Topological spaces

Generation of uniform and proximity spaces (Sections 36-40)

In: Eduard Čech (author); Zdeněk Frolík (author); Miroslav Katětov (author): Topological spaces. (English). Prague: Academia, Publishing House of the Czechoslovak Academy of Sciences, 1966. pp. 679–778.

Persistent URL: http://dml.cz/dmlcz/402632

Terms of use:

© Zdeněk Frolík

© Miroslav Katětov

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

CHAPTER VII

GENERATION OF UNIFORM AND PROXIMITY SPACES

(Sections 36 – 40)

The development of projective and inductive generation for semi-uniform spaces parallels that for closure spaces; for this reason, many details are left to the reader. While the projective and inductive generation for closure spaces were developed separately in two different sections, projective and inductive generation for semi-uniform spaces as well as for proximity spaces will be given a parallel development, so that a certain duality between the projective generation and the inductive generation, which is not formulated precisely, will be pointed out. The last section concerns projective and inductive limits of presheaves of sets, closure spaces, semi-uniform spaces and proximity spaces. The results obtained are not applied to topologized algebraic structs.

Particular attention is given to the interrelations between generations for closure spaces, semi-uniform spaces and proximity spaces. If f is a uniformly continuous mapping, then the transpose of f to a mapping for closure space, denoted by $\gamma_{\mathsf{CU}} f$, is continuous. It turns out that if $\{f_a\}$ is a projective generating family for semi-uniform spaces, then $\{\gamma_{\mathsf{CU}} f_a\}$ is a projective generating family for closure spaces. On the other hand, if $\{f_a\}$ is an inductive generating family for semi-uniform spaces, then the transposed family $\{\gamma_{\mathsf{CU}} f_a\}$ need not be an inductive generating family for closure spaces, and moreover (cf. 37 A.8) every semi-uniform space is a uniform quotient of a discrete uniform space while each quotient of a discrete space is a discrete space. We have known that the transpose of a uniformly continuous mapping f to a mapping for proximity spaces, denoted by $\gamma_{\mathsf{PU}} f$, is proximally continuous. It turns out that if $\{f_a\}$ is an inductive generating family for semi-uniform spaces then the family $\{\gamma_{\mathsf{PU}} f_a\}$ is an inductive generating family for proximity spaces; on the other hand, if $\{f_a\}$ is a projective generating family for semi-uniform spaces then the transposed family $\{\gamma_{\mathsf{PU}} f_a\}$ need not be a projective generating family for proximity spaces (e.g. if $\{\mathscr{U}_a\}$ is a family of semi-uniformities for a set P, each inducing a proximity p, then $\{\mathsf{J} : \langle P, \inf \{\mathscr{U}_a\}\rangle \to \langle P, \mathscr{U}_a\rangle$ is a projective generating family for semi-uniform spaces but $\inf \{\mathscr{U}_a\}$ need not induce p, and therefore the induced family need not be a projective generating family for proximity spaces).

36. ORDERED SETS OF SEMI-UNIFORMITIES

The present section is devoted to an investigation of the class **U** of all semi-uniformities ordered by the relation $\{\mathscr{U} \to \mathscr{V} \mid \mathscr{U}$ is uniformly finer than $\mathscr{V}\}$. The section follows the same pattern as 31 A, B and similarly it is preparatory in character. The results obtained will be applied to projective and inductive generation for semi-uniform spaces (37).

It may be appropriate to recall that semi-uniformities were first studied in 23, followed by the study of uniformities in 24, proximally coarse semi-uniformities in 25 and the introduction of some extreme semi-uniformities (the Čech uniformity of a space and the fine uniformity of a space in 28, the Wallman proximity of a space in 29); besides in a few places some other special semi-uniformities were considered.

For convenience we shall review earlier material and often try to make the older notation, terminology and results more clear and more precise.

In subsection A we shall prove that **U** is boundedly order-complete, and the canonical mapping of **U** into **C** (which assigns to each $\mathscr{U}$ the closure operation induced by $\mathscr{U}$) is completely lattice-preserving. The proof is based on a description of suprema and infima in **U**. In this connection the definition of fine semi-uniformities and coarse semi-uniformities are introduced.

Subsection B concerns the ordered class $\mathfrak{v}$**U** of all uniformities. We shall prove that $\mathfrak{v}$**U** is completely meet-stable in **U** (but not completely join-stable), and the canonical mapping of $\mathfrak{v}$**U** into $\mathfrak{v}$**C** is completely meet-preserving (but not join-preserving). We shall introduce the definition of a coarse uniformity (a fine uniformity was defined in 28) although we are not able to prove anything about it here. Coarse uniformities will be studied in 41 D.

A. ORDERED CLASS **U**

By definition 23 A.3, a semi-uniformity for a set P is a filter $\mathscr{U}$ on $P \times P$ each element of which contains the diagonal and if $U \in \mathscr{U}$ then also $U^{-1} \in \mathscr{U}$. By definition 23 C.1, a mapping f of a semi-uniform space $\langle P, \mathscr{U} \rangle$ into another one $\langle Q, \mathscr{V} \rangle$ is uniformly continuous if $(f \times f)^{-1}[V] \in \mathscr{U}$ for each V in $\mathscr{V}$. If $\mathscr{V}_0$ is a sub-base for $\mathscr{V}$ (i.e. a sub-base for the filter $\mathscr{V}$), then $(f \times f)^{-1}[V] \in \mathscr{U}$ for each V in $\mathscr{V}_0$ implies that $(f \times f)^{-1}[V] \in \mathscr{U}$ for each V in $\mathscr{V}$, and hence f is uniformly continuous

(23 C.2); this very important criterion of uniform continuity will be used without any reference. A semi-uniformity $\mathscr{U}$ is uniformly finer than a semi-uniformity $\mathscr{V}$, and $\mathscr{V}$ is uniformly coarser than $\mathscr{U}$, if both $\mathscr{U}$ and $\mathscr{V}$ are for the same set, say P, and the mapping $\mathsf{J} : \langle P, \mathscr{U}\rangle \to \langle P, \mathscr{V}\rangle$ is uniformly continuous, i.e. if $\mathscr{V} \subset \mathscr{U}$. The relation $\{\mathscr{U} \to \mathscr{V} \mid \mathscr{V}$ is uniformly coarser than $\mathscr{U}\}$ is an order for the class $\mathbf{U}$ of all semi-uniformities. We shall use the "upward" terminology, e.g. we shall speak about lower bounds. If P is a set then the symbol $\mathbf{U}(P)$ denotes the ordered subset of $\mathbf{U}$ consisting of all semi-uniformities for P (23 C.11). The letter $\mathbf{U}$ also denotes the class of all semi-uniform spaces ordered by the relation $\{\mathscr{P} \to \mathscr{Q} \mid$ the uniform structure of $\mathscr{P}$ is uniformly finer than that of $\mathscr{Q}\}$ (23 C.11).

36 A.1. Theorem. *Let P be a set. The ordered set $\mathbf{U}(P)$ of all semi-uniformities for P is order-complete; the filter $(P \times P)$ (consisting of exactly one element, $P \times P$) is the uniformly coarsest semi-uniformity for P (i.e., the greatest element of $\mathbf{U}(P)$), and (J_P) is a base for the uniformly finest semi-uniformity for P (i.e., the least element of $\mathbf{U}(P)$). If $\{\mathscr{U}_a \mid a \in A\}$ is a non-void family in $\mathbf{U}(P)$, then*

(*) $\sup \{\mathscr{U}_a \mid a \in A\} = \bigcap\{\mathscr{U}_a \mid a \in A\}$

and

(**) $\bigcup\{\mathscr{U}_a \mid a \in A\}$ *is a sub-base for* $\inf \{\mathscr{U}_a \mid a \in A\}$.

Proof. I. Obviously $(P \times P)$ is a semi-uniformity for P which is contained in each semi-uniformity for P, and hence it is the uniformly coarsest semi-uniformity for P (note that $(P \times P)$ is a filter on $P \times P$ contained in each filter in $P \times P$). Next, (J_P) is a base for a semi-uniformity for P which contains each filter in $P \times P$ whose intersection contains J_P and hence (J_P) is a base for the uniformly finest uniformity for P.

II. According to I, to prove that $\mathbf{U}(P)$ is order-complete it is sufficient to show, for instance, that each non-void family $\{\mathscr{U}_a\}$ in $\mathbf{U}(P)$ possesses a least upper bound. We shall prove that $\mathscr{U} = \bigcap\{\mathscr{U}_a\}$ is the least upper bound of a non-void family $\{\mathscr{U}_a \mid a \in A\}$. First we must show that $\mathscr{U}$ is a semi-uniformity. Since each $\mathscr{U}_a$ is a filter, necessarily $\mathscr{U}$ is a filter, and $\bigcap\mathscr{U}_a \supset \mathsf{J}_P$ for each a implies that $\bigcap\mathscr{U} \supset \mathsf{J}_P$. Finally, if $U \in \mathscr{U}$, then $U \in \mathscr{U}_a$ for each a in A, and hence, $\mathscr{U}_a$ being semi-uniformities, $U^{-1} \in \mathscr{U}_a$ for each a; it follows that $U^{-1} \in \mathscr{U}$. Thus $\mathscr{U}$ is indeed a semi-uniformity. Since $\mathscr{U} \subset \mathscr{U}_a$ for each a, $\mathscr{U}$ is uniformly coarser than each $\mathscr{U}_a$, i.e. $\mathscr{U}$ is an upper bound of $\{\mathscr{U}_a\}$. If $\mathscr{V}$ is any upper bound of $\{\mathscr{U}_a\}$, then $\mathscr{V} \subset \mathscr{U}_a$ for each a in A and hence $\mathscr{V} \subset \bigcap\{\mathscr{U}_a\} = \mathscr{U}$, which shows that $\mathscr{V}$ is uniformly coarser than $\mathscr{U}$. Thus $\mathscr{U}$ is the least upper bound, which concludes the proof of order-completeness of $\mathbf{U}(P)$ and of formula (*).

III. It remains to verify (**). By the corollary to 23 A.4, the union $\mathscr{U}_0$ of the family $\{\mathscr{U}_a\}$ is a sub-base of a semi-uniformity $\mathscr{U}$. Since $\mathscr{U}_a \subset \mathscr{U}_0 \subset \mathscr{U}$ for each a in A, $\mathscr{U}$ is a lower bound of $\{\mathscr{U}_a\}$. If $\mathscr{V}$ is any lower bound of $\{\mathscr{U}_a\}$, then $\mathscr{V}$ contains each $\mathscr{U}_a$, and hence, also the union $\mathscr{U}_0$ of $\mathscr{U}$. Therefore, since $\mathscr{U}_0$ is a sub-base of $\mathscr{U}$, $\mathscr{U}$ is uniformly coarser than $\mathscr{V}$; this shows that $\mathscr{U}$ is the greatest lower bound of $\{\mathscr{U}_a\}$.

Remark. Sometimes it is convenient to know that the collection $\bigcap\{\mathscr{U}_a\}$, where $\mathscr{U}_a$ are semi-uniformities for a set P, consists of all sets of the form $\bigcup\{U_a\}$, where $U_a \in \mathscr{U}_a$ for each a. This is an immediate consequence of the fact that $U \in \mathscr{U}_a$, $U \subset V \subset P \times P$ imply $V \in \mathscr{U}_a$.

Recall that each semi-uniformity $\mathscr{U}$ for a set P induces a closure operation u such that $[\mathscr{U}][x]$ is the neighborhood system at x in $\langle P, u\rangle$ for each x in P (23 A.3); we have

$$uX = \bigcap[\mathscr{U}][X] \,(= \bigcap\{U[X] \mid U \in \mathscr{U}\})$$

for each $X \subset P$ (by 23 B.5). By 23 B.3 a closure operation u is semi-uniformizable (i.e. it is induced by a semi-uniformity) if and only if $x \in u(y)$ implies $y \in u(x)$.

36 A.2. Definition. The symbol $\gamma_{\mathbf{CU}}$, abbreviated to γ, denotes the single-valued relation which assigns to each semi-uniformity $\mathscr{U}$ the closure induced by $\mathscr{U}$. The mapping $\gamma : \mathbf{U} \to \mathbf{C}$ is called the canonical mapping of $\mathbf{U}$ into $\mathbf{C}$, and if P is a set then the mapping $\gamma : \mathbf{U}(P) \to \mathbf{C}(P)$ is called the canonical mapping of $\mathbf{U}(P)$ into $\mathbf{C}(P)$. If u is a closure operation for a set P, then $\mathbf{U}(P, u)$ denotes the set of all semi-uniformities which induce u, i.e. $\mathbf{U}(P, u) = \gamma^{-1}[(u)]$. Thus u is semi-uniformizable if and only if $\mathbf{U}(P, u) \neq \emptyset$. The symbol $\gamma_{\mathbf{CU}}$ also denotes the relation $\{\langle P, \mathscr{U}\rangle \to \to \langle P, \gamma\mathscr{U}\rangle \mid \mathscr{U} \in \mathbf{U}\}$.

By 23 C.7, if $f : \mathscr{P} \to \mathscr{Q}$ is a uniformly continuous mapping then $f : \gamma\mathscr{P} \to \gamma\mathscr{Q}$ is a continuous mapping; in particular, if $\mathscr{U}$ is uniformly finer than $\mathscr{V}$, then $\gamma\mathscr{U}$ is finer than $\gamma\mathscr{V}$. It follows that the mapping $\gamma : \mathbf{U} \to \mathbf{C}$ is order-preserving. Now we shall prove essentially more.

36 A.3. Theorem. *The canonical mapping of* $\mathbf{U}$ *into* $\mathbf{C}$ *is completely lattice-preserving. In particular, if P is a set, then the canonical mapping of* $\mathbf{U}(P)$ *into* $\mathbf{C}(P)$ *is completely lattice-preserving; moreover*

(*) $\gamma \inf\{\mathscr{U}_a\} = \inf\{\gamma\mathscr{U}_a\}$

(**) $\gamma \sup\{\mathscr{U}_a\} = \sup\{\gamma\mathscr{U}_a\}$

for each family $\{\mathscr{U}_a\}$ *in* $\mathbf{U}(P)$ *(not necessarily non-void).*

Proof. Clearly it is sufficient to prove formulae (*) and (**). If $\{\mathscr{U}_a\}$ is empty, then $\inf\{\mathscr{U}_a\} = (P \times P)$, $\inf\{\gamma\mathscr{U}_a\}$ is the accrete closure for P and the accrete closure for P is the greatest lower bound of each empty family in $\mathbf{C}(P)$, in particular, of $\{\gamma\mathscr{U}_a\}$. Similarly, if $\{\mathscr{U}_a\}$ is empty, then $\sup\{\mathscr{U}_a\}$ and $\sup\{\gamma\mathscr{U}_a\}$ are the finest elements of $\mathbf{U}(P)$ and $\mathbf{C}(P)$ respectively, and the finest element of $\mathbf{U}(P)$ induces the finest closure for P. — I. Now let the index set be non-void. By 36 A.1 the union $\mathscr{U}$ of $\{\mathscr{U}_a\}$ is a sub-base for $\inf\{\mathscr{U}_a\}$, and hence $[\mathscr{U}][x]$ is a local sub-base at x in $\langle P, \gamma \inf\{\mathscr{U}_a\}\rangle$ for each x in P. On the other hand, $[\mathscr{U}_a][x]$ being a local base at x in $\langle P, \gamma\mathscr{U}_a\rangle$ for each x in P and a in A, by 31 A.5 the collection $\mathscr{V}_x = \bigcup\{[\mathscr{U}_a][x] \mid a \in A\}$ is a local sub-base at x in $\langle P, \inf\{\gamma\mathscr{U}_a\}\rangle$. But obviously $\mathscr{V}_x = [\mathscr{U}][x]$ for each x in P, which establishes (*). The proof of (**) is quite similar. By 36 A.1 the intersection of $\{\mathscr{U}_a\}$ is $\sup\{\mathscr{U}_a\}$ and hence the collection $[\bigcap\{\mathscr{U}_a\}][x]$ is the neighborhood system at x in $\gamma \sup\{\mathscr{U}_a\}$.

On the other hand, $[\mathcal{U}_a][x]$ being the neighborhood system at x in $\langle P, \mathcal{U}_a \rangle$, by 31 A.4 the collection $\bigcap\{[\mathcal{U}_a][x] \mid a \in A\}$ is the neighborhood system at x in $\langle P, \sup \gamma\mathcal{U}_a\} \rangle$. But obviously

$$[\bigcap\{\mathcal{U}_a\}][x] = \bigcap\{[\mathcal{U}_a][x] \mid a \in A\}$$

for each x, which establishes (**). — II. An alternate proof of formula (**) may also be obtained from the following description of induced closures: $\gamma\mathcal{U}X = \bigcap[\mathcal{U}][X]$.

Corollary (a). *If $\langle P, u \rangle$ is a semi-uniformizable space, that is $\mathbf{U}(P, u) \neq \emptyset$, then $\mathbf{U}(P, u)$ is order-complete, in particular, there exist the uniformly finest and coarsest semi-uniformities inducing the closure u.*

Proof. If $\{\mathcal{U}_a\}$ is a non-void family in $\mathbf{U}(P, u)$, then $\sup\{\mathcal{U}_a\}$ and $\inf\{\mathcal{U}_a\}$, both taken in $\mathbf{U}(P)$, belong to $\mathbf{U}(P, u)$ by the theorem, and hence they coincide with $\sup\{\mathcal{U}_a\}$ and $\inf\{\mathcal{U}_a\}$ (taken in $\mathbf{U}(P, u)$).

It may be noted that we proved essentially more than Corollary (a), namely

Corollary (b). *If $\{\mathcal{U}_a\}$ is a non-void family in $\mathbf{U}(P, u)$ then the greatest lower bounds (least upper bounds) taken in $\mathbf{U}(P)$ and $\mathbf{U}(P, u)$ coincide, i.e. $\mathbf{U}(P, u)$ is completely lattice-preserving in $\mathbf{U}(P)$.*

Remark. Notice that $\mathbf{U}(P, u)$ is a closed interval in $\mathbf{U}(P)$.

Corollary (c). *If u is any closure for P then there exists a finest semi-uniformizable closure coarser than u as well as a coarsest semi-uniformizable closure finer than u. In other words, for each u in $\mathbf{C}(P)$ there exist both upper and lower modifications of u in the set $\gamma\mathbf{U}(P)$ of all semi-uniformizable closures for P.*

Proof. Let $\mathcal{U}_1 = \inf\{\mathcal{V} \mid \gamma\mathcal{V}$ is coarser than $u\}$, and $\mathcal{U}_2 = \sup\{\mathcal{V} \mid \gamma\mathcal{V}$ is finer than $u\}$. By the theorem the closures $\gamma\mathcal{U}_1$ and $\gamma\mathcal{U}_2$ possess the required properties.

Remark. Notice that $\mathbf{U}$ is an ordered subclass of the class of all sets ordered by $\supset$; by 36 A.1 $\mathbf{U}$ is completely join-preserving (but not meet-preserving) in this ordered class.

36 A.4. Definition. A semi-uniformity $\mathcal{U}$ for a set P is said to be *fine* or to be *coarse* if $\mathcal{U}$ is, respectively, the uniformly finest or coarsest semi-uniformity from $\mathbf{U}(P, \gamma\mathcal{U})$, that is, if $\mathcal{U}$ is the uniformly finest or coarsest semi-uniformity inducing the closure $\gamma\mathcal{U}$.

It follows from Corollary (a) that, for each semi-uniformity $\mathcal{U}$, there exists exactly one fine semi-uniformity and exactly one coarse semi-uniformity inducing the same closure as $\mathcal{U}$. Now we shall describe them directly.

36 A.5. *Let $\mathcal{U}$ be a semi-uniformity for a set P. The coarse semi-uniformity $\mathcal{U}_c$ inducing the same closure as $\mathcal{U}$ consists of all relations $U \cup ((P - X) \times (P - X))$, where $U \in \mathcal{U}$ and $X \subset P$ is finite. The fine semi-uniformity $\mathcal{U}_f$ inducing the same closure as $\mathcal{U}$ consists of all $V \subset P \times P$ such that, for each x in P, there exists a U_x in $\mathcal{U}$ such that $U_x[x] \subset (V \cap V^{-1})[x]$.*

Proof. I. It is easily seen that $\mathcal{U}_c$ and $\mathcal{U}_f$ are semi-uniformities inducing the same closure as $\mathcal{U}$. — II. Let $\mathcal{V}$ be any semi-uniformity inducing the same closure as $\mathcal{U}$.

