid f[X]$ is proximal to $f[Y]$ in $\mathsf{R}\}$. It is easy to verify that each p_f is a proximity for P and p is the proximally coarsest proximity for P proximally finer than each p_f, $f \in \mathscr{F}$. Now the statement is implied by the following lemma:

25 E.7. *Let P be a set and let $\{p_a \mid a \in A\}$ be a family of proximity relations for P. There exists a proximally coarsest proximity p for P proximally finer than each p_a, $a \in A$. If $A \neq \emptyset$ then $X p Y$ if and only if $X \subset P$, $Y \subset P$ and the following condition is fulfilled*:

If $\{X_i\}$ is a finite cover of X and $\{Y_j\}$ is a finite cover of Y, then there exist indices i and j such that $X_i\, p_a Y_j$ for each a in A.

Proof. I. If $A = \emptyset$, then clearly the proximally coarsest proximity p for P has the required property.

II. Suppose that $A \neq \emptyset$ and let us consider the relation $q = \mathsf{E}\{\langle X, Y\rangle \mid a \in A \Rightarrow \Rightarrow X\, p_a\, Y\}$ (by 25 E.5 q need not be a proximity because condition (prox 4) need not be fulfilled). Let p be the relation consisting of all $\langle X, Y\rangle$ such that, for each finite cover $\{X_i\}$ of X and each finite cover $\{Y_j\}$ of Y, there exist indices i and j so that $X_i\, q\, Y_j$. Thus p fulfils the condition in the theorem. We shall prove that p is a proximity for P; conditions (prox 1), (prox 2) and (prox 3) are evident and condition (prox 4) is verified in the following way. Suppose X^k non $p\, Y$, $k = 1, 2$; we shall prove $(X^1 \cup X^2)$ non$p\, Y$. By our assumption there exist finite covers (families!) $\mathscr{X}^k$ of X^k and $\mathscr{Y}^k$ of Y, $k = 1, 2$, such that for each member X' of $\mathscr{X}^k$ and Y' of $\mathscr{Y}^k$ we have X' non $q\, Y'$. Let $\mathscr{Y}$ be a finite cover of Y refining the collection $\mathsf{E}\mathscr{Y}^1 \cup \mathsf{E}\mathscr{Y}^2$ and let $\mathscr{X} = \mathsf{E}\mathscr{X}^1 \cup \mathsf{E}\mathscr{X}^2$ (remember that $\mathsf{E}\mathscr{Y}^i$ denotes the collection of all members of $\mathscr{Y}^i$). Now if $X' \in \mathscr{X}$, $Y' \in \mathscr{Y}$ then X' belongs to $\mathsf{E}\mathscr{X}^1$ or $\mathsf{E}\mathscr{X}^2$, say $\mathsf{E}\mathscr{X}^1$, and Y' is contained in an element Y'' of $\mathsf{E}\mathscr{Y}^1$. By our assumption X' non $q\, Y''$ and therefore also, obviously, X' non $q\, Y'$ which shows that $(X^1 \cup X^2)$ non $p\, Y$.

III. Now let p' be any proximity for P proximally finer than each p_a, $a \in A$; thus $Xp'Y$ implies XqY. Fix subsets X and Y of P such that $Xp'Y$; we shall prove that XpY. This will imply that p' is proximally finer than p. Let $\{X_i\}$ be a finite cover of X and $\{Y_j\}$ be a finite cover of Y. Since $X\, p'\, Y$, by 25 A.3 there exist indices i and j so that $X_i\, p'\, Y_j$ and hence $X_i\, q\, Y_j$. As a consequence $X\, p\, Y$ by the definition of p, which completes the proof.

Now we are prepared to prove two propositions which will imply Theorem 25 E.2. We begin with a sufficient condition for a subset of $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$ to be dense in $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$.

25 E.8. *Suppose that $\mathscr{F}$ is a collection of functions on a proximity space $\mathscr{P} = \langle P, p\rangle$ satisfying the following condition*:

If X non pY and if r is a positive real, then there exists an f in $\mathscr{F}$ such that $0 \leqq fx \leqq r$ for each $x \in P$, $f[X] \subset (0)$, $f[Y] \subset (r)$.

Then for each non-negative bounded proximally continuous function g on $\mathscr{P}$ and each positive real r there exists a finite family $\{f_i\}$ in $\mathscr{F}$ such that $|gx - \Sigma\{f_i x\}| \leqq \leqq r$ for each x in P.