We must show that $\mathscr{U}_c \subset \mathscr{V} \subset \mathscr{U}_f$. The inclusion $\mathscr{V} \subset \mathscr{U}_f$ if almost self-evident. Indeed, if $V \in \mathscr{V}$, then we can choose a family $\{\mathscr{U}_x \mid x \in P\}$ in $\mathscr{U}$ such that $U_x[x] \subset \subset (V \cap V^{-1})[x]$ for each $x \in P$ (because $\mathscr{V}$ induces the same closure as $\mathscr{U}$), and hence $V \in \mathscr{U}_f$. To prove $\mathscr{U}_c \subset \mathscr{V}$, suppose that U is any symmetric element of $\mathscr{U}$ and X is any finite subset of P (thus $U_1 = U \cup ((P - X) \times (P - X))$ is symmetric); it is to be proved that $U_1 \in \mathscr{V}$. If $X = \emptyset$ then $U_1 = P \times P$ and hence $U_1 \in \mathscr{V}$. Assuming $X \neq \emptyset$ we shall find an element V of $\mathscr{V}$ such that $V \subset U_1$. Since $\mathscr{U}$ induces the same closure as $\mathscr{V}$ we can choose a family $\{V_x \mid x \in X\}$ in $\mathscr{V}$ such that $V_x[x] \subset U[x]$ for each x in X. We may and shall assume that each V_x is symmetric. Since X is finite, the intersection V of $\{V_x \mid x \in X\}$ belongs to $\mathscr{V}$ and obviously V is symmetric. We shall show that $V \subset U_1$. If $x \in X$ then $V[x] \subset V_x[x] \subset U[x]$. If $x \in (P - X)$ and $y \in \in V[x] - (P - X)$, then $y \in X$, $\langle y, x \rangle \in V$ (V is symmetric) and hence $\langle y, x \rangle \in U$; since U is symmetric, $\langle x, y \rangle \in U$ as well. Thus $V[x] \subset U_1[x]$ for each x, which proves that $V \subset U_1$.

Corollary (a). *If P is a semi-uniformizable space, then the fine semi-uniformity of P consists of all semi-neighborhoods of the diagonal of the product space $P \times P$, and the coarse semi-uniformity of P consists of all semi-neighborhoods of the diagonal containing a set of the form $(P - X) \times (P - X)$ where X is a finite subset of P.*

Proof. Recall that a semi-neighborhood of the diagonal of the product space $P \times P$ is a subset U of $P \times P$ such that $(U \cap U^{-1})[x]$ is a neighborhood of x for each x in P (or equivalently, if U is a neighborhood of the diagonal in ind $(P \times P)$). Notice that if $\mathscr{U}$ is a semi-uniformity inducing the closure of P, then U is a semi-neighborhood of the diagonal if and only if for each x in P there exists a U_x in $\mathscr{U}$ such that $U_x[x] \subset (U \cap U^{-1})[x]$ and then apply the theorem.

Corollary (b). *The uniformly finest coarse semi-uniformity for a set P consists of all sets of the form*

$$\mathrm{J}_P \cup ((P - X) \times (P - X))$$

where X varies over all finite subsets X of P. This semi-uniformity induces the discrete closure. (Notice that this semi-uniformity is a uniformity.)

Corollary (c). *If $\mathscr{U}$ is any semi-uniformity for a set P then the coarse semi-uniformity $\mathscr{U}$ inducing the same closure as $\mathscr{U}$ is the intersection (i.e. the least upper bound) of $\mathscr{U}$ and of the uniformly finest coarse semi-uniformity for P (i.e. the discrete coarse semi-uniformity for the set P).*

Corollary (d). *A semi-uniformity $\mathscr{U}$ for a set P is coarse if and only if each element of $\mathscr{U}$ contains a set of the form $(P - X) \times (P - X)$ with X finite.*

Proof. Let $\mathscr{U}_c$ be the coarse semi-uniformity inducing the same closure as $\mathscr{U}$. By 36 A.5 the condition is necessary and sufficient for $\mathscr{U}_c = \mathscr{U}$.

Let us recall that a closure space $\langle P, u \rangle$ is said to be quasi-discrete if $uX = \bigcup\{u(x) \mid x \in X\}$ for each subset X of P. By our convention a semi-uniform space $\langle P, \mathscr{U} \rangle$ is said to be *quasi-discrete* if the induced space $\langle P, \gamma\mathscr{U} \rangle$ is quasi-discrete.

36 A.6. Definition. A semi-uniformity $\mathcal{U}$ for a set P will be called *uniformly quasi-discrete* if $\mathcal{U}$ possesses a base consisting of one element, that is, if $\bigcap \mathcal{U}$ belongs to $\mathcal{U}$.

For example, the uniformly finest and the uniformly coarsest semi-uniformities for a set P are uniformly quasi-discrete.

36 A.7. *Every uniformly quasi-discrete semi-uniformity is quasi-discrete. If P is a quasi-discrete semi-uniformizable space then the fine semi-uniformity of P is the only uniformly quasi-discrete semi-uniformity inducing the closure structure of P.*

Proof. Recall that by Theorem 26 A.9 a closure space is quasi-discrete if and only if at each point there exists a one-element local base. Now the first statement is obvious and the second one is proved as follows: let P be a quasi-discrete semi-uniformizable space and let U_x be the smallest neighborhood of x in P for each x in P; the set $\Sigma \{U_x \mid x \in P\}$ is evidently the smallest element of the fine semi-uniformity of P, and clearly the fine semi-uniformity for P is the only uniformly quasi-discrete semi-uniformity unducing the closure structure of P.

Corollary *A semi-uniformity $\mathcal{U}$ is a uniformly quasi-discrete uniformity if and only if $\bigcap \mathcal{U}$ is an equivalence belonging to $\mathcal{U}$.*

36 A.8. Theorem. *Every semi-uniformity $\mathcal{U}$ for a set P is the greatest lower bound of a family of uniformly quasi-discrete semi-uniformities for P; furthermore, if $\mathcal{B}$ is a sub-base for $\mathcal{U}$ and, for each B in $\mathcal{B}$, $\mathcal{U}_B$ is the semi-uniformity $\mathbf{E}\{V \mid B \cap B^{-1} \subset V \subset P \times P\}$, then $\mathcal{U} = \inf \{\mathcal{U}_B \mid B \in \mathcal{B}\}$ and if $\mathcal{B}$ is a base, then $\mathcal{U} = \bigcup \{\mathcal{U}_B \mid B \in \mathcal{B}\}$.* – Obvious.

Remark. A uniformity need not be the greatest lower bound of a family of uniformly quasi-discrete uniformities. One can prove that a uniformity $\mathcal{U}$ is the greatest lower bound of uniformly quasi-discrete uniformities if and only if $\mathcal{U}$ is uniformly totally disconnected, that is, if the equivalences form a base for $\mathcal{U}$.

36 A.9. Theorem. *If $\{f : \langle P, \mathcal{U}_a \rangle \to \langle Q, \mathcal{V}_a \rangle\}$ is a family of uniformly continuous mappings, then the mappings $f : \langle P, \sup \{\mathcal{U}_a\} \rangle \to \langle Q, \sup \{\mathcal{V}_a\} \rangle$ and $f : \langle P, \inf \{\mathcal{U}_a\} \rangle \to \langle Q, \inf \{\mathcal{V}_a\} \rangle$ are also uniformly continuous.*

B. UNIFORMITIES

Recall that a uniformity is a semi-uniformity $\mathcal{U}$ such that each element of $\mathcal{U}$ contains $U \circ U$ for some U in $\mathcal{U}$. By 24 B.2, for each semi-uniformity $\mathcal{U}$ there exists a uniformly finest uniformity uniformly coarser than $\mathcal{U}$; this uniformity is called the uniform modification of $\mathcal{U}$ (24 B.1). Using this result we shall derive some results concerning the ordered set of all uniformities. First we shall introduce some terminology and notation.

36 B.1. Definition. Denote by $\mathfrak{v}\mathbf{U}$ the ordered subclass of $\mathbf{U}$ which consists of all uniformities. Let $\mathfrak{v}$ be the single-valued relation which assigns to each semi-uniformity $\mathcal{U}$ the uniform modification of $\mathcal{U}$; this relation is called the *uniform modification.* If P is a set then $\mathfrak{v}\mathbf{U}(P)$ denotes the ordered set $\mathfrak{v}\mathbf{U} \cap \mathbf{U}(P)$. If $\mathscr{P}$ is a closure space then $\mathfrak{v}\mathbf{U}(\mathscr{P})$ denotes the ordered set $\mathfrak{v}\mathbf{U} \cap \mathbf{U}(\mathscr{P})$ (thus $\mathfrak{v}\mathbf{U}(\mathscr{P})$ may be void, and it is non-void if and only if $\mathscr{P}$ is uniformizable). The letter $\mathfrak{v}$ also denotes the relation $\{\langle P, \mathcal{U}\rangle \rightarrow \langle P, \mathfrak{v}\mathcal{U}\rangle\}$, and hence we can write $\mathfrak{v}\langle P, \mathcal{U}\rangle = \langle P, \mathfrak{v}\mathcal{U}\rangle$. We write $\mathbf{U}(P, u)$ instead of $\mathbf{U}(\langle P, u\rangle)$. The subscripts $\mathbf{U}$, $\mathbf{C}$ or $\mathbf{P}$ to $\mathfrak{v}$ will be used to specify the notation for semi-uniform, closure or proximity spaces, respectively.

It is evident that the uniform modification of a semi-uniformity $\mathcal{U}$ is the upper modification of $\mathcal{U}$ in the ordered class $\mathfrak{v}\mathbf{U}$; thus Lemma 31 B.2 applies and we obtain the following theorem.

36 B.2. Theorem. *Let P be a set. The ordered set $\mathfrak{v}\mathbf{U}(P)$ is completely meet-stable and completely meet-preserving in $\mathbf{U}(P)$; the uniformly finest and uniformly coarsest semi-uniformities for P are uniformities, and hence they are the uniformly finest and uniformly coarsest uniformities for P. Furthermore, $\mathfrak{v}\mathbf{U}(P)$ is order-complete and $\mathfrak{v}\mathbf{U}(P) = \mathfrak{v}[\mathbf{U}(P)]$. The mapping $\mathfrak{v} : \mathbf{U}(P) \rightarrow \mathfrak{v}\mathbf{U}(P)$ is completely meet-preserving, $\mathfrak{v} \circ \mathfrak{v} = \mathfrak{v}$ and $\mathfrak{v}\mathcal{U}$ is uniformly coarser than $\mathcal{U}$ for each $\mathcal{U}$ in $\mathbf{U}(P)$. If $\{\mathcal{U}_a\}$ is any family in $\mathbf{U}(P)$ then*

$$\mathfrak{v} \sup \{\mathcal{U}_a\} = \mathfrak{v} \sup \{\mathfrak{v}\mathcal{U}_a\} = \sup \{\mathfrak{v}\mathcal{U}_a\}$$

where the last supremum is taken in $\mathfrak{v}\mathbf{U}(P)$.

Corollary. *The class $\mathfrak{v}\mathbf{U}$ is completely meet-stable and completely meet-preserving in $\mathbf{U}$, and $\mathfrak{v}\mathbf{U} = \mathbf{E}\mathfrak{v}$. The class $\mathfrak{v}\mathbf{U}$ is boundedly order-complete and contains each uniformly discrete or uniformly accrete semi-uniformity. We have $\mathfrak{v} \circ \mathfrak{v} = \mathfrak{v}$, $\mathfrak{v}\mathcal{U}$ is uniformly coarser than $\mathcal{U}$ for each $\mathcal{U}$ in $\mathbf{U}$ and the mapping $\mathfrak{v} : \mathbf{U} \rightarrow \mathfrak{v}\mathbf{U}$ is surjective and completely meet-preserving.*

Proof. As noted above, for each $\mathcal{U}$ in $\mathbf{U}(P)$ there exists an upper modification of $\mathcal{U}$ in $\mathfrak{v}\mathbf{U}(P)$. Since $\mathbf{U}(P)$ is order-complete, from lemma 31 B.2 we obtain all the statements of the theorem with the exception of one, namely that the finest semi-uniformity is a uniformity; but this is obvious because (J_P) is its base and $\mathsf{J}_P \circ \mathsf{J}_P = \mathsf{J}_P$.

According to 36 B.2 the greatest lower bounds of a family $\{\mathcal{U}_a\}$ of uniformities taken in $\mathfrak{v}\mathbf{U}(P)$ and in $\mathbf{U}(P)$ coincide. By virtue of 36 A.1 we obtain that $\bigcup\{\mathcal{U}_a\}$ is a sub-base for inf $\{\mathcal{U}_a\}$ taken in $\mathfrak{v}\mathbf{U}(P)$. On the other hand, sup $\{\mathcal{U}_a\}$ taken in $\mathbf{U}(P)$ is $\bigcap\{\mathcal{U}_a\}$ by 36 A.1, and it is easily seen that $\bigcap\{\mathcal{U}_a\}$ need not be a uniformity (see 24 ex. 6); therefore $\bigcap\{\mathcal{U}_a\}$ is not sup $\{\mathcal{U}_a\}$ taken in $\mathfrak{v}\mathbf{U}(P)$. Of course, by the foregoing theorem, $\mathfrak{v}(\bigcap\{\mathcal{U}_a\})$ is sup $\{\mathcal{U}_a\}$ taken in $\mathfrak{v}\mathbf{U}(P)$. Thus we have proved

36 B.3. *Let $\{\mathcal{U}_a\}$ be a family in $\mathfrak{v}\mathbf{U}(P)$. The union of $\{\mathcal{U}_a\}$ is a sub-base for the greatest lower bound of $\{\mathcal{U}_a\}$ in $\mathfrak{v}\mathbf{U}(P)$, and $\mathfrak{v}(\bigcap\{\mathcal{U}_a\})$ is the least upper bound of $\{\mathcal{U}_a\}$ in $\mathfrak{v}\mathbf{U}(P)$.*

The uniformly finest uniformity uniformly coarser than each member of a given family of uniformities can be described simply in terms of a uniform collection of pseudometrics. Recall that (by 24 B.8) a uniform collection of pseudometrics for a set P is a non-void collection $\mathscr{M}$ of pseudometrics for the set P such that each pseudometric d for P belongs to $\mathscr{M}$ whenever either $d = d_1 + d_2$ with d_1 and d_2 in $\mathscr{M}$ or for each positive real r there exists a d_1 in $\mathscr{M}$ and a positive real s such that $d_1(x, y) < s$ implies $d(x, y) < r$. For each $\mathscr{U}$ in $\mathbf{U}(P)$ the collection $\boldsymbol{\mu}\mathscr{U}$ of all uniformly continuous pseudometrics for $\langle P, \mathscr{U}\rangle$ is a uniform collection of pseudometrics.

36 B.4. Theorem. *Let* $\mathbf{M}$ *be the class of all uniform collections of pseudometrics ordered by the inverse inclusion* $\supset$ *and let* $\boldsymbol{\mu}$ *be the single-valued relation which assigns to each* $\mathscr{U} \in \mathbf{U}$ *the collection of all uniformly continuous pseudometrics (with respect to* $\mathscr{U}$*). Then* $\boldsymbol{\mu}\mathscr{U} \in \mathbf{M}$ *for each* $\mathscr{U}$ *in* $\mathbf{U}$*, the mapping* $\boldsymbol{\mu} : \mathbf{U} \to \mathbf{M}$ *is completely join-preserving (in particular, order-preserving) and surjective, the mapping* $\boldsymbol{\mu} : \mathfrak{v}\mathbf{U} \to \mathbf{M}$ *is an order-isomorphism, in particular,* $\mathbf{M}$ *is boundedly order-complete. For each* $\mathscr{M}$ *in* $\mathbf{M}$ *the set* $\boldsymbol{\mu}^{-1}[(\mathscr{M})]$ *contains exactly one uniformity* $\mathscr{U}$*, and* $\boldsymbol{\mu}^{-1}[(\mathscr{M})]$ *consists of those semi-uniformities* $\mathscr{V}$ *such that* $\mathfrak{v}\mathscr{V} = \mathscr{U}$*. If* $\{\mathscr{M}_a\}$ *is any family in* $\mathbf{M}$ *such that* $\sup \{\mathscr{M}_a\}$ *exists, then* $\inf \{\mathscr{M}_a\}$ *exists and*

(*) $\sup \{\mathscr{M}_a\} = \bigcap\{\mathscr{M}_a\}$;

(**) $\inf \{\mathscr{M}_a\}$ *is the smallest uniform collection of pseudometrics containing* $\bigcup\{\mathscr{M}_a\}$.

The proof, which follows from results of 24 B, is left to the reader as a useful exercise.

By 36 A.3 the canonical mapping of $\mathbf{U}$ into $\mathbf{C}$ is completely lattice-preserving. The following result concerns the mappings $\gamma : \mathfrak{v}\mathbf{U} \to \mathbf{C}$ and $\gamma : \mathfrak{v}\mathbf{U} \to \mathfrak{v}\mathbf{C}$. Recall that $\mathfrak{v}\mathbf{C}$ is the class of all uniformizable closures, and $\mathfrak{v}\mathbf{C}$ is completely meet-stable in $\mathbf{C}$ by 31 B.4.

36 B.5. Theorem. *The canonical mappings* $\gamma : \mathfrak{v}\mathbf{U} \to \mathbf{C}$ *and* $\gamma : \mathfrak{v}\mathbf{U} \to \mathfrak{v}\mathbf{C}$ *are completely meet-preserving. If* $\mathscr{U}$ *is a uniformly accrete or a uniformly discrete uniformity, then* $\gamma\mathscr{U}$ *is, respectively, an accrete or a discrete closure.*

Proof. The mapping $\gamma : \mathfrak{v}\mathbf{U} \to \mathbf{C}$ is the composite of two completely meet-preserving mappings, namely $\mathrm{J} : \mathfrak{v}\mathbf{U} \to \mathbf{U}$ (by 36 B.2) followed by $\gamma : \mathbf{U} \to \mathbf{C}$ (by 36 A.3). Thus $\gamma : \mathfrak{v}\mathbf{U} \to \mathbf{C}$ is completely meet-preserving. The range $\mathfrak{v}\mathbf{C}$ of the completely meet-preserving mapping $\gamma : \mathfrak{v}\mathbf{U} \to \mathbf{C}$ is completely meet-stable in $\mathbf{C}$ (by 31 B.2) and hence the range-restriction $\gamma : \mathfrak{v}\mathbf{U} \to \mathfrak{v}\mathbf{C}$ is completely meet-preserving. The proof of the remaining statements is evident.

Corollary. *If* P *is a set then* $\gamma \inf \{\mathscr{U}_a\}$ *with the infimum taken in* $\mathfrak{v}\mathbf{U}$, $\inf \{\gamma\mathscr{U}_a\}$ *taken in* $\mathbf{C}(P)$*, and* $\inf \{\gamma\mathscr{U}_a\}$ *taken in* $\mathfrak{v}\mathbf{C}(P)$ *coincide for each family* $\{\mathscr{U}_a\}$ *in* $\mathfrak{v}\mathbf{U}$ *(not necessarily non-void).*

36 B.6. Remark. It should be noted that the canonical mappings $\gamma : \mathfrak{v}\mathbf{U} \to \mathbf{C}$ and $\gamma : \mathfrak{v}\mathbf{U} \to \mathfrak{v}\mathbf{C}$ are not completely join-preserving. In the former case it is sufficient to keep in mind that the least upper bound in **C** of a family of uniformizable closures need not be uniformizable (if $u = \sup\{u_a\}$ is not uniformizable and if $\mathscr{U} = \sup\{\mathscr{U}_a\}$ in $\mathfrak{v}$**U** such that $\gamma\mathscr{U}_a = u_a$ for each a, then $\gamma\mathscr{U}$ is strictly coarser than u because $\gamma\mathscr{U}$ is uniformizable and coarser than u). In the latter case consider a uniformizable closure operation u for a set P and the set $\mathfrak{v}$**U** $\cap$ **U**(P, u). This set has no greatest element if $\langle P, u\rangle$ is not locally compact (41 D.6) and hence the supremum of this set in $\mathfrak{v}$**U** induces a closure strictly coarser than u.

36 B.7. *For each* $\mathscr{U}$ *in* **U** *the closure* $\gamma\mathfrak{v}\mathscr{U}$ (i.e., the closure induced by the uniform modification of $\mathscr{U}$) *is coarser than the closure* $\mathfrak{v}\gamma\mathscr{U}$ (i.e., the uniformizable modification of the closure induced by $\mathscr{U}$). In fact, $\gamma\mathfrak{v}\mathscr{U}$ is a uniformizable closure coarser than the closure $\gamma\mathscr{U}$. We shall show that $\gamma\mathfrak{v}\mathscr{U}$ may be strictly coarser than $\mathfrak{v}\gamma\mathscr{U}$ (and hence the mappings $\mathfrak{v} \circ \gamma$: **U** $\to$ **C** and $\gamma \circ \mathfrak{v}$: **U** $\to$ **C** do not coincide); in addition, the closure $\gamma\mathscr{U}$ may be uniformizable. Let $\langle P, u\rangle$ be an infinite separated uniformizable space such that each neighborhood of any point is infinite (e.g. the space of reals) and consider the coarse semi-uniformity $\mathscr{U}$ of $\langle P, u\rangle$. By 36 A.5 each element of $\mathscr{U}$ contains a set $(P - X) \times (P - X)$ with X finite. We shall prove that $\mathfrak{v}\mathscr{U}$ is uniformly accrete (and hence $\gamma\mathfrak{v}\mathscr{U}$ is accrete). It is sufficient to show that the complement of any neighborhood of any point of $\langle P, \gamma\mathfrak{v}\mathscr{U}\rangle$ is finite. Let W be a neighborhood of x in $\langle P, \gamma\mathfrak{v}\mathscr{U}\rangle$ and choose a U in $\mathfrak{v}\mathscr{U}$ such that $U \circ U[x] \subset W$. The set $U[x]$ is a neighborhood of x in $\langle P, \gamma\mathfrak{v}\mathscr{U}\rangle$ and so certainly in $\langle P, u\rangle$; thus $U[x]$ is infinite. If X is a finite set such that $(P - X) \times (P - X) \subset U$, then $U \circ U[x] = U[U[x]] \supset P - X$ because $(P - X) \cap \cap U[x] \neq \emptyset$. Thus $W \supset P - X$.

36 B.8. The example in 36 B.7 makes it possible to show that the canonical mappings $\gamma : \mathfrak{v}$**U** $\to$ **C** and $\gamma : \mathfrak{v}$**U** $\to \mathfrak{v}$**C** are not join-preserving (also see 36 B.6). The semi-uniformity $\mathscr{U}$ (in the example of 36 B.7) is the supremum in **U** of any uniformity $\mathscr{V}$ inducing u and of the coarse semi-uniformity $\mathscr{W}$ of P endowed with the discrete closure (by Corollary (c) of 36 A.5). It follows from Corollary (b) of 36 A.5 that $\mathscr{W}$ is a uniformity. The supremum of $\mathscr{V}$ and $\mathscr{W}$ in $\mathfrak{v}$**U** is the uniform modification of the supremum taken in **U**, and hence is $\mathfrak{v}\mathscr{U}$. We have proved (in 36 B.7) that $\mathfrak{v}\mathscr{U}$ is the uniformly accrete uniformity for P. Thus $\gamma\mathfrak{v}\mathscr{U}$ is the accrete closure for P. On the other hand the supremum of $\gamma\mathscr{V}$ and $\gamma\mathscr{W}$ in **C** as well as in $\mathfrak{v}$**C** is $u = \gamma\mathscr{V}$.

In 28 A.1 the definition of a fine uniformity and a fine uniformity of a space was introduced. In 36 A.4 the definitions of a fine semi-uniformity and a coarse semi-uniformity were introduced.

36 B.9. Definition. A *coarse uniformity* is a uniformity $\mathscr{U}$ with the following property: if a uniformity $\mathscr{V}$ induces the same closure as $\mathscr{U}$ then $\mathscr{V}$ is uniformly finer than $\mathscr{U}$.

36 B.10. *If* $\mathscr{U}$ *is a fine semi-uniformity, then the uniform modification* $\mathfrak{v}\mathscr{U}$ *of* $\mathscr{U}$ *is a fine uniformity and every fine uniformity is the uniform modification of a fine semi-uniformity (namely, of the fine semi-uniformity inducing the same closure).*

In particular, if a fine semi-uniformity is a uniformity, then it is a fine uniformity. – Almost evident.

36 B.11. *If $\mathscr{U}$ is a coarse semi-uniformity, then the uniform modification $\mathfrak{v}\mathscr{U}$ of $\mathscr{U}$ is a coarse uniformity, in particular, if a coarse semi-uniformity is a uniformity, then it is a coarse uniformity.*

Proof. I. By the Corollary (d) of 36 A.5 a semi-uniformity $\mathscr{U}$ is coarse if and only if each element of $\mathscr{U}$ contains a set $(P - X) \times (P - X)$ with X finite. It follows that if $\mathscr{U}$ is a coarse semi-uniformity, then each semi-uniformity uniformly coarser than $\mathscr{U}$ is also coarse. Thus, if $\mathscr{U}$ is a coarse semi-uniformity, then the uniform modification $\mathfrak{v}\mathscr{U}$ of $\mathscr{U}$ is also a coarse semi-uniformity. Obviously, if a coarse semi-uniformity $\mathscr{U}$ is a uniformity, then $\mathscr{U}$ is a coarse uniformity. The assertion follows.

II. The second statement is obvious, as stated in I, but it may be useful to notice that it is a consequence of the first statement.

While every fine uniformity is the uniform modification of some fine semi-uniformity, with some trivial exceptions a coarse uniformity is not the uniform modification of any coarse semi-uniformity. As noted in the proof of the foregoing proposition, a semi-uniformity uniformly coarser than a coarse semi-uniformity is a coarse semi-uniformity, and hence, if a coarse uniformity $\mathscr{U}$ is the uniform modification of a coarse semi-uniformity, then necessarily $\mathscr{U}$ is a coarse semi-uniformity. Since by the foregoing proposition any uniformity which is a coarse semi-uniformity is then a coarse uniformity, we obtain that a coarse uniformity $\mathscr{U}$ is the uniform modification of a coarse semi-uniformity if and only if $\mathscr{U}$ is a coarse semi-uniformity. Coarse semi-uniformities were described in 36 A.5. A similar characterization of coarse uniformities will be given in the exercises to 41. In the concluding theorem of this section all the separated closures induced by a uniformity which is a coarse semi-uniformity will be described. A similar description of closures induced by a coarse uniformity will be given in 41 D.

36 B.12. Theorem. *The closure structure of a separated space is induced by a coarse semi-uniformity which is a uniformity if and only if the following condition is fulfilled: either $\mathscr{P}$ is discrete, or $\mathscr{P}$ has exactly one accumulation point, say x, and the complements of neighborhoods of x are finite.*

Proof. I. If $\mathscr{P}$ is a discrete space, then the coarse semi-uniformity $\mathscr{U}$ inducing the closure structure of $\mathscr{P}$ has the collection of all relations $V_X = \mathrm{J}_{|\mathscr{P}|} \cup ((|\mathscr{P}| - X) \times (|\mathscr{P}| - X))$, X finite, for a base (Corollary (b) of 36 A.5), and hence $\mathscr{U}$ is a uniformity ($V_X \circ V_X = V_X$). If $\mathscr{P}$ is a space with exactly one accumulation point, say x, and all the complements of neighborhoods of x are finite, and if $\mathscr{U}$ is the semi-uniformity having the collection of all relations $V_U = (U \times U) \cup \mathrm{J}_{|\mathscr{P}|}$, U varying over all neighborhoods of x, for a base, then clearly $\mathscr{U}$ is a uniformity, $\mathscr{U}$ induces the closure structure of $\mathscr{P}$, and $\mathscr{U}$ is a coarse semi-uniformity because each element of $\mathscr{U}$ contains a set $(|\mathscr{P}| - X) \times (|\mathscr{P}| - X)$ with X finite. – II. To prove "only if", suppose that the closure structure of a separated non-discrete space $\mathscr{P}$ is induced by a coarse semi-uniformity $\mathscr{U}$ which is a uniformity. We have proved in 36 B.7 that if each neighborhood of a point x is infinite, then the complement of each neighborhood of x is finite. Since $\mathscr{P}$ is semi-separated, if a point x is not isolated, then each neighborhood of x is infinite and the complement of each neighborhood of x is finite. It follows that there exists at most one point of $\mathscr{P}$ which is not isolated ($\mathscr{P}$ is separated). On the other hand, by our assumption there exists at least one point which is not isolated.

Remark. In the exercises we shall describe all spaces whose closure structure is induced by a coarse semi-uniformity which is a uniformity.

37. PROJECTIVE AND INDUCTIVE GENERATION FOR SEMI-UNIFORM SPACES

In 32 we investigated closure spaces projectively generated by a family of mappings ranging in closure spaces, and in 33 we investigated closure spaces inductively generated by a family of mappings the domain carriers of which were closure spaces. In the present section we shall do the same for semi-uniform spaces. Although projective and inductive constructions for closure spaces were studied separately, the projective and inductive construction for semi-uniform spaces will be treated in a parallel manner as far as possible, and moreover, corresponding results for projective and inductive constructions will be similarly labelled.

As in the case of closure spaces, projectively generated semi-uniform spaces inherit many properties of the range carriers of generating mappings, in particular the property of being a uniform space. Further, if a semi-uniform space $\langle P, \mathscr{U} \rangle$ is projectively generated by a family $\{f_a : \langle P, \mathscr{U} \rangle \to \langle Q_a, \mathscr{V}_a \rangle\}$, then the induced closure space $\langle P, \gamma\mathscr{U} \rangle$ is projectively generated by the family $\{f_a : \langle P, \gamma\mathscr{U} \rangle \to \langle Q_a, \gamma\mathscr{V}_a \rangle\}$. On the other hand, inductively generated semi-uniformities inherit very few properties of the domain carriers of generating mappings. It will be shown (37 A.8) that every semi-uniform space is inductively generated by a mapping the domain carrier of which is a discrete uniform space. It follows that the property of being a uniform space is not inherited, and if a semi-uniformity $\mathscr{U}$ is inductively generated by a mapping f, then the induced closure $\gamma\mathscr{U}$ need not be inductively generated by f transposed to a mapping of induced closure spaces. The points in which the projective and inductive constructions differ will be discussed separately. Because of the importance of uniform spaces we shall introduce the notion of a uniformity inductively generated in the uniform sense by a family of mappings; this is defined to be the finest uniformity making all given mappings uniformly continuous, and this is easily seen to be the uniform modification of the semi-uniformity inductively generated by the same family of mappings. In 33 D we defined the inductive product and topological inductive product of closure spaces. The corresponding concepts for semi-uniform spaces will be introduced in the exercises only.

A. GENERALITIES

37 A.1. Definition. A semi-uniformity $\mathcal{U}$ for a set P is said to be *projectively generated by a family of mappings* $\{f_a\}$, and $\{f_a\}$ is said to be a *projective generating family for semi-uniform spaces with domain carrier* P, if P is the common domain carrier of all f_a and $\mathcal{U}$ is the uniformly coarsest semi-uniformity for P such that all mappings $f_a : \langle P, \mathcal{U} \rangle \to \mathbf{E}^* f_a$ are uniformly continuous. Similarly, a semi-uniformity $\mathcal{U}$ for a set P is said to be *inductively generated by a family of mappings* $\{f_a\}$ and $\{f_a\}$ is said to be an *inductive generating family for semi-uniform spaces with common range carrier* P, if P is the common range carrier of all f_a and $\mathcal{U}$ is the uniformly finest semi-uniformity such that all mappings $f_a : \mathbf{D}^* f_a \to \langle P, \mathcal{U} \rangle$ are uniformly continuous. A semi-uniform space $\langle P, \mathcal{U} \rangle$ is said to be *projectively (inductively) generated by a family of mappings* $\{f_a\}$, and $\{f_a\}$ is said to be a *projective (inductive) generating family for* $\langle P, \mathcal{U} \rangle$, if $\langle P, \mathcal{U} \rangle$ is the common domain carrier (range carrier) of all f_a and $\mathcal{U}$ is the semi-uniformity for P projectively generated (inductively generated) by the family $\{\operatorname{gr} f_a : P \to \mathbf{E}^* f_a\}$ ($\{\operatorname{gr} f_a : \mathbf{D}^* f_a \to P\}$). Finally, a *projective (inductive) generating family of mappings for semi-uniform spaces* is a family $\{f_a\}$ with a common domain carrier (range carrier) which is projectively (inductively) generated by the family $\{f_a\}$. The definitions just stated are carried over to collections of mappings and single mappings in such a way that a collection $\mathcal{F}$ has a property $\mathfrak{P}$ if and only if the family $\{f \mid f \in \mathcal{F}\}$ has the property $\mathfrak{P}$ and a mapping f has a property $\mathfrak{P}$ if and only if the singleton (f) has the property $\mathfrak{P}$.

We begin with existence, uniqueness and a description of generated semi-uniformities.

37 A.2 proj. Theorem. *Any projective family of mappings for semi-uniform spaces projectively generates exactly one semi-uniformity. If a semi-uniformity* $\mathcal{U}$ *is projectively generated by a non-void family* $\{f_a\}$ *and each* $\mathcal{U}_a$ *is projectively generated by* f_a, *then* $\mathcal{U} = \inf \{\mathcal{U}_a\}$, *that is,* $\bigcap\{\mathcal{U}_a\}$ *is a sub-base for* $\mathcal{U}$. *If a semi-uniformity* $\mathcal{U}$ *is generated by a mapping* $f : P \to \langle Q, \mathcal{V} \rangle$, *then the set of all* $(f \times f)^{-1}[V]$, $V \in \mathcal{V}$, *is a base for* $\mathcal{U}$. *It follows that if a semi-uniformity* $\mathcal{U}$ *is projectively generated by a non-void family of mappings* $\{f_a : P \to \langle Q_a, \mathcal{V}_a \rangle \mid a \in A\}$, *then the set of all* $(f_a \times f_a)^{-1}[V]$, $a \in A$, $V \in \mathcal{V}_a$, *is a sub-base for* $\mathcal{U}$.

37 A.2 ind. Theorem. *Any inductive family of mappings for semi-uniform spaces inductively generates exactly one semi-uniformity. If a semi-uniformity* $\mathcal{U}$ *is inductively generated by a family* $\{f_a\}$ *and* $\mathcal{U}_a$ *is the semi-uniformity inductively generated by* f_a, *then* $\mathcal{U} = \sup \{\mathcal{U}_a\}$ $(= \bigcap\{\mathcal{U}_a\})$. *If a semi-uniformity* $\mathcal{U}$ *is inductively generated by a mapping* $f : \langle Q, \mathcal{V} \rangle \to P$, *then* $\mathcal{U}$ *is the collection of all vicinities* U *of the diagonal* $\rfloor_P$ *such that* $(f \times f)^{-1}[U] \in \mathcal{V}$, *that is,*

$$(1)\quad \mathcal{U} = \mathbf{E}\{U \mid \rfloor_P \subset U \subset P \times P,\ (f \times f)^{-1}[U] \in \mathcal{V}\}.$$

It follows that if a semi-uniformity $\mathcal{U}$ *is inductively generated by a family of map-*

pings $\{f_a : \langle Q_a, \mathcal{V}_a \rangle \to P \mid a \in A\}$, *then* $\mathcal{U}$ *consists of all vicinities U of the diagonal* J_P *such that* $(f_a \times f_a)^{-1}[U] \in \mathcal{V}_a$ *for each a in A, that is,*

(2) $\mathcal{U} = \mathbf{E}\{U \mid J_P \subset U \subset P \times P, (f_a \times f_a)^{-1}[U] \in \mathcal{V}_a$ *for each* $a \in A\}$.

Proof of 37 A.2 proj. I. Uniqueness is self-evident.

II. Existence: Let $\{f_a : P \to \langle Q_a, \mathcal{V}_a \rangle \mid a \in A\}$ be a projective family for semi-uniform spaces and let us consider the set Ψ of all $\mathcal{W} \in \mathbf{U}(P)$ such that each mapping $f_a : \langle P, \mathcal{W} \rangle \to \langle Q_a, \mathcal{V}_a \rangle$ is uniformly continuous. By 36 A.9 sup $\Psi \in \Psi$ and by definition obviously sup Ψ is the projectively generated semi-uniformity.

III. Now let Ψ_a, $a \in A$, be the set of all $\mathcal{W} \in \mathbf{U}(P)$ such that the mapping f_a : $: \langle P, \mathcal{W} \rangle \to \langle Q_a, \mathcal{V}_a \rangle$ is uniformly continuous. By II sup Ψ_a is the semi-uniformity projectively generated by f_a and obviously $\Psi = \bigcap\{\Psi_a \mid a \in A\}$. It follows that sup $\Psi =$ $=$ inf $\{$sup $\Psi_a\}$ and this is the required formula $\mathcal{U} = \inf\{\mathcal{U}_a\}$. The fact that $\bigcup\{\mathcal{U}_a \mid a \in A\}$ is a sub-base of $\mathcal{U}$ follows from 36 A.1.

IV. It remains to verify the description of the semi-uniformity projectively generated by a single mapping $f : P \to \langle Q, \mathcal{V} \rangle$ in terms of $\mathcal{V}$. Let $\mathcal{U}_1$ be the collection of all $(f \times f)^{-1}[V]$, $V \in \mathcal{V}$. It is easily proved that $\mathcal{U}_1$ is a base of a semi-uniformity $\mathcal{U}$; clearly $\mathcal{U}_1$ is a filter base and each element of $\mathcal{U}_1$ contains the diagonal J_P, and finally, if $V \in \mathcal{V}$ is symmetric then the set $(f \times f)^{-1}[V]$ is also symmetric. The mapping $f : \langle P, \mathcal{U} \rangle \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous and if $f : \langle P, \mathcal{W} \rangle \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous, then each $(f \times f)^{-1}[V]$, $V \in \mathcal{V}$, must belong to $\mathcal{W}$, i.e. $\mathcal{W} \supset \mathcal{U}_1$; this shows that the semi-uniformity $\mathcal{U}$ with the base $\mathcal{U}_1$ is the uniformly coarsest semi-uniformity making f continuous.

Proof of 37 A.2 ind. I. Uniqueness is again clear and the existence and the formula $\mathcal{U} = \sup\{\mathcal{U}_a\}$ is similar to the corresponding proofs for the projective construction; if Φ and Φ_a are sets of all semi-uniformities making respectively all f_a, $a \in A$, or f_a continuous, then inf $\Phi = \mathcal{U}$, inf $\Phi_\alpha = \mathcal{U}_a$ and $\Phi = \bigcap\{\Phi_a\}$ which gives the formula.

II. By 36 A.1 $\sup\{\mathcal{U}_a\} = \bigcap\{\mathcal{U}_a\}$ and consequently formula (1) implies (2).

III. It remains to prove (1); we shall show that $\mathcal{U}$ given by (1) is inductively generated by $f : \langle Q, \mathcal{V} \rangle \to P$. The reader will find no difficulty in showing that $\mathcal{U}$ is a semi-uniformity and that the mapping $f : \langle Q, \mathcal{V} \rangle \to \langle P, \mathcal{U} \rangle$ is uniformly continuous. If $\mathcal{W}$ is any semi-uniformity for P such that the mapping $f : \langle Q, \mathcal{V} \rangle \to \langle P, \mathcal{W} \rangle$ is uniformly continuous, then $(f \times f)^{-1}[W] \in \mathcal{V}$ for each W in $\mathcal{W}$ and thus each element of $\mathcal{W}$ belongs to $\mathcal{U}$; i.e. $\mathcal{W} \subset \mathcal{U}$, that is, $\mathcal{W}$ is uniformly coarser than $\mathcal{U}$.

Corollaries proj.: (a) *A semi-uniformity for a set P projectively generated by an empty family is the uniformly coarsest semi-uniformity for P, that is, the uniformly accrete semi-uniformity for P.*

(b) *Let* $\{f_a \mid a \in A\}$ *be a family of mappings of a semi-uniform space* $\mathcal{P}$ *into semi-uniform spaces and let* A_1 *be a subset of A such that the range carrier of each* f_a *with a in* $A - A_1$ *is a uniformly accrete space. Then* $\mathcal{P}$ *is projectively generated by the family* $\{f_a \mid a \in A\}$ *if and only if it is projectively generated by the family* $\{f_a \mid a \in A_1\}$.

(c) *Let* $\{f_a \mid a \in A\}$, $A \neq \emptyset$, *be a family of mappings of a semi-uniform space* $\langle P, \mathscr{U}\rangle$ *into semi-uniform spaces and* $\{\mathscr{V}'_a\}$ *be a family such that* $\mathscr{V}'_a$ *is a sub-base for the semi-uniformity of* $\mathbf{E}^*f_a$. *Then* $\langle \mathscr{P}, \mathscr{U}\rangle$ *is projectively generated by the family* $\{f_a\}$ *if and only if the set of all* $(f_a \times f_a)^{-1}[V]$, $V \in \mathscr{V}'_a$, $a \in A$, *is a sub-base for* $\mathscr{U}$.

(d) *The product of a family of semi-uniform spaces is projectively generated by the family of all projections.*

(e) *If* $\mathscr{Q}$ *is a subspace of a semi-uniform space* $\mathscr{P}$, *then* $\mathsf{J} : \mathscr{Q} \to \mathscr{P}$ *is a projective generating mapping.*

Proof. Corollary (a) is evident and Corollary (b) follows from the fact that every mapping of a semi-uniform space into a uniformly accrete semi-uniform space is uniformly continuous. Corollary (c) is a straightforward consequence of the description of projectively generated semi-uniformities. Finally, the product semi-uniformity was defined (23 D.10) to be the semi-uniformity whose sub-base is the collection of all $(\pi_a \times \pi_a)^{-1}[V]$, where π_a is a projection into the a-th-coordinate space and V is any element of the semi-uniform structure of the a-th-coordinate space; according to the description of projectively generated semi-uniformities, this collection is a sub-base of the semi-uniformity projectively generated by the family of all projections. It is to be noted that proposition 23 D.11 states explicitly that the product semi-uniformity is the uniformly coarsest semi-uniformity for the product of the underlying sets making all projections uniformly continuous; hence Corollary (d) has already been proved. Statement (e) is evident.