Corollary. *If a linear subspace $\mathscr{F}$ of* $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$ *fulfils the above condition, then $\mathscr{F}$ is dense in the normed space* $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$.

Proof. I. To prove the corollary it is sufficient to notice that $\mathscr{F}$ contains all constant functions. Given an $r > 0$, there exists an f in $\mathscr{F}$ such that $f[\emptyset] \subset (0)$, $f[P] \subset \subset (r)$ (because $\emptyset$ non p P), and hence $fx = r$ for each x in P. — II. Let g be a non-negative bounded proximally continuous function on $\mathscr{P}$ and let $r > 0$. Let k be the smallest positive integer such that $gx \leqq kr$ for each $x \in P$. For each $i \leqq k$ let $X_i = = \mathbf{E}\{x \mid gx \leqq ir\}$. If $1 \leqq i \leqq k$, then the sets X_{i-1} and $P - X_i$ are distant in $\mathscr{P}$ and therefore we can choose an f_i in $\mathscr{F}$ such that $0 \leqq f_i x \leqq r$ for each $x \in P$ and $f_i[X_{i-1}] \subset (0)$, $f_i[P - X_i] \subset (r)$. It is easy to verify that $|gx - \Sigma\{f_i x | 1 \leqq i \leqq k\}| \leqq \leqq r$ for each x in P.

25 E.9. Lemma. *Suppose that $\mathscr{P} = \langle P, p\rangle$ is a proximity space and $\mathscr{F}$ is a sub-lattice-module of* $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$ *containing all constant functions, and projectively generating $\mathscr{P}$. Then for each X* non pY *and each positive real r there exists an f in $\mathscr{F}$ so that f is 0 on X, r on Y and $0 \leqq fx \leqq r$ for each $x \in P$.*

Proof. I. It will suffice to prove that, given X non p Y and $r > 0$, there exist finite families $\{X_i\}$ and $\{Y_j\}$ such that $X = \bigcup\{X_i\}$, $Y = \bigcup\{Y_j\}$, and for each of the indices i and j there exists a required function f_{ij} for X_i and Y_j, i.e. f_{ij} is 0 on X_i, r on Y_j and $0 \leqq f_{ij}x \leqq r$ for each x in P. Indeed, $f = \inf_i \sup_j \{f_{ij}\}$ is then a required function for X and Y.

II. Suppose X non p Y, $X \neq \emptyset \neq Y$ and let f be an element of $\mathscr{F}$ such that the distance from $f[X]$ to $f[Y]$ is positive, say r (such an element need not exist). Choose a finite decomposition $\{X_i\}$ of X and $\{Y_j\}$ of Y such that the diameter of each set $f[X_i]$ as well as each $f[Y_j]$ is less than $\frac{1}{2}r$; this is possible because f is bounded. We may and shall assume that $X_i \neq \emptyset \neq Y_j$ for each i and j. If $x' \in X_i$, $y' \in Y_j$ and $fx' < fy'$, then $fx < fy$ for each x in X_i and $y \in Y_j$; indeed, since the distance from $f[X_i]$ to $f[Y_j]$ is at least that from $f[X]$ to $f[Y]$, i.e. r, and $|fx - fx'| < < \frac{1}{2}r$, $|fy - fy'| < \frac{1}{2}r$, we obtain $fx < (fx' + \frac{1}{2}r) \leqq (fy' - \frac{1}{2}r) < fy$. Similarly, if $fx' > fy'$ for some $x' \in X_i$, $y' \in Y_j$, then $fx > fy$ for each x in X_i and each y in Y_j. If $fx < fy$ for each $x \in X_i$ and $y \in Y_j$, then put

$$h_{ij} = \{z \to \min(fz, \inf f[Y_j])\} : \mathscr{P} \to \mathsf{R},$$
$$g_{ij} = \{z \to \max(h_{ij}z, \sup f[X_i])\} : \mathscr{P} \to \mathsf{R},$$
$$f_{ij} = \{z \to (g_{ij}z - \sup f[X_i])\} : \mathscr{P} \to \mathsf{R}.$$

Clearly the function h_{ij}, and hence g_{ij}, and finally f_{ij} all belong to $\mathscr{F}$, f_{ij} is zero on X_i and dist $(f[X_i], f[Y_j]) \geqq r$. Now given a positive real s, for an appropriate real t, $t \cdot f_{ij}$ is s on Y_j and zero on X_i. Similarly, if $fx > fy$ for $x \in X_i$ and $y \in Y_j$, then the same construction leads to a function $f \in \mathscr{F}$ which is zero on Y_j and s on X_i.

III. Now suppose that X non p Y. Since $\mathscr{F}$ generates $\mathscr{P}$, by 25 E.6 there exist finite decompositions $\{X_i\}$ of X and $\{Y_j\}$ of Y such that for each i and j there exists an f in $\mathscr{F}$ so that the distance from $f[X_i]$ to $f[Y_j]$ is positive. Applying II to each pair

X_i, Y_j we obtain finite decompositions $\{Z_k\}$ of X and $\{T_l\}$ of Y such that for each k and l there exists a function in $\mathscr{F}$ which is zero on Z_k, s on Y_l and its range is contained in the interval $[\![0, s]\!]$. The proof is complete.

25 E.10. Proof of Theorem 25 E.2. Suppose that a proximity space $\mathscr{P}$ is projectively generated by a collection $\mathscr{M}$ of bounded functions, and let $\mathscr{F}$ be the smallest algebra containing $\mathscr{M}$ and the constant function $\{x \to 1\} : \mathscr{P} \to \mathsf{R}$ (and hence all constant functions). Let us consider the closure $\mathscr{G}$ of $\mathscr{F}$ in $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$. Since $\mathscr{M} \subset \subset \mathbf{P}^*(\mathscr{P}, \mathsf{R})$, $(\{x \to 1\} : \mathscr{P} \to \mathsf{R}) \in \mathbf{P}^*(\mathscr{P}, \mathsf{R})$ and $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$ is a closed subalgebra of $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ (by 25 D.2), we have $\mathscr{G} \subset \mathbf{P}^*(\mathscr{P}, \mathsf{R})$. Clearly $\mathscr{G}$ is closed in $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ (the closure structure of $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ is topological!) and $\mathscr{G}$ is an algebra by 19 D.5 because it is the closure of an algebra, namely of $\mathscr{F}$. Since $\mathscr{G}$ is a closed algebra, $\mathscr{G}$ is a lattice (by 19 D.16). Since $\mathbf{P}^*(\mathscr{P}, \mathsf{R}) \supset \mathscr{G} \supset \mathscr{M}$ and $\mathscr{M}$ projectively generates $\mathscr{P}$, $\mathscr{G}$ also projectively generates $\mathscr{P}$, and therefore, by 25 E.9, $\mathscr{G}$ is dense in $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$. Since $\mathscr{G}$ is closed, $\mathscr{G} = \mathbf{P}^*(\mathscr{P}, \mathsf{R})$.

The concluding theorems are intended to clarify the relations between proximities and sets of bounded functions. We shall need the following description of the proximity of bounded subsets of R which is also a corollary of a result of Section 41 on compactness.

25 E.11. Theorem. *A bounded subset X of* R *is proximal to a subset Y of* R *if and only if $\overline{X} \cap \overline{Y} \neq \emptyset$.*

Proof. If $\overline{X} \cap \overline{Y} \neq \emptyset$, then the distance from X to Y is zero and hence the sets X and Y are proximal (without any supposition on X). Conversely, assuming that a bounded set X is proximal to a set Y, i.e. the distance from X to Y is zero, we can take sequences $\{x_n\}$ in X and $\{y_n\}$ in Y such that the sequence $\{|x_n - y_n|\}$ converges to zero. Since X is bounded, some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to a point x (Corollary to 15 B.24). Clearly $x \in \overline{X}$. Since $|x - y_{n_i}| \leqq |x - x_{n_i}| + |x_{n_i} - y_{n_i}|$, the sequence $\{y_{n_i}\}$ also converges to x. Thus $x \in \overline{Y}$ and hence $x \in \overline{X} \cap \overline{Y}$.

25 E.12. Theorem. *Let $\mathscr{P} = \langle P, p \rangle$ be a uniformizable proximity space and let A be a closed linear subspace of* $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ *containing the constant function* $\{x \to 1\} : \mathscr{P} \to \mathsf{R}$. *The following statements are equivalent*:

(a) $A = \mathbf{P}^*(\mathscr{P}, \mathsf{R})$.