Corollaries ind.: (a) *A semi-uniformity for a set* P *inductively generated by an empty family is the uniformly finest semi-uniformity for* P, *that is, the uniformly discrete semi-uniformity (in fact, uniformity) for* P.

(b) *Let* $\{f_a \mid a \in A\}$ *be a family of mappings of semi-uniform spaces into a semi-uniform space* $\mathscr{P}$ *and let* A_1 *be a subset of* A *such that the domain carrier of each* f_a *with* a *in* $A - A_1$ *is a uniformly discrete semi-uniform space (in fact, uniform space). Then the space* $\mathscr{P}$ *is inductively generated by the family* $\{f_a \mid a \in A\}$ *if and only if* $\mathscr{P}$ *is inductively generated by the family* $\{f_a \mid a \in A_1\}$.

(c) *The sum of a family of semi-uniform spaces (Definition* 23 D.8*) is inductively generated by the family of all canonical embeddings.*

Proof. Corollary (a) is obvious and Corollary (b) follows from the fact that every mapping of a uniformly discrete semi-uniform space into any semi-uniform space is uniformly continuous. Finally, Corollary (c) is obtained by comparing the definition of the sum semi-uniformity (23 D.8) with the description of inductively generated semi-uniformities. It may be noted that, in fact, Corollary (c) has already been proved in 23 D.9.

Let $\{f_a\}$ be a family of uniformly continuous mappings with a common domain carrier $\langle P, \mathscr{U}\rangle$ and let us consider the semi-uniformity $\mathscr{V}$ for P projectively generated by a family of mappings $\{\operatorname{gr} f_a : P \to \mathbf{E}^*f_a\}$. (The existence of $\mathscr{V}$ follows from

37 A.2 proj.) Let h be the identity mapping of $\langle P, \mathscr{U} \rangle$ onto $\langle P, \mathscr{V} \rangle$ and for each a let g_a be the mapping $\langle \operatorname{gr} f_a, \langle P, \mathscr{V} \rangle, \mathbf{E}^* f_a \rangle$. Clearly h is a one-to-one uniformly continuous surjective mapping, $\{g_a\}$ is a projective generating family for semi-uniform spaces and $f_a = g_a \circ h$ for each a. In particular, every uniformly continuous mapping admits of a factorization $f = g \circ h$ where h is a one-to-one uniformly continuous surjective mapping and g is a projective generating mapping for semi-uniform spaces. It may be in place to note that, in accordance with the general rule for the use of square parentheses, the family $\{g_a \circ h\}$ can be written as $[\{g_a\}] \circ h$ and hence, as in the case of a single mapping, we obtain formally that every projective family $\{f_a\}$ of uniformly continuous mappings admits a factorization $\{f_a\} = [\{g_a\}] \circ h$ where $\{g_a\}$ is a projective generating family for semi-uniform spaces and h is a one-to-one uniformly continuous surjective mapping; such a decomposition will be called a projective decomposition of $\{f_a\}$. Similarly, we shall show that every inductive family $\{f_a\}$ of uniformly continuous mappings with a common range carrier $\langle P, \mathscr{U} \rangle$ admits an inductive factorization $\{f_a\} = h \circ [\{g_a\}]$, where h is a one-to-one uniformly continuous surjective mapping and $\{g_a\}$ is an inductive generating family for semi-uniform spaces. The existence of an inductive factorization is proved as follows: let $\mathscr{V}$ be the semi-uniformity for P inductively generated by the family $\{\operatorname{gr} f_a : \mathbf{D}^* f_a \to P\}$ and put

$$h = \langle \mathrm{J}_P, \langle P, \mathscr{V} \rangle, \langle P, \mathscr{U} \rangle \rangle, \quad g_a = \langle \operatorname{gr} f_a, \mathbf{D}^* f_a, \langle P, \mathscr{V} \rangle \rangle .$$

Clearly h and $\{g_a\}$ possess the required properties. Thus we have proved:

37 A.3. Theorem. Projective factorization: *Every projective family $\{f_a \mid a \in A\}$ of uniformly continuous mappings admits a projective factorization, that is, there exists a uniformly continuous bijective mapping h and a projective generating family for semi-uniform spaces $\{g_a\}$ such that $g_a \circ h = f_a$ for each a in A; this can be written as $\{f_a\} = [\{g_a\}] \circ h$. If h is an identity mapping then this factorization is called the* canonical projective factorization.

Inductive factorization: *For every inductive family $\{f_a\}$ of uniformly continuous mappings there exists an inductive factorization of $\{f_a\}$, that is, a uniformly continuous bijective mapping h and an inductive generating family $\{g_a\}$ for semi-uniform spaces such that $f_a = h \circ g_a$ for each a in A, that is, $\{f_a\} = h \circ [\{g_a \mid a \in A\}]$; if h is an identity mapping then this factorization is called the* canonical inductive factorization.

37 A.4 proj. Theorem. *Suppose that $\{f_a \mid a \in A\}$ is a family of mappings of a semi-uniform space $\mathscr{P}$ into semi-uniform spaces and the range carrier of each f_a is projectively generated by a family of mappings $\{g_{ab} \mid b \in B_a\}$. Then the space $\mathscr{P}$ is projectively generated by the family $\{f_a\}$ if and only if it is projectively generated by the family $\{g_{ab} \circ f_a \mid a \in A, b \in B_a\}$.*

37 A.4 ind. Theorem. *Suppose that $\{f_a \mid a \in A\}$ is a family of mappings into a semi-uniform space $\mathscr{P}$ and the domain carrier of each f_a is a semi-uniform space* *inductively generated by a family of mappings* $\{g_{ab} \mid b \in B_a\}$. *Then the space* $\mathscr{P}$ *is inductively generated by the family* $\{f_a \mid a \in A\}$ *if and only if* $\mathscr{P}$ *is inductively generated by the family* $\{f_a \circ g_{ab} \mid a \in A, b \in B_a\}$.

The proof of both theorems is a matter of a simple calculation based on the description of generated semi-uniformities and therefore the details will be left to the reader.

Proof of 37 A.4 proj.: I. First suppose that the set A and also all sets B_a, $a \in A$, are non-void. For brevity denote by $\mathscr{U}$, $\mathscr{V}_a$ and $\mathscr{W}_{ab}$ the semi-uniform structures of $\mathscr{P}$, $\mathbf{E}^*f_a$ $(= \mathbf{D}^*g_{ab})$ and $\mathbf{E}^*g_{ab}$ respectively. By 37 A.2 proj., it follows from our assumptions that

(3) for each a the set $\mathscr{V}'_a$ of all $(g_{ab} \times g_{ab})^{-1} [W]$, $W \in \mathscr{W}_{ab}$, $b \in B_a$, is a sub-base for $\mathscr{V}_a$.

Now, again by 37 A.2 proj., the fact that $\mathscr{U}$ is projectively generated by the family $\{g_{ab} \circ f_a\}$ is equivalent to the following assertion:

(4) the set of all $(g_{ab} \circ f_a \times g_{ab} \circ f_a)^{-1} [W]$, $W \in \mathscr{W}_{ab}$, $b \in B_a$, $a \in A$, is a sub-base for $\mathscr{U}$.

By corollary (c) of 37 A.2 proj., the fact that $\mathscr{U}$ is projectively generated by $\{f_a\}$ is equivalent to the assertion (keep in mind that each $\mathscr{V}'_a$ is a sub-base of $\mathscr{V}_a$ by (3)):

(5) the set of all $(f_a \times f_a)^{-1} [V]$, $V \in \mathscr{V}'_a$, $a \in A$, is a sub-base for $\mathscr{U}$.

Since $(g_{ab} \circ f_a \times g_{ab} \circ f_a)^{-1} [X] = (f_a \times f_a)^{-1} [(g_{ab} \times g_{ab})^{-1} [X]]$ for each X, the equivalence of (4) and (5) follows from (3).

II. If $A = \emptyset$ then the families $\{f_a\}$ and $\{g_{ab} \circ f_a\}$ are both empty and the statement follows from corollary (a) of 37 A.2 proj.

III. The general case will be reduced to cases I and II. Let A_1 be the subset of A consisting of all a such that $B_a \neq \emptyset$. By corollary (a) of 37 A.2 proj. the range space of each f_a with a in $A - A_1$ is uniformly accrete and consequently, by corollary (b), $\mathscr{P}$ is projectively generated by the family $\{f_a \mid a \in A\}$ if and only if it is projectively generated by $\{f_a \mid a \in A_1\}$. However, the families $\{g_{ab} \circ f_a \mid b \in B_a, a \in A\}$ and $\{g_{ab} \circ \circ f_a \mid b \in B_a, a \in A_1\}$ coincide. As a consequence, if $A_1 \neq \emptyset$ then the statement follows from I and in the other case from II.

Proof of 37 A.4 ind. The proof is simpler than that of 37 A.4 proj. because we need not examine separately the case where A or some B_a are empty. By our assumption and 37 A.2 ind., for each a in A, a vicinity U of the diagonal in $\mathbf{D}^*f_a$ belongs to the semi-uniformity of $\mathbf{D}^*f_a$ if and only if $(g_{ab} \times g_{ab})^{-1} [U]$ belongs to the semi-uniformity of $\mathbf{D}^*g_{ab}$ for each $b \in B_a$. Now again by 37 A.2 ind., stating in symbols that $\mathscr{P}$ is inductively generated by $\{f_a\}$ or $\{f_a \circ g_{ab}\}$, we obtain the theorem immediately.

From the theorems 37 A.4 proj. and ind. we can show that a semi-uniform space projectively or inductively generated by a non-void family of mappings is projectively or inductively generated by a single mapping, namely by the reduced product or the reduced sum of the family in question. Clearly also every empty generating family may be replaced by any constant mapping. As a consequence, projective and inductive constructions can be reduced to the corresponding construction for a single mapping and to the construction of the reduced product of mappings or the reduced sum of mappings respectively.

Corollary proj. *Let f be the reduced product of a non-void projective family $\{f_a \mid a \in A\}$ of mappings for semi-uniform spaces with a common domain carrier $\mathscr{P}$ which is a space, i.e. $\mathbf{D}^*f = \mathbf{D}^*f_a = \mathscr{P}$, $\mathbf{E}^*f = \Pi\{\mathbf{E}^*f_a\}$ and $fx = \{f_a x \mid a \in A\}$. Then $\mathscr{P}$ is projectively generated by $\{f_a\}$ if and only if $\mathscr{P}$ is projectively generated by f.*

Corollary ind. *Let f be the reduced sum of a non-void inductive family $\{f_a\}$ of mappings for semi-uniform spaces such that the common range carrier is a space $\mathscr{P}$, i.e. $\mathbf{D}^*f = \Sigma\{\mathbf{D}^*f_a\}$, $\mathbf{E}^*f = \mathbf{E}^*f_a = \mathscr{P}$ and $f\langle a, x\rangle = f_a x$. Then $\mathscr{P}$ is inductively generated by $\{f_a\}$ if and only if $\mathscr{P}$ is inductively generated by f.*

Proof of Corollary proj. According to Corollary (d) of 37 A.2 proj. the family $\{\mathrm{pr}_a : \mathbf{E}^*f \to \mathbf{E}^*f_a\}$ is a projective generating family. Since

$$f_a = (\mathrm{pr}_a : \mathbf{E}^*f \to \mathbf{E}^*f_a) \circ f$$

for each a, 37 A.4 proj. applies.

Proof of Corollary ind. According to Corollary (c) of 37 A.2 ind. the family $\{\mathrm{inj}_a : \mathbf{D}^*f_a \to \mathbf{D}^*f\}$ is an inductive generating family. Since

$$f_a = f \circ (\mathrm{inj}_a : \mathbf{D}^*f_a \to \mathbf{D}^*f)$$

for each a, 37 A.4 ind. applies.

37 A.5 proj. Theorem. *A semi-uniform space $\mathscr{P}$ is projectively generated by a family of mappings $\{f_a \mid a \in A\}$ of $\mathscr{P}$ into semi-uniform spaces if and only if the following condition is fulfilled:*

A mapping f of a semi-uniform space $\mathscr{Q}$ into $\mathscr{P}$ is uniformly continuous if and only if all composites $f_a \circ f$, $a \in A$, are uniformly continuous.

37 A.5 ind. Theorem. *A semi-uniform space $\mathscr{P}$ is inductively generated by a family of mappings $\{f_a \mid a \in A\}$ of semi-uniform spaces into $\mathscr{P}$ if and only if the following condition is fulfilled:*

A mapping f of $\mathscr{P}$ into a semi-uniform space $\mathscr{Q}$ is uniformly continuous if and only if all composites $f \circ f_a$, $a \in A$, are uniformly continuous.

The proof is again a matter of a simple calculation based on the description of generated semi-uniformities and therefore the details will be left to the reader.

Proof of 37 A.5 proj. I. The statement is trivial if A is empty. – II. Suppose that $\mathscr{P}$ is projectively generated by the family $\{f_a\}$ and $A \neq \emptyset$. If f is uniformly continuous then all the mappings $f_a \circ f$ are uniformly continuous as composites of uniformly continuous mappings. Conversely, let all compositions $f_a \circ f$ be uniformly continuous. If U is any element of the semi-uniform structure of $\mathscr{P}$, then we can choose a finite subset A' of A and a family $\{V_a \mid a \in A'\}$ such that $\bigcap\{(f_a \times f_a)^{-1}[V_a] \mid a \in A'\} \subset U$ and each V_a is an element of the semi-uniform structure of the range carrier of f_a (by 37 A.2). Clearly

$$(f \times f)^{-1}[U] \supset \bigcap\{(f_a \circ f \times f_a \circ f)^{-1}[V_a] \mid a \in A'\}.$$

Since the composites $f_a \circ f$, $a \in A'$, are uniformly continuous, the right side belongs to the semi-uniform structure of $\mathbf{D}^*f$, and consequently, the left side also belongs to the semi-uniform structure of $\mathbf{D}^*f$. Since U was chosen arbitrarily in the semi-uniform structure of $\mathscr{P}$, f is uniformly continuous. — III. Now suppose that the condition is fulfilled. If f is the identity mapping of $\mathscr{P}$ onto $\mathscr{P}$, then f is a uniformly continuous mapping and hence, according to the condition, all composites $f_a \circ f$ are uniformly continuous; but $f_a \circ f = f_a$ for each a and hence each f_a is uniformly continuous. Let $[\{g_a\}] \circ h$ be the canonical projective factorization of $\{f_a\}$. It is enough to prove that h is a uniform homeomorphism. Since h is a one-to-one uniformly continuous mapping onto, it remains to show that h^{-1} is uniformly continuous. Put $f = h^{-1}$ in the condition. Since $f_a \circ h^{-1} = g_a$ for each a and all g_a are uniformly continuous, h^{-1} is necessarily uniformly continuous by the condition; this concludes the proof.

Proof of 37 A.5 ind. I. The statement is trivial if $A = \emptyset$. — II. Suppose that $\mathscr{P}$ is inductively generated by a non-void family $\{f_a\}$. If f is uniformly continuous then all the mappings $f \circ f_a$ are continuous as composites of uniformly continuous mappings. Conversely, if all composites $f \circ f_a$ are uniformly continuous, then from the description 37 A.2 of inductively generated semi-uniformities it follows at once that f is uniformly continuous (compare with the corresponding fact in the proof in 37 A.5 proj.). — III. Now assume the condition. Substituting $f = \langle J_P, \mathscr{P}, \mathscr{P}\rangle$ we obtain from the condition that all $f \circ f_a$, and hence all $f_a = f \circ f_a$, are uniformly continuous. Consider the canonical inductive factorization $h \circ [\{g_a\}]$ of $\{f_a\}$. Substituting $f = h^{-1}$ we find that h^{-1} is uniformly continuous because all $h^{-1} \circ f_a = g_a$ are uniformly continuous. It follows that h is a uniform homeomorphism and hence $f_a = g_a$ for each a; this concludes the proof.

The next pair of theorems corresponds to theorems 32 A.13 and 33 A.7 for closure spaces.

37 A.6 proj. Theorem on commutativity. *If a semi-uniform space $\mathscr{P}$ is projectively generated by a family of mappings $\{f_a\}$, then each subspace $\mathscr{Q}$ of $\mathscr{P}$ is projectively generated by the family $\{g_a\}$ where each g_a is the domain-restriction of f_a to $\mathscr{Q}$, and also by the family $\{h_a\}$ where each h_a is the restriction to a mapping of $\mathscr{Q}$ into the subspace $\mathbf{E}g_a$ of $\mathbf{E}^*f_a$.*

37 A.6 ind. Theorem on partial commutativity. *If a semi-uniform space $\mathscr{P}$ is inductively generated by a family of mappings $\{f_a\}$, then each subspace $\mathscr{Q}$ of $\mathscr{P}$ is inductively generated by the family $\{g_a\}$, where each g_a is the restriction of f_a to a mapping of the subspace $f_a^{-1}[|\mathscr{Q}|]$ of $\mathbf{D}^*f_a$ into the subspace $\mathscr{Q}$ of $\mathscr{P}$.*

Proof of 37 A.6 proj. The identity mapping f of $\mathcal{Q}$ into $\mathcal{P}$ is a projective generating mapping (by Corollary (e) of 37 A.2 proj.). Since $g_a = f_a \circ f$ for each a, $\{g_a\}$ is a projective generating family by 37 A.4 proj. Let k_a be the identity mapping of the subspace $\mathbf{E}g_a$ of $\mathbf{E}^*f_a$ into $\mathbf{E}^*f_a$. Each of the mappings k_a is a projective generating mapping (again by Corollary (e) of 37 A.2 proj.) and $g_a = k_a \circ h_a$ for each a. Again by 37 A.4 proj. the family $\{h_a\}$ is a projective generating family.

Proof of 37 A.6 ind. Let $R_a = f_a^{-1}[|\mathcal{Q}|]$. If $U \subset |\mathbf{D}^*f_a| \times |\mathbf{D}^*f_a|$, then

$$(g_a \times g_a)[U \cap (R_a \times R_a)] = (f_a \times f_a)[U] \cap (|\mathcal{Q}| \times |\mathcal{Q}|).$$

Now 37 A.6 ind. follows from the description of inductively generated uniformities (37 A.2 ind.) and the definition of a subspace (23 D.1).

Remark. One can easily prove 37 A.6 proj. without any reference to 37 A.4 proj. and Corollary (e) of 37 A.2 proj. On the other hand 37 A.6 ind. cannot be derived from the foregoing general results. The reason for this is that a subspace is defined "projectively", not "inductively".

37 A.7. Up to now the theory of projectively and inductively generated semi-uniformities have been parallel. Now we shall state two distinctions in the theory of these concepts.

(a) If $\{f_a\}$ is a projective generating family of mappings for semi-uniform spaces and the range carrier of each f_a is a uniform space, then the common domain carrier is a uniform space; stated in other words, a projectively generated semi-uniform space inherits the property of being a uniform space from the range spaces (see 37 B.1). On the other hand, if f is an inductive generating mapping for semi-uniform spaces and the domain carrier of f is a uniform space, then the range carrier of f need not be a uniform space.

(b) If $\{\langle f_a, \langle P, \mathcal{U}\rangle, \langle Q_a, \mathcal{V}_a\rangle\rangle \mid a \in A\}$ is a projective generating family for semi-uniform spaces, then the family $\{\langle f_a, \langle P, \gamma\mathcal{U}\rangle, \langle Q_a, \gamma\mathcal{V}_a\rangle\rangle \mid a \in A\}$ is a projective generating family for closure spaces; in other words, if a semi-uniformity $\mathcal{U}$ for a set P is projectively generated by a family $\{f_a : P \to \langle Q, \mathcal{V}_a\rangle\}$, then the closure $\gamma\mathcal{U}$ induced by $\mathcal{U}$ is projectively generated by the family $\{f_a : P \to \langle Q_a, \gamma\mathcal{V}_a\rangle\}$ (see 37 B.6). For inductive generation a similar result does not hold.

Therefore in the following we shall study projective and inductive generation separately. We shall begin with the projective generation. Nevertheless it may be in place to present a general example showing that statements (a) and (b) concerning inductive constructions are actually true.

37 A.8. Theorem. *Every semi-uniform space is inductively generated by a surjective mapping whose domain is a discrete uniform space; stated in other words, if $\langle P, \mathcal{U}\rangle$ is a semi-uniform space, then there exists a discrete uniform space $\langle Q, \mathcal{V}\rangle$ and a surjective mapping f of $\langle Q, \mathcal{V}\rangle$ into $\langle P, \mathcal{U}\rangle$ such that f is an inductively generating mapping, i.e. $\langle P, \mathcal{U}\rangle$ is inductively generated by f.*

Proof. I. Suppose that $\langle R, \mathscr{W}\rangle$ is a semi-uniform space and (X_1, X_2) is a disjoint cover of R such that $W[X_1] \cap X_2 \neq \emptyset$ for each W in $\mathscr{W}$ (and hence $X_1 \cap W[X_2] \neq \emptyset$ for each W in $\mathscr{W}$). It is clear that for a given $\langle R, \mathscr{W}\rangle$ such a cover need not exist. On the other hand there exist $\langle R, \mathscr{W}\rangle$ and X_1, X_2 such that $\langle R, \mathscr{W}\rangle$ is a discrete uniform space. For example, if R is an infinite set (for instance N) and $\mathscr{W}$ is the uniformly coarsest uniformity for R which induces the discrete closure for R (that is, the sets of the form $((R - X) \times (R - X)) \cup \mathsf{J}_R$, X finite, form a base for $\mathscr{W}$) then each disjoint cover (X_1, X_2) consisting of infinite sets has the required property.