(b) *A is a subalgebra of* $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ *(i.e. $g_1, g_2 \in A \Rightarrow g_1 \cdot g_2 \in A$), if $X \, p \, Y$ and $f \in A$, then $\overline{f[X]} \cap \overline{f[Y]} \neq \emptyset$, and if X* non *pY then there exists an f in A, $0 \leqq f \leqq 1$, which is zero on X and one on Y.*

(b′) *A is a sublattice of* $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ *(i.e. $g \in A \Rightarrow |g| \in A$, or equivalently, $g_1, g_2 \in \in A \Rightarrow \sup(g_1, g_2) \in A$, $\inf(g_1, g_2) \in A$), if $X p Y$ and $f \in A$ then $\overline{f[X]} \cap \overline{f[Y]} \neq \emptyset$, and if X* non *pY then there exists an f in A, $0 \leqq f \leqq 1$, which is zero on X and one on Y.*

(c) *A is a subalgebra of* $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ *and $X \, p \, Y$ if and only if $\overline{f[X]} \cap \overline{f[Y]} \neq \emptyset$ for each f in A.*

(c′) *A is a sublattice of* $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ *and* $X\,p\,Y$ *if and only if* $\overline{f[X]} \cap \overline{f[Y]} \neq \emptyset$ *for each* f *in* A.

(d) *A is a subalgebra of* $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ *and A projectively generates* $\mathscr{P}$.

(d′) *A is a sublattice of* $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ *and A projectively generates* $\mathscr{P}$.

Proof. Evidently (b) implies (c), and (b′) implies (c′). By 25 E.11 (c) implies (d), and (c′) implies (d′). Every closed subalgebra is a sublattice (by 19 D.16) and therefore (d) implies (d′). By the proof 25 E.10 of the Stone-Weierstrass Theorem, (d′) implies (a). It remains to show that (a) implies both (b) and (b′). This follows from 25 C.5 and 25 D.2.

Remark. Notice that the preceding theorem states that, under certain assumptions, A is stable under multiplication if and only if A is lattice-stable (compare (b) and (b′) or (c) and (c′) or (d) and (d′)).

A proximity space $\mathscr{P}$ is uniquely determined by any mapping of $\mathscr{P}$, in particular, by any function on $\mathscr{P}$. Indeed, if f is a mapping of $\mathscr{P}$ into any struct, then $\mathscr{P} = \mathbf{D}^*f$. By 25 C.5 a uniformizable proximity space $\mathscr{P} = \langle P, p\rangle$ is uniquely determined by graphs of bounded proximally continuous functions, namely X non p Y if and only if there exists a bounded proximally continuous function f such that $(\mathrm{gr}\, f)\,[X] \subset (0)$ and $(\mathrm{gr}\, f)\,[Y] \subset (1)$. If $\mathscr{F}$ is a class of mappings, then $\mathrm{gr}\,[\mathscr{F}]$ will denote the class of all $\mathrm{gr}\, f$, $f \in \mathscr{F}$. Theorem 25 D.2 states that $\mathrm{gr}\,[\mathbf{P}^*(\mathscr{P}, \mathsf{R})]$ is a closed subalgebra of the normed algebra $\mathrm{gr}\,[\mathbf{F}^*(\mathscr{P}, \mathsf{R})]$ of all bounded real-valued relations on $\mathscr{P}$, and evidently it contains all constant relations. In the converse direction the foregoing results lead to the following important theorem.

25 E.13. Theorem. *Let P be a set and A a closed subalgebra of* $\mathrm{gr}[\mathbf{F}^*(P, \mathsf{R})]$ *containing all constant relations. There exists a unique uniformizable proximity p for P such that* $\mathrm{gr}\,[\mathbf{P}^*(\langle P, p\rangle, \mathsf{R})] = A$. *The proximity p is described by any of conditions* (b), (c) *or* (d) from 25 E.12.

Proof. By 25 E.4 there exists a unique proximity projectively generated by the collection A. The remainder follows from 25 E.12. Another formulation may be in place.

25 E.14. The relation $\{\mathscr{P} \to \mathrm{gr}\,[\mathbf{P}^*(\mathscr{P}, \mathsf{R})] \mid \mathscr{P}$ is a uniformizable proximity space$\}$ is a one-to-one relation ranging on the class of all closed algebras of bounded real-valued relations containing all constant relations.

Of course, by a closed algebra of bounded real-valued relations we mean a closed subalgebra of a normed algebra $\mathrm{gr}\,[\mathbf{F}^*(\mathscr{P}, \mathsf{R})]$, where $\mathscr{P}$ is a struct.