II. Now let $\langle P, \mathscr{U}\rangle$ be a semi-uniform space. Consider the sum Q of the constant family $\{R \mid z \in P \times P\}$ and the single-valued relation f on Q to P which assigns to each $\langle z, r\rangle \in Q$, where $z = \langle x_1, x_2\rangle$, the point x_i if $r \in X_i$. Preceding the construction of the required $\mathscr{V}$ for Q, we denote by R'_z, $z \in P \times P$, the set $\mathsf{E}\{\langle z, r\rangle \mid r \in R\}$, by f'_z the restriction of f to R'_z and by $\mathscr{W}'_z$ the semi-uniformity for R'_z inductively generated by the canonical mapping $\{r \to \langle z, r\rangle\}$ of R onto R'_z, that is, $\mathscr{W}'_z$ is the collection of all sets $W'_z = \mathsf{E}\{\langle\langle z, x\rangle, \langle z, y\rangle\rangle \mid \langle x, y\rangle \in W\}$, $W \in \mathscr{W}$.

III. Construction of $\mathscr{V}$. For each U in $\mathscr{U}$ and each family $\{W'_z \mid z \in P \times P\}$, where $W'_z \in \mathscr{W}'_z$, put $V(U, \{W'_z\}) = \bigcup\{X_z \mid z \in P \times P\}$ where $X_z = W'_z$ if $z \in U$ and X_z is the diagonal of $R'_z \times R'_z$ otherwise. It is easily seen that

$$(*)\ (f \times f)\,[V(U, \{W'_z\})] = U \cup U^{-1}$$

for each U in $\mathscr{U}$ and each family $\{W'_z\}$. Next, if ${}^1W'_z \circ {}^2W'_z = {}^3W'_z$ for each z, then evidently $V(U, \{{}^3W'_z\}) = (V(U, \{{}^1W'_z\})) \circ (V(U, \{{}^2W'_z\}))$, and if all W'_z are symmetric then also $V(U, \{W'_z\})$ is symmetric. Thus the collection of all $V(U, \{W'_z\})$ is a base of a semi-uniformity $\mathscr{V}$ for Q, and $\mathscr{V}$ is a uniformity if $\mathscr{W}$ is a uniformity. From $(*)$ it is clear that $\langle f, \langle Q, \mathscr{V}\rangle, \langle P, \mathscr{U}\rangle\rangle$ is an inductively generating mapping.

IV. The reader can easily verify that the relativization of $\mathscr{V}$ to R'_z is $\mathscr{W}'_z$ if $z \in \bigcap \mathscr{U}$ and is the finest uniformity for R'_z otherwise.

V. Clearly the set $\bigcup\{R'_z \times R'_z \mid z \in P \times P\}$ belongs to $\mathscr{V}$. It follows from IV that if $\langle R, \mathscr{W}\rangle$ is discrete, then the space $\langle Q, \mathscr{V}\rangle$ is discrete.

We shall introduce the terminology which enables us to formulate 37 A.7 more precisely.

37 A.9. Definition. The *projective progeny* (*inductive progeny*) *of a class* K *of semi-uniform spaces*, denoted by $\operatorname{proj}_{\mathsf{U}} K$ or simply $\operatorname{proj} K$ ($\operatorname{ind}_{\mathsf{U}} K$ or simply $\operatorname{ind} K$) is the class of all semi-uniform spaces projectively (inductively) generated by a family of mappings with range carriers (domain carriers) in K. A *class* K *of semi-uniform spaces* is said to be *projective-stable* or *inductive-stable* if, respectively, $\operatorname{proj} K = K$ or $\operatorname{ind} K = K$. As in the case of closure spaces the terminology introduced is applied to classes of semi-uniformities.

Statement (a) of 37 A.7 can be formulated as follows: the class $\mathfrak{u}\mathsf{U}$ is projective-stable but not inductive-stable. Let K be any class of semi-uniform spaces. It follows from 37 A.7 (b) that $\gamma_{\mathsf{CU}}\,[\operatorname{proj}_{\mathsf{U}} K]$ is the projective progeny of the class $\gamma_{\mathsf{CU}}[K]$ of closure spaces (37 B.7), but $\gamma_{\mathbf{CU}}[\mathrm{ind}_{\mathbf{U}} K]$ need not be the inductive progeny of the class $\gamma_{\mathbf{CU}}[K]$ (substitute $K = \mathfrak{v}\mathbf{U}$).

37 A.10. Theorem. *The projective (inductive) progeny of any class of semi-uniform spaces is projective-stable (inductive-stable), that is to say,*

$$\mathrm{proj}\,\mathrm{proj}\,K = \mathrm{proj}\,K\,, \quad \mathrm{ind}\,\mathrm{ind}\,K = \mathrm{ind}\,K$$

for any class K of semi-uniform spaces.

Proof. 37 A.4.

37 A.11. Theorem. *Let K be a class of semi-uniform spaces and let L be the class consisting of semi-uniform structures of spaces of K. Then K is projective-stable (inductive-stable) if and only if the following two conditions are fulfilled:*

(a) *L is completely meet-stable (completely join-stable) in* $\mathbf{U}$ *and contains all uniformly accrete (uniformly discrete) semi-uniformities.*

(b) *If f is a projective (inductive) generating mapping for semi-uniform spaces and* $\mathbf{E}^*f \in K$ $(\mathbf{D}^*f \in K)$ *then* $\mathbf{D}^*f \in K$ $(\mathbf{E}^*f \in K)$.

The proof is left to the reader.

Remark. Notice that condition (a) is equivalent to the statement that every semi-uniformity has an upper (lower) modification in L.

B. PROJECTIVE GENERATION

37 B.1. Theorem. *Every semi-uniformity projectively generated by a family of mappings into uniform spaces is a uniformity, i.e., the class* $\mathfrak{v}\mathbf{U}$ *is projective-stable.*

Proof. Suppose that a semi-uniformity $\mathscr{U}$ for a set P is projectively generated by a family of mappings $\{f_a \mid a \in A\}$ into uniform spaces. If $\mathscr{U}_a$ is the semi-uniformity projectively generated by f_a, $a \in A$, then $\mathscr{U} = \inf\{\mathscr{U}_a\}$ by 37 A.2 proj. Since the greatest lower bound of a family of uniformities is a uniformity (36 B.2), to prove that $\mathscr{U}$ is a uniformity it will suffice to show that a semi-uniformity $\mathscr{U}$ for a set P projectively generated by a single mapping f into a uniform space $\langle Q, \mathscr{V}\rangle$ is a uniformity. By 37 A.2 proj. the collection of all $(f \times f)^{-1}[V]$, $V \in \mathscr{V}$, is a base for $\mathscr{U}$. Consequently, to prove that $\mathscr{U}$ is a uniformity it will suffice to show that each $V' = (f \times f)^{-1}[V]$, $V \in \mathscr{V}$, contains a $V_1' \circ V_1'$ for some $V_1' = (f \times f)^{-1}[V_1]$ with V_1 in $\mathscr{V}$. Evidently, if $V_1 \circ V_1 \subset V$ then $V_1' \circ V_1' \subset V'$, which completes the proof.

37 B.2. Theorem. *In order that a mapping f of a uniform space* $\langle P, \mathscr{U}\rangle$ *onto another* $\langle Q, \mathscr{V}\rangle$ *be a projective generating mapping it is necessary and sufficient that a pseudometric d for* $\langle P, \mathscr{U}\rangle$ *be uniformly continuous if and only if* $d = d_1 \circ (f \times f)$ *for some uniformly continuous pseudometric* d_1 *for* $\langle Q, \mathscr{V}\rangle$, *or stated in other words, if* $\boldsymbol{\mu}\langle P, \mathscr{U}\rangle$ *is the range of the relation* $\{d_1 \to d_1 \circ (f \times f) \mid d_1 \in \boldsymbol{\mu}\langle Q, \mathscr{V}\rangle\}$.

This is a straightforward consequence of 37 A.2 proj. and earlier results.

Corollary. *A semi-uniform space is a uniform space if and only if it is projectively generated by a family of mappings into metrizable uniform spaces.*

37 B.3. Theorem. *A semi-uniformity projectively generated by a family of mappings into proximally coarse semi-uniform spaces is proximally coarse.*

Proof. A semi-uniformity $\mathscr{U}$ is proximally coarse if and only if the finite square elements of $\mathscr{U}$ form a base for $\mathscr{U}$. Using this fact, it is easily shown that a semi-uniformity projectively generated by a single mapping into a proximally coarse semi-uniform space is a proximally coarse semi-uniformity, and then the theorem follows from Theorem 37 A.2 proj. asserting that the semi-uniformity projectively generated by a family of mappings $\{f_a\}$ is the greatest lower bound of semi-uniformities projectively generated by single mappings f_a, and from the description of infima in Theorem 36 A.1 (see also 38 B.12).

Corollary. *The product of a family of proximally coarse semi-uniform spaces is a proximally coarse semi-uniform space.*

37 B.4. Theorem. *Each of the following three conditions is necessary and sufficient for a semi-uniformity $\mathscr{U}$ for a set P to be a totally bounded uniformity:*

(a) *$\mathscr{U}$ is projectively generated by the family of mappings $\{f \mid f \in \mathbf{U}^*(\langle P, \mathscr{U}\rangle, \mathsf{R})\}$*

(b) *$\mathscr{U}$ is projectively generated by a family of bounded functions.*

(c) *$\mathscr{U}$ is projectively generated by a family of mappings into proximally coarse uniform spaces.*

Proof. Evidently (a) implies (b), and it follows from 37 B.1 and 37 B.3 that (c) is sufficient. We shall show that (b) implies (c), and (a) is necessary. — I. (b) $\Rightarrow$ (c): Assume that $\langle P, \mathscr{U}\rangle$ is projectively generated by a family $\{f_a\}$ of bounded functions and consider the family $\{g_a\}$ where each g_a is the range-restriction of f_a to the subspace $\mathbf{E}f_a$ of R. By 37 A.6 the space $\langle P, \mathscr{U}\rangle$ is projectively generated by $\{g_a\}$. Each set $\mathbf{E}f_a$ is bounded in R and therefore each space $\mathbf{E}^*g_a$ is proximally coarse (by 25 B.16). — II. (a) is necessary: Assume that $\mathscr{U}$ is a proximally coarse uniformity and consider the semi-uniformity $\mathscr{V}$ projectively generated by the family $\{f : P \to \mathsf{R} \mid f \in \mathbf{U}^*(\langle P, \mathscr{U}\rangle, \mathsf{R})\}$. By 37 B.1 and 37.B.3, $\mathscr{V}$ is a proximally coarse uniformity. We shall prove that $\mathscr{V} = \mathscr{U}$. Both uniformities are proximally coarse and therefore it is sufficient to show that the proximity p induced by $\mathscr{U}$ coincides with the proximity q induced by $\mathscr{V}$ (by 25 B.9). Both proximities are uniformizable and therefore it is sufficient to show that a bounded $f : \langle P, \mathscr{U}\rangle \to \mathsf{R}$ is proximally continuous if and only $f : \langle P, \mathscr{V}\rangle \to \mathsf{R}$ is proximally continuous, or equivalently, a bounded $f : \langle P, \mathscr{U}\rangle \to \mathsf{R}$ is uniformly continuous if and only if $f : \langle P, \mathscr{V}\rangle \to \mathsf{R}$ is uniformly continuous. Evidently $\mathscr{V} \subset \mathscr{U}$ (i.e. $\mathscr{V}$ is uniformly coarser than $\mathscr{U}$) and hence "if" is obvious. On the other hand, if a bounded $f : \langle P, \mathscr{U}\rangle \to \mathsf{R}$ is uniformly continuous then $f : \langle P, \mathscr{V}\rangle \to \mathsf{R}$ is uniformly continuous because $\mathscr{V}$ is, by definition of $\mathscr{V}$, projectively generated by such functions.

Remark. The necessity of (a) will be an immediate consequence of 39 B.7.

If $\{\langle f_a, P, \langle Q_a, \mathscr{V}_a\rangle\rangle\}$ is a projective family for semi-uniform spaces, then we can consider the projective family $\{\langle f_a, P, \langle Q_a, \gamma\mathscr{V}_a\rangle\rangle\}$ for closure spaces, and it is natural to ask whether the closure induced by the semi-uniformity projectively generated by the former family coincides with the closure projectively generated by the latter family. The answer is positive. Before presenting the proof some notation and terminology may be in place. Recall that $\gamma_{\mathbf{CU}}\mathscr{U}$, or simply $\gamma\mathscr{U}$, denotes the closure induced by $\mathscr{U}$ and see 7 B.6.

37 B.5. Definition. $\gamma_{\mathbf{CU}}$, abbreviated to γ, denotes the single-valued relation which assigns to each mapping f for semi-uniform spaces the mapping $f : \gamma_{\mathbf{CU}}\mathbf{D}^*f \to \to \gamma_{\mathbf{CU}}\mathbf{E}^*f$ for closure spaces which is said to be the *transpose of f to a mapping for closure spaces*. If $\{f_a\}$ is a family of mappings for semi-uniform spaces then $\{\gamma f_a\}$ is the transpose of $\{f_a\}$ to a family of mappings for closure spaces.

Evidently, if f is uniformly continuous, then the transposed mapping f is continuous. Now we shall prove

37 B.6. Theorem. *If $\{f_a\}$ is a projective generating family for a semi-uniform space $\mathscr{P}$, then the transposed family $\{\gamma_{\mathbf{CU}}f_a\}$ is a projective generating family for the closure space $\gamma_{\mathbf{CU}}\mathscr{P}$.*

Proof. It will suffice to prove the theorem for a single mapping. Indeed, if $\mathscr{U}_a$ is projectively generated by f_a and u_a is projectively generated by γf_a, then $\inf\{\mathscr{U}_a\}$ is projectively generated by $\{f_a\}$ (by 37 A.2 proj.), $\inf\{u_a\}$ is projectively generated by $\{\gamma f_a\}$, (32 A.4) and by 36 A.3, if $\gamma\mathscr{U}_a = u_a$ for each a, then $\gamma \inf\{\mathscr{U}_a\} = \inf\{u_a\}$. Suppose that f is a projective generating mapping for semi-uniform spaces; write $\langle P, \mathscr{U}\rangle = \mathbf{D}^*f$, $\langle Q, \mathscr{V}\rangle = \mathbf{E}^*f$, $u = \gamma\mathscr{U}$, $v = \gamma\mathscr{V}$. To prove that u is projectively generated by $f : P \to \langle Q, v\rangle$ one may merely compare the description of $\mathscr{U}$ by means of $\mathscr{V}$ (37 A.2 proj.) with the description of neighborhoods relatively to a closure projectively generated by a mapping by means of neighborhoods in the range carrier (32 A.6).

From 37 B.6 we immediately obtain the following result.

37 B.7. Theorem. *If K is a class of semi-uniform spaces then $\gamma_{\mathbf{CU}}[\text{proj}_{\mathbf{U}} K]$ is the projective progeny of the class $\gamma_{\mathbf{CU}}[K]$ of closure spaces.*

Corollary. *If a class K of semi-uniform spaces is projective-stable then the class $\gamma_{\mathbf{CU}}[K]$ of closure spaces is projective-stable.*

Example. The class $\mathfrak{v}\mathbf{U}$ is projective-stable and hence the class $\gamma_{\mathbf{CU}}[\mathfrak{v}\mathbf{U}] = \mathfrak{v}\mathbf{C}$ is projective-stable.

37 B.8. Theorem. *Let K be a class of semi-uniform spaces and let K_1 be the class of all uniformly accrete semi-uniform spaces. A semi-uniform space $\mathscr{P}$ belongs to* proj K *if and only if $\mathscr{P}$ is homeomorphic to a subspace of the product of a family of spaces from $K \cup K_1$.*

Corollary. *A class K of semi-uniform spaces is projective-stable if and only if K contains all uniformly accrete spaces, K is hereditary and completely pro-*

ductive, and, of course, K contains all the uniform homeomorphs of each of its elements.

Proof. Let K_2 be the class of all spaces satisfying the condition. Clearly $K_2 \subset \subset \operatorname{proj} K$. Suppose that $\mathscr{P} \in \operatorname{proj} K$ and $\{f_a\}$ is a projective generating family for $\mathscr{P}$ such that $\mathbf{E}^* f_a \in K$ for each a. If A is empty then $\mathscr{P}$ is a uniformly accrete space and hence $\mathscr{P} \in K_2$. Assuming $A \neq \emptyset$ consider the reduced product f of $\{f_a\}$; by Corollary of 37 A.4, f is a projective generating mapping. Let $\mathscr{Q}$ be the set $|\mathscr{P}|$ endowed with a uniformly accrete semi-uniformity and let g be the reduced product of $\mathrm{J} : \mathscr{P} \to \mathscr{Q}$ and f. Clearly g is a projective generating mapping. Since g is injective, g is a uniform embedding.

C. INDUCTIVE GENERATION

By 37 A.8 the class $\mathfrak{v}\mathbf{U}$ of all uniform spaces is not inductive-stable, and moreover, $\mathbf{U}$ is the inductive progeny of $\mathfrak{v}\mathbf{U}$ (37 A.8 implies that $\mathbf{U}$ is contained in the inductive progeny of the class of all discrete uniform spaces). Thus if $\{f_a\}$ is an inductive generating family for semi-uniform spaces then $\{\gamma_{\mathbf{CU}} f_a\}$ need not be an inductive generating family for closure spaces. Let $\{f_a\}$ be an inductive generating family for a semi-uniform space $\langle P, \mathscr{U} \rangle$ and let $\mathscr{U}_a$ be inductively generated by $f_a : \mathbf{D}^* f_a \to P$; from 37 A.2 we have $\mathscr{U} = \sup \{\mathscr{U}_a\}$. Consider the closure u_a inductively generated by $f_a : \gamma_{\mathbf{CU}} \mathbf{D}^* f_a \to P$ and the closure u inductively generated by $\{f_a : \gamma_{\mathbf{CU}} \mathbf{D}^* f_a \to P\}$; from 33 A.4 we have $u = \sup \{u_a\}$. If $u_a = \gamma_{\mathbf{CU}} \mathscr{U}_a$ for each a then $u = \gamma_{\mathbf{CU}} \mathscr{U}$ because $\gamma_{\mathbf{CU}} : \mathbf{U} \to \mathbf{C}$ is completely lattice preserving (36 A.3). It follows that if the transpose of each mapping $f_a : \mathbf{D}^* f_a \to \langle P, \mathscr{U}_a \rangle$ to a mapping for closure spaces is an inductive generating mapping for closure spaces, then the transpose $\{\gamma_{\mathbf{CU}} f_a\}$ of the family $\{f_a\}$ is an inductive generating family for closure spaces. Thus the fact that the transpose of an inductive generating family of mappings for semi-uniform spaces to a family of mappings for closure spaces need not be an inductive generating family for closure spaces lies in the fact that the transpose $\gamma_{\mathbf{CU}} f$ of an inductive generating mapping f for semi-uniform spaces need not be an inductive generating mapping for closure spaces. Inductive generating mappings for semi-uniform spaces will be studied in 37 D, to which we also leave the discussion of transposed families of mappings.

Considering that uniform spaces form the most important class of semi-uniform spaces, we shall introduce the concept of an inductive generating family for uniform spaces. The results concerning inductive generating families for uniform spaces will be proved directly and also by a reduction to analogous results for inductive generating families for semi-uniform spaces; in the first case the proofs are similar to those concerning the corresponding results for inductive generating families for semi-uniform spaces, and in the latter case the reduction is dependent upon the properties of the uniform modification and the almost self-evident proposition 37 C.1. Compare this development with the similar one for inductive generating families for topological spaces in 33 B.

37 C.1. If $\mathscr{U}$ is the uniformly finest semi-uniformity for a set P rendering all given mappings $f_a : \langle Q, \mathscr{V}_a \rangle \to P$ uniformly continuous, then the uniform modification $\mathfrak{v}\mathscr{U}$ of $\mathscr{U}$ is the uniformly finest uniformity for P rendering all the above mappings uniformly continuous.

37 C.2. Definition. A *uniformity* $\mathscr{U}$ for a set P is said to be *inductively generated in the uniform sense by a family of mappings* $\{f_a\}$ if P is the common range carrier of all f_a, the domain carrier of each f_a is a semi-uniform space and $\mathscr{U}$ is the uniformly finest uniformity for P such that all enriched mappings $\langle \operatorname{gr} f_a, \mathbf{D}^* f_a, \langle P, \mathscr{U} \rangle \rangle$ are uniformly continuous. A *uniform space* is said to be *inductively generated in the uniform sense by a family* $\{f_a\}$, and $\{f_a\}$ is said to be an *inductive generating family in the uniform sense* for $\langle P, \mathscr{U} \rangle$, if $\langle P, \mathscr{U} \rangle$ is the common range carrier of all f_a and the uniformity $\mathscr{U}$ is inductively generated in the uniform sense by the family $\{\langle \operatorname{gr} f_a, \mathbf{D}^* f_a, P \rangle\}$. An *inductive generating family for uniform spaces* is a family $\{f_a\}$ with a common range carrier $\langle P, \mathscr{U} \rangle$ which is inductively generated in the uniform sense by the family $\{f_a\}$ (thus $\mathscr{U}$ is a uniformity). All the terminology introduced is carried over to collections of mappings and single mappings.

Now 37 C.1 can be restated as follows.

37 C.3. *If* $\{\langle f_a, \mathscr{Q}_a, \mathscr{P} \rangle\}$ *is an inductive generating family for semi-uniform spaces, then* $\{\langle f_a, \mathscr{Q}_a, \mathfrak{v}\mathscr{P} \rangle\}$ *is an inductive generating family for uniform spaces; in other words, if* $\mathscr{P}$ *is inductively generated by the former family then the uniform modification* $\mathfrak{v}\mathscr{P}$ *of* $\mathscr{P}$ *is inductively generated in the uniform sense by the latter family.*

Corollary. *If* $\{f_a\}$ *is an inductive generating family for semi-uniform spaces and the common range carrier of the* f_a *is a uniform space, then* $\{f_a\}$ *is also an inductive generating family for uniform spaces.*

It should be noted that an inductive generating family $\{f_a\}$ for uniform spaces need not be an inductive generating family for semi-uniform spaces. Indeed if f is the identity mapping of a semi-uniform space onto its uniform modification $\mathfrak{v}\mathscr{P}$, then clearly f is an inductive generating mapping for uniform spaces but f is not an inductive generating mapping for semi-uniform spaces provided that $\mathscr{P} \neq \mathfrak{v}\mathscr{P}$.

Before proceeding, some comments on the definitions may be in place.

Remarks. (a) Now we are in the same situation as in Section 33 when we had proved the fundamental theorems about inductive generation for closure spaces and we had noticed that the property of being a topological space is not inherited by inductively generated spaces. Because of the importance of topological spaces we introduced the definition of "a closure operation topologically inductively generated by a family of mappings" as the finest topological closure rendering all given mappings continuous. Similarly we introduced the definitions of "a topological inductive generating family" and "a topological inductive generating family for a space". In accordance with our previous terminology it would be more consistent to say "a uniformity uniformly inductively generated by a family $\{f_a\}$", instead of "a uniformity inductively generated in the uniform sense by a family $\{f_a\}$" and a "uniform inductive generating family for a space $\mathscr{P}$" instead of "an inductive generating family in the uniform sense for a space $\mathscr{P}$". However, there are serious reasons for avoiding this terminology. The words uniform and uniformly are used to indicate that a notion relates to semi-uniform spaces and not merely to uniform spaces, for example, "uniformly continuous mapping", "uniformly discrete semi-uniform space". Next, and this is in accordance with the current use of the word uniform, we desire that "$\mathscr{P}$ is uniformly $\mathfrak{P}$" should imply "$\mathscr{P}$ is $\mathfrak{P}$", e.g. if P is uniformly discrete, then P is discrete. On the other hand if a uniform space $\mathscr{P}$ is inductively generated by a family $\{f_a\}$, then $\mathscr{P}$ is inductively generated in the uniform sense by $\{f_a\}$ but the converse is not true. Finally, it is convenient to leave the term uniform generating family for a special type of generating families.

(b) The theory which follows, and also the corresponding terminology, would be more lucid if only uniform spaces were considered. Nevertheless, the notion of a semi-uniform space is basic and in the most general situation it expresses the intuitive content of the notion of a structure describing "uniformness". Furthermore, a semi-uniformity $\mathscr{U}$ inductively generated by a family $\{f_a\}$ can be easily obtained from the semi-uniform structures of the domain carriers of f_a; we can say that $\mathscr{U}$ is the image of these semi-uniformities. For uniformities no such a single description exists; however, the theory can easily be reduced to that for semi-uniform spaces. On the other hand uniformities form the most important class of semi-uniform spaces because the class of all uniformities is the greatest class having some important additional properties (extension theorem 27 B.15 and so on). We shall study the inductive generation of uniform spaces from this point of view.

(c) If $\{f_a\}$ is an inductive generating family for uniform spaces, then the common range carrier of each f_a is a uniform space, but the domain carriers are not required to be uniform spaces. It is worth noticing that the following proposition is true.

37 C.4. *A family $\{f_a\}$ is an inductive generating family for uniform spaces if and only if the family $\{\langle \operatorname{gr} f_a, \mathfrak{v}\mathbf{D}^*f_a, \mathbf{E}^*f_a\rangle\}$ is an inductive generating family for uniform spaces.*

37 C.5. Theorem. *Suppose that P is a set and $\{f_a \mid a \in A\}$ is a family of mappings of semi-uniform spaces into P. Then there exists exactly one uniformity, say $\mathscr{U}$, inductively generated in the uniform sense by $\{f_a\}$: a pseudometric d for $\langle P, \mathscr{U}\rangle$ is uniformly continuous if and only if the pseudometric $d \circ (f_a \times f_a)$ is a uniformly continuous pseudometric for $\mathbf{D}^*f_a$ for each a in A. If $\mathscr{U}_a$ is inductively generated by f_a, then $\mathscr{U}$ is the uniformly finest uniformity uniformly coarser than each $\mathscr{U}_a$, that is, $\mathscr{U}$ is the least upper bound of $\{\mathfrak{v}\mathscr{U}_a\}$ in $\mathfrak{v}\mathbf{U}(P)$.*

Proof. The uniqueness is evident. Now let $\mathscr{V}$ and $\mathscr{V}_a$, $a \in A$, be semi-uniformities inductively generated by the family $\{f_a\}$ or by the mapping f_a, respectively. By 37 C.1 $\mathscr{U} = \mathfrak{v}\mathscr{V}$ is inductively generated in the uniform sense by $\{f_a\}$ and $\mathscr{U}_a = \mathfrak{v}\mathscr{V}_a$ is inductively generated in the uniform sense by f_a; in particular we obtain the existence of uniformities inductively generated in the uniform sense. By 37 A.2 ind. the semi-uniformity $\mathscr{V}$ is the least upper bound of the family $\{\mathscr{V}_a\}$ in $\mathbf{U}(P)$. By virtue of 36 B.2 we find that $\mathscr{U} = \mathfrak{v}\mathscr{V}$ is the least upper bound of $\{\mathscr{U}_a\}$ in $\mathfrak{v}\mathbf{U}(P)$. Finally, the description of uniformly continuous pseudometrics for $\langle P, \mathscr{U}\rangle$ follows from the description 37 A.2 ind of $\mathscr{V}$ and the fact that d is uniformly continuous for $\langle P, \mathscr{V}\rangle$ if and only if it is uniformly continuous for $\langle P, \mathfrak{v}\mathscr{V}\rangle = \langle P, \mathscr{U}\rangle$.

Direct proof. Let $\mathscr{M}$ be the set of all pseudometrics for P such that each $d \circ (f_a \times f_a)$ is a uniformly continuous pseudometric for $\mathbf{D}^* f_a$ for each a in A. It is easily seen that $\mathscr{M}$ is a uniform collection of pseudometrics for P and the corresponding uniformity is inductively generated in the uniform sense by $\{f_a\}$. Now let $\mathscr{M}_a$, $a \in A$, be the collection of all pseudometrics d for P such that $d \circ (f_a \times f_a)$ is uniformly continuous for $\mathbf{D}^* f_a$. Since we have proved that $\mathscr{M}_a = \mu\mathscr{U}_a$ for each a and clearly $\mathscr{M} = \bigcap\{\mathscr{M}_a\}$, it follows that $\mathscr{U} = \sup\{\mathscr{U}_a\}$ in $\mathfrak{v}\mathbf{U}(P)$.

Corollary. *Let $\{f_a \mid a \in A\}$ be a family of mappings of semi-uniform spaces into a semi-uniform space $\mathscr{P}$ and let A_1 be a subset of A such that the domain carrier of each f_a, $a \in A - A_1$, is a uniformly discrete uniform space. Then $\{f_a \mid a \in A\}$ is an inductive generating family for uniform spaces if and only if $\{f_a \mid a \in A_1\}$ is such.*

37 C.6. Let us consider a family $\{f_a\}$ of mappings of semi-uniform spaces ranging in a semi-uniform space $\langle P, \mathscr{U}\rangle$ and let $\mathscr{V}$ be the uniformity for P inductively generated in the uniform sense by the family $\{\operatorname{gr} f_a : \mathbf{D}^* f_a \to P\}$. Put

$$h = \mathrm{J} : \langle P, \mathscr{V}\rangle \to \langle P, \mathscr{U}\rangle\,, \quad g_a = f_a : \mathbf{D}^* f_a \to \langle P, \mathscr{V}\rangle\,.$$

Thus $\{g_a\}$ is an inductive generating family for uniform spaces, h is a bijective mapping and $f_a = h \circ g_a$ for each a; this could be written as

$$(*) \qquad \{f_a\} = h \circ [\{g_a\}].$$

This factorization $(*)$ will be called the *canonical inductive factorization in the uniform sense* of the family $\{f_a\}$. If $\langle P, \mathscr{U}\rangle$ is a uniform space, then obviously the mapping h is uniformly continuous if and only if all f_a are uniformly continuous. Of course, the assumption that $\mathscr{U}$ is a uniformity is essential. Let us also consider the canonical inductive factorization (see 37 A.3)

$$(**) \qquad \{f_a\} = h' \circ [\{g'_a\}]$$

of $\{f_a\}$. By 37 C.3 we have

$$g_a = \langle \mathrm{J}_P, \mathbf{E}^* g'_a, \quad \mathfrak{v}\mathbf{E}^* g'_a\rangle \circ g'_a$$

for each a and

$$h = h' \circ \langle \mathrm{J}_P, \mathfrak{v}\mathbf{D}^* h', \mathbf{D}^* h'\rangle\,.$$

37 C.7. Theorem. *In order that a family $\{f_a \mid a \in A\}$ of mappings of semi-uniform spaces into a uniform space $\langle P, \mathscr{U}\rangle$ be an inductive generating family for uniform spaces it is necessary and sufficient that*

a mapping f of $\langle P, \mathscr{U}\rangle$ into a uniform space $\langle Q, \mathscr{V}\rangle$ be uniformly continuous if and only if each composite $f \circ f_a$, $a \in A$, is uniformly continuous.

Proof. Let us consider the canonical inductive factorization $\{f_a\} = h \circ [\{g_a\}]$. By 37 A.5 a mapping $f \circ h$ into a semi-uniform space is uniformly continuous if and only if all mappings $(f \circ h) \circ g_a$ are uniformly continuous. But $f \circ h \circ g_a = f \circ f_a$ and consequently

(*) A mapping $f \circ h$ is uniformly continuous if and only if all mappings $f \circ f_a$ are uniformly continuous.

According to 37 C.1, $\{f_a\}$ is an inductive generating family for uniform spaces if and only if the common range carrier $\langle P, \mathscr{U} \rangle$ ($= \mathbf{E}^*h$) of all f_a is the uniform modification of the common range carrier of each g_a, say $\langle P, \mathscr{V} \rangle$, which coincides with $\mathbf{D}^*h$. However, in order that $\langle P, \mathscr{U} \rangle$ be the uniform modification of $\langle P, \mathscr{V} \rangle$ it is necessary and sufficient that a mapping f of $\langle P, \mathscr{U} \rangle$ into a uniform space be uniformly continuous if and only if the mapping $f \circ h$ is uniformly continuous. Combining this with (*) we obtain the statement.

Direct proof. I. Necessity. Suppose that $\{f_a\}$ is an inductive generating family for uniform spaces. If $f : \langle P, \mathscr{U} \rangle \to \langle Q, \mathscr{V} \rangle$ is uniformly continuous, then each mapping $f \circ f_a$ is uniformly continuous as the composite of two uniformly continuous mappings (notice that $\langle Q, \mathscr{V} \rangle$ need not be a uniform space here). Conversely, suppose that all composites $f \circ f_a$ are uniformly continuous and $\langle Q, \mathscr{V} \rangle = \mathbf{E}^*f$ is a uniform space. To prove that f is uniformly continuous it is enough to show that $d \circ (f \times f)$ is a uniformly continuous pseudometric for $\langle P, \mathscr{U} \rangle$ provided that d is uniformly continuous pseudometric for $\langle Q, \mathscr{V} \rangle$. Let d be any uniformly continuous pseudometric for $\langle Q, \mathscr{V} \rangle$. Since $d \circ (f \times f)$ is a pseudometric for $\langle P, \mathscr{U} \rangle$ and $\langle P, \mathscr{U} \rangle$ is inductively generated in the uniform sense by the family $\{f_a\}$, to prove that $d \circ (f \times f)$ is uniformly continuous it is enough to show (by 37 C.5) that each $(d \circ (f \times f)) \circ (f_a \times f_a)$ is a uniformly continuous pseudometric for $\mathbf{D}^*f_a$. However, $(d \circ (f \times f)) \circ (f_a \times f_a)$ is equal to $d \circ (f \circ f_a \times f \circ f_a)$, and the latter pseudometric is uniformly continuous because d is a uniformly continuous pseudometric and $f \circ f_a$ is a uniformly continuous mapping.

II. Sufficiency. Suppose that the condition is fulfilled and let us consider the canonical inductive factorization in the uniform sense (see 37 C.6) $\{f_a\} = h \circ [\{g_a\}]$ of $\{f_a\}$. If f is the identity mapping of the common range carrier $\langle P, \mathscr{U} \rangle$ of all f_a onto itself, then the condition yields that all $f \circ f_a = f_a$ are uniformly continuous, and consequently by 37 C.6, h is uniformly continuous. If we put $f = h^{-1}$ then $f \circ f_a = g_a$ and hence all $f \circ f_a$ are uniformly continuous. By the condition, $f = h^{-1}$ is uniformly continuous. Since both h and h^{-1} are uniformly continuous, h is a uniform homeomorphism, and the graph of h being the identity relation J_P, we obtain $f_a = g_a$ for each a.

Remarks. (a) Notice that it follows from the preceding result that the following condition is necessary and sufficient for a uniform space $\langle P, \mathscr{U} \rangle$ be the uniform modification of a uniform space $\langle P, \mathscr{V} \rangle$: a mapping f of the space $\langle P, \mathscr{V} \rangle$ into a uniform space $\mathscr{R}$ is uniformly continuous if and only if the mapping $f : \langle P, \mathscr{U} \rangle \to \mathscr{R}$

is uniformly continuous. (b) It should be noted that we can introduce the definition of an inductive generating family in a subclass K of $\mathbf{U}$; then the concept of an inductive generating family for uniform spaces would be a special case $K = \mathfrak{v}\mathbf{U}$. We leave to the reader the task of verifying that the preceding theorem follows from the fact that $\mathfrak{v}\mathbf{U}$ is projective-stable (compare with 33 B).

37 C.8. Theorem. *Let $\{f_a \mid a \in A\}$ be a family of mappings of semi-uniform spaces which range in a semi-uniform space $\langle P, \mathscr{U} \rangle$ and let the domain carrier of each f_a be inductively generated in the uniform sense by a family $\{g_{ab} \mid b \in B_a\}$. Then $\{f_a\}$ is an inductive generating family for uniform spaces if and only if $\{f_a \circ g_{ab}\}$ is such.*

Proof. Let us consider the following canonical inductive factorizations (not in the uniform sense!):

$$\{g_{ab} \mid b \in B_a\} = h_a \circ [\{g'_{ab} \mid b \in B_a\}] ,$$
$$\{f_a \circ h_a \mid a \in A\} = h \circ [\{f'_a \mid a \in A\}] .$$

It follows that $h \circ f'_a \circ g'_{ab} = f_a \circ g_{ab}$ for each a in A and b in B. Since $\{f'_a\}$ and all $\{g'_{ab} \mid b \in B_a\}$ and hence also $\{f'_a \circ g'_{ab}\}$ are inductive generating families for semi-uniform spaces, by 37 C.3 the family $\{f_a \circ g_{ab}\}$ is an inductive generating family for uniform spaces if and only if the space $\langle P, \mathscr{U} \rangle$ ($= \mathbf{E}^* f_a \circ g_{ab}$) is the uniform modification of the common range carrier $\langle P, \mathscr{V} \rangle$ of all $f'_a \circ g'_{ab}$. By our assumption and 37 C.1 each mapping h_a is the identity mapping of each $\mathbf{E}^* g'_{ab}$, $b \in B_a$, onto its uniform modification. It follows that $\{f_a\}$ is an inductive generating family for uniform spaces if and only if $\{f_a \circ h_a\}$ is such. Finally, by 37 C.3 and 37 C.4 we find that $\{f_a \circ h_a\}$ is an inductive generating family for uniform spaces if and only if $\langle P, \mathscr{U} \rangle$ is the uniform modification of $\langle P, \mathscr{V} \rangle$, which completes the proof.

Direct proof. According to the preceding theorem it is sufficient to show that the statement "$\langle P, \mathscr{U} \rangle$ is a uniform space, and a mapping f of $\langle P, \mathscr{U} \rangle$ into a uniform space is uniformly continuous if and only if all the mappings $f \circ f_a$ are uniformly continuous" is equivalent to the statement "$\langle P, \mathscr{U} \rangle$ is a uniform space, and a mapping f of $\langle P, \mathscr{U} \rangle$ into a uniform space is uniformly continuous if and only if all the mappings $f \circ (f_a \circ g_{ab})$ are uniformly continuous". For each a the family $\{g_{ab}\}$ is an inductive generating family for uniform spaces and therefore, again by the preceding theorem, the mapping $f \circ f_a$ is uniformly continuous if and only if all the mappings $(f \circ f_a) \circ g_{ab}$, $b \in B_a$, are uniformly continuous (notice that this conclusion remains valid if we assume that $\{g_{ab} \mid b \in B_a\}$ is an inductive generating family; we must use 37 A.5). Since $f \circ (f_a \circ g_{ab}) = (f \circ f_a) \circ g_{ab}$, the proof is complete.

37 C.9. *In 37 C.8 the assumption that $\{g_{ab} \mid b \in B_a\}$ is an inductive generating family for uniform spaces can be replaced by the following formally weaker assumption: If $a \in A$, then $\{g_{ab} \mid b \in B_a\}$ is an inductive generating family for uniform spaces or an inductive generating family for semi-uniform spaces.*

Proof. The direct proof of the preceding theorem applies in this more general situation. On the other hand we shall show that 37 C.9 is an immediate corollary of the preceding theorem. Put

$$g'_{ab} = g_{ab} : \mathbf{D}^* g_{ab} \to \mathfrak{v}\mathbf{E}^* g_{ab}\,, \quad f'_a = f_a : \mathfrak{v}\mathbf{D}^* f_a \to \mathbf{E}^* f_a\,.$$

By 37 C.4 the family $\{f_a\}$ is an inductive generating family for uniform spaces if and only if $\{f'_a\}$ is such. Apply 37 C.8 to $\{g'_{ab}\}$ and $\{f'_a\}$ and notice that $f_a \circ g_{ab} = f'_a \circ g'_{ab}$.

As a straightforward consequence of 37 C.9 we obtain the following theorem which shows that the inductive generation for uniform spaces by a non-void family can be reduced to the construction of the sum semi-uniformity and the inductive generation for uniform spaces by a single mapping.

37 C.10. Theorem. *Let $\{f_a \mid a \in A\}$ be a non-void family of mappings of semi-uniform spaces into a semi-uniform space $\langle P, \mathscr{U}\rangle$. Then $\{f_a\}$ is an inductive generating family for uniform spaces if and only if the reduced sum f of $\{f_a\}$ is an inductive generating mapping for uniform spaces.*

Proof. We have $\mathbf{D}^* f = \Sigma\{\mathbf{D}^* f_a\}$ and $f\langle a, x\rangle = f_a x$. Thus $f \circ i_a = f_a$ for each a where i_a is the canonical embedding of $\mathbf{D}^* f_a$ into $\mathbf{D}^* f$. Since $\{i_a\}$ is an inductive generating family for semi-uniform spaces, the statement follows from 37 C.9.

37 C.11. If $\langle Q, \mathscr{V}\rangle$ is a subspace of a semi-uniform space $\langle P, \mathscr{U}\rangle$, $\{f_a\}$ is an inductive generating family for $\langle P, \mathscr{U}\rangle$ and $\mathscr{R}_a$ is the subspace of $\mathbf{D}^* f_a$ whose underlying set is $f_a^{-1}[Q]$, then $\langle Q, \mathscr{V}\rangle$ is inductively generated by the family $\{f_a : \mathscr{R}_a \to \to \langle Q, \mathscr{V}\rangle\}$. A similar result does not hold for the inductive generation for uniform spaces. E.g. let $\langle Q, \mathscr{V}\rangle$ be a subspace of a semi-uniform space $\langle P, \mathscr{U}\rangle$ such that $\langle Q, \mathfrak{v}\mathscr{V}\rangle$ is not a subspace of $\langle P, \mathfrak{v}\mathscr{U}\rangle$. Then $\mathsf{J} : \langle Q, \mathscr{V}\rangle \to \langle Q, \mathfrak{v}\mathscr{V}\rangle$ and $\mathsf{J} : : \langle P, \mathscr{U}\rangle \to \langle P, \mathfrak{v}\mathscr{U}\rangle$ are inductive generating mappings for uniform spaces and so on.