25 E.15. Examples. (a) Let $\mathscr{P}$ be the sum of a non-void family $\{\mathscr{P}_a \mid a \in A\}$ of uniformizable proximity spaces and let B be the set of all bounded functions f on $\mathscr{P}$ such that all functions $f_a = f \circ \mathrm{inj}_a : \mathscr{P}_a \to \mathsf{R}$ are proximally continuous and all f_a except for a finite number of a's are zero-functions. It is easily seen that B projectively generates $\mathscr{P}$. Clearly B is an algebra. By the Stone-Weierstrass Theorem, B is dense in $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$. On the other hand, it is easily seen that $B \neq \mathbf{P}^*(\mathscr{P}, \mathsf{R})$.

(b) Let $\mathcal{Q}$ be a subspace of a uniformizable proximity space $\mathcal{P}$ and let ϱ be the mapping of $\mathbf{P}^*(\mathcal{P}, \mathrm{R})$ into $\mathbf{P}^*(\mathcal{Q}, \mathrm{R})$ which assigns to each f the domain-restriction to $\mathcal{Q}$. It is almost evident that $\mathbf{E}\varrho$ projectively generates $\mathcal{Q}$. Since $\mathbf{E}\varrho$ is an algebra, $\mathbf{E}\varrho$ is dense in $\mathbf{P}^*(\mathcal{P}, \mathrm{R})$ by the Stone-Weierstrass Theorem. In the next subsection we shall show essentially more, namely $\mathbf{E}\varrho = \mathbf{P}^*(\mathcal{P}, \mathrm{R})$, stated in other words, each bounded proximally continuous function on a subspace $\mathcal{Q}$ of a uniformizable proximity space is the domain-restriction of a bounded proximally continuous function on the whole space, i.e., it permits a bounded uniformly continuous extension on the whole space.

25 E.16. Remark. The Stone-Weierstrass Theorem does not hold for complex-valued functions (see ex. 15).

F. EXTENSION OF UNIFORMLY CONTINUOUS PSEUDOMETRICS

The purpose of this subsection is to prove the following two rather profound results.

25 F.1. Theorem. *Let $\mathcal{Q}$ be a subspace of a uniform space $\mathcal{P}$. Every bounded uniformly continuous function on $\mathcal{Q}$ has a uniformly continuous domain-extension on $\mathcal{P}$, i.e. the mapping*

$$\{f \to f|\mathcal{Q}\} : \mathbf{U}^*(\mathcal{P}, \mathrm{R}) \to \mathbf{U}^*(\mathcal{Q}, \mathrm{R})$$

is surjective.

25 F.2. Theorem. *Let $\mathcal{Q}$ be a subspace of a uniform space $\mathcal{P}$. Every bounded uniformly continuous pseudometric for $\mathcal{Q}$ is the relativization of a bounded uniformly continuous pseudometric for $\mathcal{P}$.*

First we shall prove that 25 F.1 implies 25 F.2. The proof will be given in two propositions which follow.

25 F.3. *If d is a uniformly continuous pseudometric for a subspace $\langle Q, \mathcal{V}\rangle$ of a semi-uniform space $\langle P, \mathcal{U}\rangle$ and if there exists a uniformly continuous pseudometric D for $\langle P, \mathcal{U}\rangle$ such that $d\langle x, y\rangle \leqq D\langle x, y\rangle$ for each $\langle x, y\rangle \in Q \times Q$, then there exists a uniformly continuous pseudometric d^* for $\langle P, \mathcal{U}\rangle$ such that d is the relativization of d^*.*

Proof. Let $f\langle x, y\rangle = d\langle x, y\rangle$ if $\langle x, y\rangle \in Q \times Q$ and $f\langle x, y\rangle = D\langle x, y\rangle$ if $\langle x, y\rangle \in ((P \times P) - (Q \times Q))$. It is evident that $f = \{\langle x, y\rangle \to f\langle x, y\rangle \mid \langle x, y\rangle \in P \times P\}$ is a uniformly continuous semi-pseudometric for $\langle P, \mathcal{U}\rangle$. Let d^* be the largest pseudometric for P such that $d^* \leqq f$. Clearly d^* is a uniformly continuous pseudometric $(\mathrm{J} : \langle P, f\rangle \to \langle P, d^*\rangle$ is Lipschitz continuous and hence uniformly continuous) and we shall prove that d is the relativization of d^*. Fix a point $\langle x, y\rangle$ of $Q \times Q$. Since $d^* \leqq f$, we have $d^*\langle x, y\rangle \leqq d\langle x, y\rangle$; to prove the inverse inequality we shall use proposition 18 B.4 which states that $d^*\langle x, y\rangle$ is the infimum of all the numbers