D. QUOTIENTS

As in the case of closure spaces and topological spaces, we shall introduce the definitions of a quotient, and of a quotient in the uniform sense, of a semi-uniform space under a mapping or an equivalence.

37 D.1. Definition. Let $\langle P, \mathscr{U}\rangle$ be a semi-uniform space. If f is a mapping whose domain carrier is $\langle P, \mathscr{U}\rangle$, then the *quotient* of $\langle P, \mathscr{U}\rangle$ under f, denoted by $\langle P, \mathscr{U}\rangle/f$ (the *quotient in the uniform sense* of $\langle P, \mathscr{U}\rangle$ under f, denoted by $\langle P, \mathscr{U}\rangle/_{\mathfrak{v}} f$), is the set $\mathbf{E} f$ endowed with the semi-uniformity inductively generated (with the uniformity inductively generated in the uniform sense) by the mapping $f : \langle P, \mathscr{U}\rangle \to \mathbf{E} f$. If ϱ is an equivalence on $\langle P, \mathscr{U}\rangle$ then the *quotient* of $\langle P, \mathscr{U}\rangle$ *under* ϱ, denoted by $\langle P, \mathscr{U}\rangle/\varrho$, (the *quotient in the uniform sense* of $\langle P, \mathscr{U}\rangle$ *under* ϱ, denoted by $\langle P, \mathscr{U}\rangle/_{\mathfrak{v}}\varrho$) is the space $\langle P, \mathscr{U}\rangle/\pi$ ($\langle P, \mathscr{U}\rangle/_{\mathfrak{v}}\pi$), where π is the canonical mapping of $\langle P, \mathscr{U}\rangle$ onto P/ϱ. More specifically we shall often write *uniform quotient* instead of quotient.

As a corollary of 37 C.1 we obtain:

37 D.2. *If $\mathscr{P}$ is any semi-uniform space, f is a mapping with $\mathbf{D}^*f = \mathscr{P}$ and ϱ is an equivalence on $\mathscr{P}$, then*

$$\mathscr{P}/_{\mathfrak{v}}f = \mathfrak{v}(\mathscr{P}/f)$$

$$\mathscr{P}/_{\mathfrak{v}}\varrho = \mathfrak{v}(\mathscr{P}/\varrho).$$

The quotient of a semi-uniform space under an equivalence was defined as the quotient under a certain mapping, and hence any investigation of quotients under equivalences reduces to an investigation of quotients under mappings. On the other hand the theorem which follows shows that any investigation of quotients under mappings also reduces to an investigation of quotients under equivalences. Hence we shall state various results for $\mathscr{P}/\varrho$ or $\mathscr{P}/f$, leaving the formulation for the other quotient to the reader.

37 D.3. *Let f be a mapping of a semi-uniform space $\mathscr{P}$ into any struct and let ϱ be the equivalence $\{x \to y \mid fx = fy\}$. Then there exist uniform homeomorphisms h and $h_{\mathfrak{v}}$ such that*

(∗) $\langle \operatorname{gr} f, \mathscr{P}, \mathscr{P}/f \rangle = h \circ \pi$

(∗∗) $\langle \operatorname{gr} f, \mathscr{P}, \mathscr{P}/_{\mathfrak{v}}f \rangle = h_{\mathfrak{v}} \circ \pi_{\mathfrak{v}}$

where π and $\pi_{\mathfrak{v}}$ are the canonical mappings of $\mathscr{P}$ onto $\mathscr{P}/\varrho$ and $\mathscr{P}/_{\mathfrak{v}}\varrho$. In particular, $\mathscr{P}/f$ is uniformly homeomorphic to $\mathscr{P}/\varrho$ and $\mathscr{P}/_{\mathfrak{v}}f$ is uniformly homeomorphic to $\mathscr{P}/_{\mathfrak{v}}\varrho$.

Proof. Equation (∗) defines exactly one mapping h which is bijective, and also (∗∗) defines exactly one mapping $h_{\mathfrak{v}}$ which is bijective; moreover, if we denote the left side of (∗) and (∗∗) by g and $g_{\mathfrak{v}}$ then also $\pi = h^{-1} \circ g$, $\pi_{\mathfrak{v}} = h_{\mathfrak{v}}^{-1} \circ g_{\mathfrak{v}}$. Since π and g are inductive generating mappings for semi-uniform spaces and $\pi_{\mathfrak{v}}$ and $g_{\mathfrak{v}}$ are inductive generating mappings for uniform spaces, we obtain from 37 A.5 and 37 C.7 respectively that the mappings h and h^{-1}, $h_{\mathfrak{v}}$ and $h_{\mathfrak{v}}^{-1}$ are uniformly continuous; this completes the proof.

To the projective concept "a uniform embedding into a space", i.e. a projective generating mapping for semi-uniform spaces which is also injective, there corresponds the inductive concept "a mapping of a space $\mathscr{P}$ onto a quotient of $\mathscr{P}$", i.e. an inductive generating mapping for semi-uniform spaces with $\mathbf{D}^*f = \mathscr{P}$ such that f is surjective. As an example we shall describe the inductive progeny of a class (compare with 37 B.8).

37 D.4. Theorem. *Let K be a class of semi-uniform spaces and let K_1 be the class of all uniformly discrete spaces. The inductive progeny of K consists of quotients of sums of families in $K \cup K_1$.*

Proof. Let K_2 be the class which consists of all spaces satisfying the condition. Clearly $K_2 \subset \operatorname{ind} K$. Let $\mathscr{P}$ be any space of ind K and let $\{f_a\}$ be an inductive generating family for $\mathscr{P}$ with domain carriers in K. The reduced sum f of $\{f_a\}$ is an inductive generating mapping for $\mathscr{P}$. Let $\mathscr{Q}$ be the uniformly discrete space such that $|\mathscr{P}| = |\mathscr{Q}|$.

The reduced sum g of $\mathsf{J} : \mathscr{Q} \to \mathscr{P}$ and f is an inductive generating mapping for $\mathscr{P}$, and clearly g is surjective. Thus $\mathscr{P} \in K_2$.

37 D.5 Corollary. *A class K of semi-uniform spaces is inductive-stable if and only if K contains all uniformly discrete spaces, K is closed under arbitrary sums and all quotients of spaces of K belong to K.*

37 D.6. By 37 A.8 every semi-uniform space is the quotient of a discrete uniform space. It follows that, in general, $\mathscr{P}/\varrho \neq \mathscr{P}/_{\upsilon}\varrho$ even if $\mathscr{P}$ is a uniform space, and $\gamma_{\mathbf{CU}}(\mathscr{P}/\varrho) \neq \gamma_{\mathbf{CU}}\mathscr{P}/\varrho$. Two questions naturally arise: under what necessary and sufficient conditions on $\mathscr{P} = \langle P, \mathscr{U}\rangle$ and ϱ do the equalities

$$\langle P, \mathscr{U}\rangle/\varrho = \langle P, \mathscr{U}\rangle/_{\upsilon}\varrho$$

$$\gamma_{\mathbf{CU}}(\langle P, \mathscr{U}\rangle/\varrho) = \gamma_{\mathbf{CU}}\langle P, \mathscr{U}\rangle/\varrho$$

hold. The most important sufficient conditions require compactness and other related concepts, and therefore will be not treated here (consult the exercises to 41). At present we shall restrict ourselves to a general discussion. It should be noted that according to the introduction to 37 D the solution of the problem whether the transpose of an inductive generating family for semi-uniform spaces to a family of mappings for closure spaces is an inductive generating family is easily reduced to the problem whether $\gamma_{\mathbf{CU}}(\mathbf{D}^*f_a/f_a) = \gamma_{\mathbf{CU}}\mathbf{D}^*f_a/f_a$ for each a.

37 D.7. Theorem. *In order that the quotient of a semi-uniform space $\langle P, \mathscr{U}\rangle$ under an equivalence ϱ be a uniform space it is necessary and sufficient that for each U in $\mathscr{U}$ there exist a U_1 in $\mathscr{U}$ such that $U_1 \circ \varrho \circ U_1 \subset \varrho \circ U \circ \varrho$.*

Proof. Write $\langle Q, \mathscr{V}\rangle = \langle P, \mathscr{U}\rangle/\varrho$. First we shall show that

(*) $\mathscr{V}$ consists of all $(\pi \times \pi)[U]$, $U \in \mathscr{U}$.

We know that $\mathscr{V}$ consists of all vicinities V of the diagonal of $Q \times Q$ such that $(\pi \times \pi)^{-1}[V] \in \mathscr{U}$. Since π is surjective, $\mathscr{V}$ coincides with all $V \subset Q \times Q$ such that $(\pi \times \pi)^{-1}[V] \in \mathscr{U}$. Since $(\pi \times \pi)[(\pi \times \pi)^{-1}[V]] = V$ for each $V \subset P \times P$, the auxiliary statement follows.

Let us consider the semi-uniformity $\mathscr{U}'$ projectively generated by the mapping $\pi : P \to \langle Q, \mathscr{V}\rangle$. Evidently, if $\mathscr{U}'$ is a uniformity then $\mathscr{V}$ is a uniformity, and it follows from the fact that $\upsilon\mathbf{U}$ is projective-stable, that if $\mathscr{V}$ is a uniformity then $\mathscr{U}'$ is a uniformity. Thus $\mathscr{U}'$ is a uniformity if and only if $\mathscr{V}$ is such. Now prove that $\mathscr{U}'$ is a uniformity if and only if the condition of 37 D.7 is fulfilled. First we shall show that

(**) $\mathscr{U}'$ has for a base the collection of all $\varrho \circ U \circ \varrho$, $U \in \mathscr{U}$.

The semi-uniformity $\mathscr{U}'$ has the collection of all $(\pi \times \pi)^{-1}[V]$, $V \in \mathscr{V}$, for a base, and hence by (*), the collection of all $(\pi \times \pi)^{-1}[(\pi \times \pi)[U]]$, $U \in \mathscr{U}$, is a base for $\mathscr{U}'$. On the other hand

$$(\pi \times \pi)^{-1}[(\pi \times \pi)[U]] = \bigcup\{\varrho[x] \times \varrho[y] \mid \langle x, y\rangle \in U\} = \varrho \circ U \circ \varrho$$

because ϱ is symmetric. The proof of (**) is complete.

Now $\mathcal{U}'$ is a uniformity if and only if for each U in $\mathcal{U}$ there exists a U_1 in $\mathcal{U}$ such that

$$(\varrho \circ U_1 \circ \varrho) \circ (\varrho \circ U_1 \circ \varrho) \subset \varrho \circ U \circ \varrho ,$$

i.e. $\varrho \circ U_1 \circ \varrho \circ U_1 \circ \varrho \subset \varrho \circ U \circ \varrho$ (we have $\varrho \circ \varrho = \varrho$). However if $W \subset \varrho \circ U \circ \varrho$ then $\varrho \circ W \circ \varrho \subset \varrho \circ U \circ \varrho$ because $\varrho \circ \varrho \circ U \circ \varrho \circ \varrho = \varrho \circ U \circ \varrho$. The proof is complete.

The quotient of a uniform space need not be a uniform space, in other words, the condition of 37 D.7 need not be fulfilled. Nevertheless, Theorem 37 D.7 is rather general, and therefore it seems to be useful to illustrate this theorem with simple example. It will be based on the following proposition.

37 D.8. *A quotient of a semi-pseudometrizable semi-uniform space is semi-pseudometrizable. In addition, if d semi-pseudometrizes $\langle P, \mathcal{U}\rangle$, and ϱ is an equivalence on $\langle P, \mathcal{U}\rangle$, then the quotient $\langle P, \mathcal{U}\rangle/\varrho$ is semi-pseudometrized by*

$$D = \{\langle X, Y\rangle \to \operatorname{dist}(X, Y)\}$$

Proof. As above let $\langle Q, \mathcal{V}\rangle$ stand for $\langle P, \mathcal{U}\rangle/\varrho$, and π for the canonical mapping of $\langle P, \mathcal{U}\rangle$ onto $\langle Q, \mathcal{V}\rangle$. Obviously D is a semi-pseudometric for $\langle Q, \mathcal{V}\rangle$. For each positive real r put $U_r = \mathbf{E}\{\langle x, y\rangle \mid d(x, y) < r\}$ and $V_r = \mathbf{E}\{\langle X, Y\rangle \mid D(X, Y) < r\}$. Since evidently $D\langle X, Y\rangle < r$ if and only if $d(x, y) < r$ for some x in X and y in Y, we have $(\pi \times \pi)[U_r] = V_r$ for each positive real r. The semi-uniformity $\mathcal{V}$ consists of all $(\pi \times \pi)[U]$, $U \in \mathcal{U}$, and consequently, $\{U_r \mid r > 0, r \in \mathrm{R}\}$ being a base of $\mathcal{U}$, $\{V_r \mid r > 0, r \in \mathrm{R}\}$ is a base of $\mathcal{V}$; in other words, D semi-pseudometrizes $\mathcal{V}$.

37 D.9. Example. Suppose that a semi-uniform space $\langle P, \mathcal{U}\rangle$ is semi-pseudometrized by d and let D be the semi-pseudometric for $\mathcal{P}/\varrho$ defined in 37 D.8. By 24 A.3 D is a uniformly equivalent to a pseudometric if and only if for each $r > 0$ there exists an $s > 0$ such that $D\langle X, Y\rangle < s$, $D\langle Y, Z\rangle < s$ imply that $D\langle X, Z\rangle < r$. Notice that this condition coincides with the condition of 37 D.7.

We now proceed to an examination of the validity of the second formula of 37 D.6. In what follows, unless otherwise stated, $\langle P, \mathcal{U}\rangle$ will be a semi-uniform space, ϱ will be an equivalence on $\langle P, \mathcal{U}\rangle$, π will be the canonical mapping of $\langle P, \mathcal{U}\rangle$ onto the quotient space $\langle P, \mathcal{U}\rangle/\varrho$ which will be denoted by $\langle Q, \mathcal{V}\rangle$. We have shown that

(1) $$\mathcal{V} = \mathbf{E}\{(\pi \times \pi)[U] \mid U \in \mathcal{U}\} .$$

By the definition of induced closures ,

(2) $$\mathbf{E}\{((\pi \times \pi)[U])[(X)] \mid U \in \mathcal{U}\}$$

is the neighborhood system at the point X in $\langle Q, \gamma_{\mathbf{CU}}\mathcal{V}\rangle$.

It is easily seen that $((\pi \times \pi)[U])[(X)] = \pi[U[X]]$ for each $X \in Q$ and $U \subset P \times P$ (notice that $X \in Q \Rightarrow \pi^{-1}[(X)] = X$), and hence we have that

(3) $\mathbf{E}\{\pi[U[X]] \mid U \in \mathcal{U}\}$ is the neighborhood system at X in $\langle Q, \gamma_{\mathbf{CU}}\mathcal{V}\rangle$.

It follows from (3) that the closure $\gamma_{\mathbf{CU}}\mathscr{V}$ depends only on the proximity induced by $\mathscr{U}$; more precisely,

37 D.10. *If $\mathscr{U}_1$ and $\mathscr{U}_2$ are two proximally equivalent semi-uniformities for a set P, then the closure spaces $\gamma_{\mathbf{CU}}(\langle P, \mathscr{U}_1\rangle / \varrho)$ and $\gamma_{\mathbf{CU}}(\langle P, \mathscr{U}_2\rangle \mid \varrho)$ coincide for each equivalence ϱ on P.*

Proof. If $\mathscr{U}_1$ and $\mathscr{U}_2$ are proximally equivalent, then the collections $\mathbf{E}\{U[X] \mid U \in \in \mathscr{U}_1\}$ and $\mathbf{E}\{U[X] \mid U \in \mathscr{U}_2\}$ coincide for each $X \subset P$.

Now let us return to our problem. Denote by v the closure inductively generated by the mapping $\pi : \langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle \to Q$, in other words, v is defined by $\langle Q, v\rangle = = \langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle / \varrho$. By the description of neighborhoods in quotients of closure spaces,

(4) the collection $\mathbf{E}\{\pi[G] \mid G$ is a neighborhood of the set X in $\langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle\}$ is the neighborhood system at the point $X \in Q$ in $\langle Q, v\rangle$.

It follows from (3) and (4) that $\gamma_{\mathbf{CU}}\mathscr{V} = v$ provided that every neighborhood G of each $X \in Q$ contains a $U[X]$ for some $U \in \mathscr{U}$ (remember that every $U[X]$, $U \in \mathscr{U}$, is a neighborhood of X). Stated in other words, $\gamma_{\mathbf{CU}}\mathscr{V} = v$ provided that every neighborhood of each set $X \in Q$ in $\langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle$ is a proximal neighborhood relative to the proximity induced by $\mathscr{U}$. It is useful to state the result which has just been proved as a theorem.

37 D.11. Theorem. *In order that $\gamma_{\mathbf{CU}}(\langle P, \mathscr{U}\rangle / \varrho) = \gamma_{\mathbf{CU}}\langle P, \mathscr{U}\rangle / \varrho$ it is sufficient that every neighborhood in $\langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle$ of every set $\varrho[x]$, $x \in P$, be a proximal neighborhood, i.e. contain a set of the form $U[\varrho[x]]$ for some U in $\mathscr{U}$, in other words,*

$$\mathbf{E}\{(U \circ \varrho)[x] \mid U \in \mathscr{U}\}$$

be a base for the neighborhood system of the set $\varrho[x]$ in $\langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle$ for each $x \in P$.

Corollary. *If the sets $\varrho[x]$, $x \in P$, are closed in $\langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle$, $\langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle$ is a normal space and $\mathscr{U}$ is a semi-uniformity inducing the Čech proximity for $\langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle$, then the equality holds. In particular, if the sets $\varrho[x]$, $x \in P$ are closed, $\langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle$ is a normal space and $\mathscr{U}$ is the fine or the Čech uniformity for $\langle P, \gamma_{\mathbf{CU}}\mathscr{U}\rangle$, then the* equality *holds.*

Remark. The condition in Theorem 37 D.11 is not necessary. A more detailed discussion is given in 39 D.

38. ORDERED SETS OF PROXIMITIES

This section is concerned with the development of order properties of the ordered class **P** of all proximities. The results of this section will be applied to projective and inductive generation for proximity spaces in 39.

In subsection A we shall prove that **P** is boundedly order-complete and we shall describe infima and suprema in **P**. In subsection B we shall prove that the canonical mapping of **U** into **P** (which assigns to each semi-uniformity $\mathscr{U}$ the proximity induced by $\mathscr{U}$) is completely join-preserving (but not meet-preserving), and the canonical mapping of **P** into **C** (which assigns to each proximity p the closure induced by p) is completely lattice-preserving. In this connection the following concepts will be introduced: proximally fine semi-uniformity (proximally coarse semi-uniformities were introduced in 25 B), fine proximity and coarse proximity. Subsection C concerns the class $\mathfrak{v}$**P** of all uniformizable proximities. We shall prove that $\mathfrak{v}$**P** is completely meet-preserving and completely meet-stable in **P**, the canonical mapping of $\mathfrak{v}$**U** into $\mathfrak{v}$**P** is completely join-preserving (but not meet-preserving), and the canonical mapping of $\mathfrak{v}$**P** into $\mathfrak{v}$**C** is completely meet-preserving (but not join-preserving). The following concepts will be introduced: proximally fine (coarse) uniformity, fine (coarse) uniformizable proximity.

A. ORDERED CLASS P

By definition 25 A.1, a proximity for a set P is a relation p for $\exp P$ and $\exp P$ satisfying certain conditions. A proximity space is a pair $\langle P, p\rangle$ such that P is a set and p is a proximity for P. If p is a proximity for P then $\bigcup \mathbf{D} p = P$. By definition 25 A.7, a mapping f of a proximity space $\langle P, p\rangle$ into another one $\langle Q, q\rangle$ is said to be proximally continuous if XpY implies $f[X]\, q\, f[Y]$. A proximity p is said to be proximally coarser than a proximity q, and q is said to be proximally finer than p, if both proximities are for the same set, say P, and the identity mapping of $\langle P, q\rangle$ onto $\langle P, p\rangle$ is proximally continuous; stated in other words, p is proximally coarser than q if and only if XqY implies XpY (i.e. $q \subset p$) and $\bigcup \mathbf{D} q = \bigcup \mathbf{D} p$.

The relation $\{p \rightarrow q \mid p \text{ is proximally finer than } q\}$ is an order; the class of all proximities ordered by this relation is denoted by **P**, and if P is a set then $\mathbf{P}(P)$ denotes the ordered subset of **P** consisting of all proximities for P.