(*) $$\Sigma\{f\langle x_i, x_{i+1}\rangle \mid i \leqq n - 1\},\ n \geqq 1$$

where $\{x_i\}$ varies over all finite chains from x to y, i.e. $x_0 = x$, $x_n = y$. We shall prove that each of the numbers (*) is greater or equal to $d\langle x, y\rangle$. If all the x_i belong to Q, then $f\langle x_i, x_{i+1}\rangle = d\langle x_i, x_{i+1}\rangle$ and the required inequality follows from the triangle inequality for d. If all the $x_1, \ldots, x_{n-1}$ belong to $P - Q$, then $f\langle x_i, x_{i+1}\rangle = D\langle x_i, x_{i+1}\rangle$ and the triangle inequality for D yields that the number (*) is at least $D\langle x, y\rangle \geqq d\langle x, y\rangle$. The general case reduces to the preceding two cases. Indeed, let $\{i_j \mid j \leqq m\}$ be the increasing sequence of those i for which $x_i \in Q$. The sum (*) can be written as follows:

(**) $$\Sigma\{\Sigma\{f\langle x_i, x_{i+1}\rangle \mid i_j \leqq i < i_{j+1}\} \mid j \leqq m - 1\}$$

Each of the numbers $\Sigma\{f\langle x_i, x_{i+1}\rangle \mid i_j \leqq i < i_{j+1}\}$ is at least $d\langle x_{i_j}, x_{i_{j+1}}\rangle$ by the second of the above mentioned particular cases, and hence (**) is at least $d\langle x, y\rangle$ by the first of these cases.

According to 25 F.3, to prove 25 F.2 it is sufficient to verify the following proposition whose proof will be based on 25 F.1.

25 F.4. *If d is a bounded uniformly continuous pseudometric for a subspace $\langle Q, \mathscr{V}\rangle$ of a uniform space $\langle P, \mathscr{U}\rangle$, then there exists a uniformly continuous pseudometric D for $\langle P, \mathscr{U}\rangle$ such that $d\langle x, y\rangle \leqq D\langle x, y\rangle$ for each $\langle x, y\rangle \in Q \times Q$.*

Proof. By 23 D.19 the function $d : \langle Q, \mathscr{V}\rangle \times \langle Q, \mathscr{V}\rangle \to \mathsf{R}$ is uniformly continuous and, of course, bounded. Since $\langle Q, \mathscr{V}\rangle \times \langle Q, \mathscr{V}\rangle$ is a subspace of $\langle P, \mathscr{U}\rangle \times \langle P, \mathscr{U}\rangle$, by 25 F.1 there exists a bounded uniformly continuous function f on $\langle P, \mathscr{U}\rangle \times \langle P, \mathscr{U}\rangle$ the restriction of which to $\langle Q, \mathscr{V}\rangle \times \langle Q, \mathscr{V}\rangle$ is d. For each x and y in P let $D\langle x, y\rangle = \sup\{|f\langle x, z\rangle - f\langle y, z\rangle| \mid z \in P\}$. It is easily seen that $D = \{\langle x, y\rangle \to D\langle x, y\rangle \mid \langle x, y\rangle \in P \times P\}$ is a pseudometric for P. If $\langle x, y\rangle \in Q \times Q$, then $D\langle x, y\rangle \geqq |f\langle x, y\rangle - f\langle y, y\rangle| = d\langle x, y\rangle$. It remains to show that D is uniformly continuous. Let r be a positive real. We must find a U in $\mathscr{U}$ such that $\langle x, y\rangle \in U$ implies $D\langle x, y\rangle \leqq r$. Since f is uniformly continuous in $\langle P, \mathscr{U}\rangle \times \langle P, \mathscr{U}\rangle$ we can choose a U in $\mathscr{U}$ such that $\langle x, y\rangle \in U$, $\langle x', y'\rangle \in U$ implies $|f\langle x, x'\rangle - f\langle y, y'\rangle| \leqq r$. Now if $\langle x, y\rangle \in U$, $|f\langle x, z\rangle - f\langle y, z\rangle| \leqq r$ because $\langle z, z\rangle \in U$, and hence $D\langle x, y\rangle \leqq r$.

It remains to prove Theorem 25 F.1. Its proof will be performed by means of a device introduced by P. Urysohn in his proof of the theorem on continuous extension of functions on normal spaces.

25 F.5. Proof of 25 F.1. Let f_0 be a bounded uniformly continuous function on a subspace $\mathscr{Q}$ of a uniform space $\mathscr{P}$. The required uniformly continuous extension g of f_0 on $\mathscr{P}$ will be given in the form

(1) $$g = \Sigma\{g_n \mid n \in \mathsf{N}\}$$

where the series converges uniformly and g_n are uniformly continuous functions on $\mathscr{P}$.

The sequence $\{g_n\}$ will be constructed together with a sequence $\{f_n\}$ in $\mathbf{U}^*(\mathcal{Q}, \mathrm{R})$ such that

(2) $$f_{n+1} = f_n - g_n \mid \mathcal{Q}, \quad \text{i.e.} \quad f_n - f_{n+1} = g_n \mid \mathcal{Q},$$

and

(3) $$\{f_n x\} \quad \text{converges to zero for each} \quad x \in |\mathcal{Q}|.$$

First we shall show that if such sequences exist, then g is a uniformly continuous extension of f_0. The function g is uniformly continuous as the uniform limit of a sequence, namely $\{\Sigma\{g_n \mid n \leqq k\} \mid k \in \mathrm{N}\}$, of uniformly continuous functions. If $x \in |\mathcal{Q}|$, then (by (2))

$$g_0 x + \ldots + g_n x = (f_0 x - f_1 x) + \ldots + (f_n x - f_{n+1} x) = f_0 x - f_{n+1} x$$

and hence, by (3), $gx = f_0 x$. Existence of $\{f_n\}$ and $\{g_n\}$ is provided by induction. By our assumption there exists a real number K such that $|f_0 x| \leqq K$ for each $x \in |\mathcal{Q}|$. Put

$$r_n = \frac{K}{2}\left(\frac{2}{3}\right)^{n+1}.$$

We shall prove that there exist sequences $\{f_n\}$ in $\mathbf{U}^*(\mathcal{Q}, \mathrm{R})$ and $\{g_n\}$ in $\mathbf{U}^*(\mathcal{P}, \mathrm{R})$ such that (1) holds and

(4) $$|f_n x| \leqq 3r_n \quad \text{for each} \quad x \in |\mathcal{Q}|,$$

(5) $$|g_n x| \leqq r_n \quad \text{for each} \quad x \in |\mathcal{P}|.$$

Clearly (4) implies (3) and (5) implies that the series (1) is uniformly convergent. Evidently $|f_0 x| \leqq K = 3r_0$. The inductive step consists in showing that, given an $f_n \in \mathbf{U}^*(\mathcal{Q}, \mathrm{R})$ satisfying (4), there exists a $g_n \in \mathbf{U}^*(\mathcal{P}, \mathrm{R})$ such that (5) holds and the function f_{n+1} defined by (2) fulfils condition (4) (with n replaced by $n + 1$). Consider the sets

$$X = \{x \mid x \in |\mathcal{Q}|, f_n x \leqq -r_n\}, \quad Y = \{x \mid x \in |\mathcal{Q}|, f_n x \geqq r_n\}.$$

The sets X and Y are proximally distant in $\mathcal{Q}$ and hence in $\mathcal{P}$ (because the proximity induced by the uniform structure of $\mathcal{Q}$ is the relativization of the proximity induced by the uniform structure of $\mathcal{P}$, see 25 A.20). Therefore, by 25 C.5, there exists a proximally continuous function g_n on $\mathcal{P}$ such that (5) holds and g_n is $-r_n$ on X and r_n on Y. By our assumption (4) the values of f_n on X lie between $-3r_n$ and $-r_n$; on Y they lie between r_n and $3r_n$. Thus $|f_{n+1}x| \leqq 2r_n$ for each x in $X \cup Y$. If $x \in |\mathcal{Q}| - (X \cup Y)$, then $|f_{n+1}x| \leqq |f_n x| + |g_n x| \leqq 2r_n$. Thus always $|f_{n+1}x| \leqq 2r_n = 3r_{n+1}$. Since g_n is proximally continuous and bounded, g_n is uniformly continuous by 25 B.21.

25 F.6. Remark. In the exercises it is shown that theorems 25 F.1 and 25 F.2 are not true for unbounded functions and pseudometrics.