It is to be noted that the order structure of **P** coincides with the restriction of the inclusion $\subset$, and hence $\mathbf{P}(P)$ is an ordered subset of $\langle \exp(\exp P \times \exp P), \subset \rangle$.

38 A.1. Theorem. *Let P be a set. The ordered set $\mathbf{P}(P)$ is order-complete. The proximally finest proximity for P consists of all $\langle X, Y \rangle$ such that $X \subset P$, $Y \subset P$, $X \cap Y \neq \emptyset$. The proximally coarsest proximity for P consists of all pairs $\langle X, Y \rangle$ such that $P \supset X \neq \emptyset \neq Y \subset P$. If $\{p_a \mid a \in A\}$ is a non-void family in $\mathbf{P}(P)$, then*

$$\sup \{p_a \mid a \in A\} = \bigcup \{p_a \mid a \in A\} .$$

Proof. I. Let us consider the relations $p' = \{X \to Y \mid X \subset P, Y \subset P, X \cap Y \neq \emptyset\}$ and $p'' = \{X \to Y \mid P \supset X \neq \emptyset \neq Y \subset P\}$. It follows from the definition of proximities that $p' \subset p \subset p''$ for each proximity p for P ($p' \subset p$ follows from (prox 3) and $p \subset p''$ from (prox 1)). As a consequence, to prove that p' is the proximally finest proximity for P and p'' is the proximally coarsest proximity for P it remains to show that both p' and p'' are proximities. The verification of the corresponding conditions (prox i) is simple and may be left to the reader. – II. Now let $\{p_a \mid a \in A\}$ be a non-void family of proximities for P and consider the relation $p = \bigcup \{p_a \mid a \in A\}$. The reader may show without difficulty that p is a proximity for P. Clearly p is the least upper bound of $\{p_a\}$ in $\mathbf{P}(P)$. – III. The ordered set $\mathbf{P}(P)$ is order-complete because it possesses a least element and a greatest element by I, and every non-void family has a least upper bound.

38 A.2. Corollary. *The set $\mathbf{P}(P)$ is completely join-stable in the ordered set $\langle \exp(\exp P \times \exp P), \subset \rangle$.*

38 A.3. On the other hand $\mathbf{P}(P)$ is not meet-stable in $\langle \exp(\exp P \times \exp P), \subset \rangle$ whenever P has at least three points. Indeed, if x_0, x_1, x_2 are distinct elements of P and p_i, $i = 1, 2$, are proximities for P such that $(x_0)\, p_i\, (x_i)$ and (x_0) non $p_i\, (x_j)$ for $i \neq j$, $j = 1, 2$, and $p = p_1 \cap p_2$, then (x_0) non $p\, (x_i)$, $i = 1, 2$, but $(x_0)\, p\, (x_1, x_2)$. Thus p does not fulfil (prox 4) and hence p is not a proximity.

38 A.4. *If $\{p_a\}$ is a non-void family in $\mathbf{P}(P)$ and $p = \bigcap \{p_a\}$ is a proximity, then p is the infimum of $\{p_a\}$ in $\mathbf{P}(P)$.*

It is easily seen that p fulfils conditions (prox 1)–(prox 3). By 38 A.3 condition (prox 4) need not be fulfilled. If the family $\{p_a\}$ is range down-directed, then (prox 4) is fulfilled (and hence p is a proximity). In fact, if X non $p\, X_i$, $i = 1, 2$, then X non $p_{a_i} X_i$ for some a_i, and $\{p_a\}$ being down-directed, there exists an a so that p_a is proximally finer than p_{a_i}, $i = 1, 2$; clearly X non $p_a\, X_i$, $i = 1, 2$, and hence X non $p_a\, (X_1 \cup X_2)$ and hence X non $p\, (X_1 \cup X_2)$.

For the sake of completeness we recall a direct description of inf $\{p_a\}$ which was established in 25 E.7.

38 A.5. *If p is the infimum of a non-void family $\{p_a\}$ in $\mathbf{P}(P)$ then XpY if and only if $X \subset P$, $Y \subset P$ and the following condition is fulfilled: If $\{X_i\}$ is a finite decomposition of X and $\{Y_j\}$ is a finite decomposition of Y, then $X_i p_a Y_j$ for some i, j and each a.*

38 A.6. Theorem. *Let p be the supremum of a non-void family $\{p_a\}$ in $\mathbf{P}(P)$ and let X be a subset of P. A set U is a proximal neighborhood of X in $\langle P, p\rangle$ if and only if U is a proximal neighborhood of X for each a. Stated in other words, if $\mathcal{U}$ is the set of all proximal neighborhoods of X in $\langle P, p\rangle$ and if $\mathcal{U}_a$ is the set of all proximal neighborhoods of X in $\langle P, p_a\rangle$ for each a, then $\mathcal{U} = \bigcap\{\mathcal{U}_a\}$.*

Proof. By 38 A.1 we have $(P - U)$ non $p\,X$ if and only if $(P - U)$ non $p_a\,X$ for each a.

If $\mathcal{U}$ is the set of all proximal neighborhoods of a set X in $\langle P, p\rangle$, then $\mathcal{U}$ is a filter on P which is proper if and only if $X \neq \emptyset$.

38 A.7. Theorem. *Let p be the infimum of a non-void family $\{p_a\}$ in $\mathbf{P}(P)$ and let $X \subset P$. If $\mathcal{U}_{aY}$ is the collection of all neighborhoods of an Y in $\langle P, p_a\rangle$ and $\mathcal{U}_Y = \bigcup\{\mathcal{U}_{aY}\}$, then the set of all $\bigcup\{U_i\}$ with $U_i \in \mathcal{U}_{X_i}$ and $\{X_i\}$ a finite cover of X is a sub-base for the filter consisting of all proximal neighborhoods of X in $\langle P, p\rangle$.*

This is an immediate consequence of 38 A.5. In conclusion we shall state an important result, leaving the proof to the reader.

38 A.8. *Let $\{p_a\}$ be a family in $\mathbf{P}(P)$, $\{q_a\}$ be a family in $\mathbf{P}(Q)$ and let f be a mapping of P into Q. If all the mappings $f : \langle P, p_a\rangle \to \langle Q, q_a\rangle$ are proximally continuous, then the mappings $f : \langle P, \sup\{p_a\}\rangle \to \langle Q, \sup\{q_a\}\rangle$ and $f : \langle P, \inf\{p_a\}\rangle \to \langle Q, \inf\{q_a\}\rangle$ are also proximally continuous.*

B. INTERRELATIONS BETWEEN U, P AND C

If $\mathcal{U}$ is a semi-uniformity for a set P, then

$$p = \mathbf{E}\{\langle X, Y\rangle \mid X \subset P, Y \subset P, U \in \mathcal{U} \Rightarrow U[X] \cap Y \neq \emptyset\}$$

is a proximity for P which is said to be induced by $\mathcal{U}$ (cf. 25 A.1). If p is a proximity for a set P then the relation $\{X \to uX \mid X \subset P\}$, where $uX = \mathbf{E}\{x \mid (x)\, p\, X\}$, is a closure operation for P which is said to be induced by p (25 A.1). Finally, let us recall that every semi-uniformity $\mathcal{U}$ for P induces the closure $\{X \to uX\}$, where

$$uX = \mathbf{E}\{x \mid U \in \mathcal{U} \Rightarrow U[(x)] \cap X \neq \emptyset\},$$

which is denoted by $\gamma_{\mathbf{CU}}\mathcal{U}$ or merely $\gamma\mathcal{U}$, and the relation $\{\mathcal{U} \to \gamma\mathcal{U}\}$ is denoted by $\gamma_{\mathbf{CU}}$ or merely γ. The mapping $\gamma : \mathbf{U} \to \mathbf{C}$ which is termed the canonical mapping of $\mathbf{U}$ into $\mathbf{C}$ was examined in 36 A.

38 B.1. Definition. Let $\gamma_{\mathbf{PU}}$ be the single-valued relation which assigns to each semi-uniformity the proximity induced by $\mathcal{U}$, and let $\gamma_{\mathbf{CP}}$ be the single-valued relation which assigns to each proximity p the closure induced by p. The symbols $\gamma_{\mathbf{PU}}$ and $\gamma_{\mathbf{CP}}$ will also be used to denote the relations $\{\langle P, \mathcal{U}\rangle \to \langle P, \gamma_{\mathbf{PU}}\mathcal{U}\rangle\}$ and $\{\langle P, p\rangle \to \langle P, \gamma_{\mathbf{CP}}p\rangle\}$. The restrictions of the mappings $\gamma_{\mathbf{PU}} : \mathbf{U} \to \mathbf{P}$ and $\gamma_{\mathbf{CP}} : \mathbf{P} \to \mathbf{C}$ will be called the *canonical mappings*; e.g. if P is a set then the mapping $\gamma_{\mathbf{PU}} : \mathbf{U}(P) \to \mathbf{P}(P)$ will be called the canonical mapping of $\mathbf{U}(P)$ into $\mathbf{P}(P)$ and $\gamma_{\mathbf{CP}} : \mathbf{P}(P) \to \mathbf{C}(P)$ will be called the canonical mapping of $\mathbf{P}(P)$ into $\mathbf{C}(P)$.

For convenience our earlier results will be restated as follows:

38 B.2. Theorem. *The range of* $\gamma_{\mathbf{PU}}$ *is the class of all proximities and* $\gamma_{\mathbf{CU}} = \gamma_{\mathbf{CP}} \circ \gamma_{\mathbf{PU}}$. *In particular, given a set* P, *the canonical mapping of* $\mathbf{U}(P)$ *into* $\mathbf{P}(P)$ *is surjective and the canonical mapping of* $\mathbf{U}(P)$ *into* $\mathbf{C}(P)$ *is the composition of the canonical mapping of* $\mathbf{U}(P)$ *onto* $\mathbf{P}(P)$ *followed by the canonical mapping of* $\mathbf{P}(P)$ *into* $\mathbf{C}(P)$.

Proof. The fact that each proximity is induced by a semi-uniformity was explicitly stated in 25 B.9. It is to be noted that the proof of this was performed by the construction of the proximally coarse semi-uniformity inducing a given proximity. The equality $\gamma_{\mathbf{CU}} = \gamma_{\mathbf{CP}} \circ \gamma_{\mathbf{PU}}$ was explicitly stated in 25 A.2; its proof is almost self-evident.

It has already been proved that the canonical mappings $\gamma_{\mathbf{PU}}$ and $\gamma_{\mathbf{CP}}$ are order-preserving. Now we shall prove essentially more for $\gamma_{\mathbf{CP}}$ as well as for $\gamma_{\mathbf{PU}}$. Recall that, by 36 A.3, the canonical mapping of $\mathbf{U}(P)$ into $\mathbf{C}(P)$ is completely lattice-preserving. The next theorem asserts that the same is true for $\gamma_{\mathbf{CP}}$ and the theorem following it asserts that $\gamma_{\mathbf{PU}} : \mathbf{U}(P) \to \mathbf{P}(P)$ is completely join-preserving. On the other hand, $\gamma_{\mathbf{PU}}$ does not preserve infima since, in general, there exists no finest semi-uniformity inducing a given proximity.

38 B.3. Theorem. *If* P *is a set then the canonical mapping of* $\mathbf{P}(P)$ *into* $\mathbf{C}(P)$ *is completely lattice-preserving. The canonical mapping of* $\mathbf{P}$ *into* $\mathbf{C}$ *is completely lattice-preserving.*

Proof. Evidently the two statements are equivalent. We shall prove first statement. Let P be a set, $\{p_a\}$ be a non-void family in $\mathbf{P}(P)$, u_a be induced by p_a. We must show that $\sup\{u_a\}$ is induced by $\sup\{p_a\}$ and $\inf\{u_a\}$ is induced by $\inf\{p_a\}$. First let $p = \sup\{p_a\}$, $u = \sup\{u_a\}$. If $(x)\, p\, X$, then $(x)\, p_a X$ for some a (by 38 A.1), hence $x \in u_a X$ which implies $x \in uX$. Conversely, if $x \in uX$, then $x \in u_a X$ for some a (by 31 A.2) and hence $(x)\, p_a X$ which implies $(x)\, p\, X$. Thus $(x)\, p\, X$ if and only if $x \in uX$. Now let $p = \inf\{p_a\}$, $u = \inf\{u_a\}$ and let x be any element of P. Remember that proximal neighborhoods of a point coincide with neighborhoods in the induced closure space. Let $\mathscr{U}_a$ be the neighborhood system of x in $\langle P, u_a\rangle$ for each a; thus $\mathscr{U}_a$ is the set of proximal neighborhoods of x in $\langle P, p_a\rangle$. By 31 A.5 the union $\mathscr{U}$ of $\{\mathscr{U}_a\}$ is a local sub-base at x in $\langle P, u\rangle$ and by 38 A.7 the same collection is a sub-base for the filter of all proximal neighborhoods of x in $\langle P, p\rangle$. Consequently $U \subset P$ is a proximal neighborhood of x in $\langle P, p\rangle$ if and only if $U \subset P$ is a neighborhood of x in $\langle P, u\rangle$, and therefore u is induced by p.

38 B.4. Theorem. *Let* P *be a set. The canonical mapping of* $\mathbf{U}(P)$ *into* $\mathbf{P}(P)$ *is completely join-preserving. The canonical mapping of* $\mathbf{U}$ *into* $\mathbf{P}$ *is completely join-preserving.*

Proof. Evidently both statements are equivalent. We shall prove the first. Suppose that $\{\mathscr{U}_a\}$ is a non-void family in $\mathbf{U}(P)$, $\mathscr{U} = \sup \{\mathscr{U}_a\}$, p_a is induced by $\mathscr{U}_a$ for each a and p is the supremum of $\{p_a\}$ in $\mathbf{P}(P)$. We must show that p is induced by $\mathscr{U}$, i.e. XpY if and only if $X \subset P$, $Y \subset P$ and $U[X] \cap Y \neq \emptyset$ for each U in $\mathscr{U}$. By 36 A.1 we have $\mathscr{U} = \bigcap\{\mathscr{U}_a\}$ and by 38 A.1 $p = \bigcup\{p_a\}$. If XpY, then Xp_aY for some a and hence $U[X] \cap Y \neq \emptyset$ for each U in $\mathscr{U}_a$, and hence, each U in $\mathscr{U} \subset \mathscr{U}_a$. If X non pY, then X non p_a Y for each a, and hence there exists a family $\{U_a\}$ such that $U_a \in \mathscr{U}_a$, $U_a[X] \cap Y \neq \emptyset$; clearly $U = \bigcup\{U_a\}$ belongs to $\mathscr{U}$ and $U[X] \cap Y = \bigcup\{U_a[X] \cap Y\} = \emptyset$. The proof is complete.

Recall that if $\mathscr{P}$ is a closure space then the symbol $\mathbf{U}(\mathscr{P})$ denotes the ordered set of all semi-uniformities inducing the closure structure of $\mathscr{P}$. A similar notation will be introduced for proximities on closure spaces and semi-uniformities on proximity spaces.

38 B.5. Remark. Let P be a set. If u is a closure for P then $\mathbf{P}(P, u)$ denotes the ordered set of all proximities inducing u. If p is a proximity for P, then $\mathbf{U}(P, p)$ denotes the ordered set of all semi-uniformities which induce p. If $\mathscr{P} = \langle P, u\rangle$ or $\mathscr{P} = \langle P, p\rangle$ then $\mathbf{P}(\mathscr{P})$ or $\mathbf{U}(\mathscr{P})$ stands for $\mathbf{P}(P, u)$ or $\mathbf{U}(P, p)$, respectively.

The set $\mathbf{U}(P, p)$ is always non-void because each proximity is induced by a semi-uniformity. On the other hand, a set $\mathbf{P}(P, u)$ is non-void if and only if $\mathbf{U}(P, u)$ is non-void i.e. u is semi-uniformizable. In addition, by 38 B.2,

$$|\mathbf{U}(P, u)| = \bigcup\{|\mathbf{U}(P, p)| \mid p \in \mathbf{P}(P, u)\}$$

for each closure u for P.

38 B.6. Theorem. *If $\mathscr{P} = \langle P, u\rangle$ is a closure space, then the set $\mathbf{P}(\mathscr{P})$ is completely lattice-stable in $\mathbf{P}(P)$ and hence completely lattice-preserving in $\mathbf{P}(P)$.*

Proof. If $\{p_a \mid a \in A\}$ is a non-void family in $\mathbf{P}(P, u)$ then p_a induces u for each a, and by 38 B.3, $\sup \{p_a\}$ induces $\sup \{u \mid a \in A\} = u$ and $\inf \{p_a\}$ induces $\inf \{u \mid a \in A\} = u$, and hence $\sup \{p_a\}$ as well as $\inf \{p_a\}$ belong to $\mathbf{P}(P, u)$.

Corollary. *For each closure space $\mathscr{P}$ the ordered set $\mathbf{P}(\mathscr{P})$ is order-complete. In particular, if $\mathbf{P}(\mathscr{P}) \neq \emptyset$, i.e. $\mathscr{P}$ is semi-uniformizable, then $\mathbf{P}(\mathscr{P})$ has a smallest and a greatest element.*

Recall that a semi-uniformity $\mathscr{U}$ is said to be fine (coarse) if $\mathscr{U}$ is the uniformly finest (uniformly coarsest) semi-uniformity inducing the closure $\gamma\mathscr{U}$. Similarly we shall introduce the concepts of a fine and a coarse proximity.

38 B.7. Definition. A *proximity* p is said to be *fine* (*coarse*) if p is the proximally finest (proximally coarsest) proximity inducing the closure γp.

It follows from the corollary to 38 B.6 that if $\mathscr{P}$ is a semi-uniformizable closure space then the greatest element (least element) of $\mathbf{P}(\mathscr{P})$ is the unique coarse (fine) element of $\mathbf{P}(\mathscr{P})$. We shall prove somewhat more.

38 B.8. *If u is a closure for P then there exists a proximally finest proximally fine (coarse) continuous proximity for $\langle P, u\rangle$.*

Proof. Let v be the semi-uniformizable modification of u. The unique fine element of $\mathbf{P}(P, v)$ and the unique coarse element of $\mathbf{P}(P, v)$ have the required properties.

38 B.9. Theorem. *In order that a proximity p for a set P be fine it is necessary and sufficient that XpY imply that either $(x)\,pY$ for some $x \in X$ or $Xp\,(y)$ for some $y \in Y$. In order that a proximity p for a set P be coarse it is necessary and sufficient that every two infinite subsets of P be proximal in $\langle P, p\rangle$.*

Proof. Let p be a proximity for a set P and let u be the closure induced by p, i.e. $x \in uX$ if and only if $(x)p\,X$. Let q_1 consist of all $\langle X, Y\rangle$ such that X and Y are not semi-separated in $\langle P, u\rangle$ and let q_2 consist of all $\langle X, Y\rangle$ such that $X \subset P$, $Y \subset P$ and either both sets X and Y are infinite or $X\, q_1 Y$. It is easily seen that q_1 is the proximally finest continuous proximity for $\langle P, u\rangle$ and q_2 is the proximally coarsest proximity inducing u. It is easily seen that $p = q_1$ $(p = q_2)$ if and only if the condition of the first (the second) statement is fulfilled.

38 B.10. Theorem. *If p is a proximity for a set P, then the set $\mathbf{U}(P, p)$ is completely join-stable and completely lattice-preserving in $\mathbf{U}(P)$.*

Proof. Let $\{\mathscr{U}_a \mid a \in A\}$ be a non-void family in $\mathbf{U}(P, p)$. Each $\mathscr{U}_a$ induces p and hence, by 38 B.4, $\sup\{\mathscr{U}_a\}$ induces $\sup\{p \mid a \in A\} = p$.

To prove that $\mathbf{U}(P, p)$ is completely lattice-preserving in $\mathbf{U}(P)$ it is sufficient to show that $\mathbf{U}(P, p)$ is interval-like in $\mathbf{U}(P)$; but this is evident.

Corollary. *If $\mathscr{P}$ is a proximity space then the ordered set $\mathbf{U}(\mathscr{P})$ is boundedly order-complete and has a greatest element.* – Remember that $\mathbf{U}(\mathscr{P})$ is non-void.

We have introduced the concept of a proximally coarse semi-uniformity (25 B.8). By 25 B.7 a semi-uniformity $\mathscr{U}$ is proximally coarse if and only if $\mathscr{U}$ is the greatest element of $\mathbf{U}(P, p)$ where p is the proximity induced by $\mathscr{U}$. For the sake of completeness we shall state the characterization 25 B.8 of proximally coarse semi-uniformities.

38 B.11. Theorem. *A semi-uniformity $\mathscr{U}$ is proximally coarse if and only if the finite square elements of $\mathscr{U}$ form a base for $\mathscr{U}$.*

38 B.12. Theorem. *Let P be a set. The set T of all proximally coarse elements of $\mathbf{U}(P)$ is completely meet-stable in $\mathbf{U}(P)$, and the canonical mapping of T into $\mathbf{P}(P)$ is an order-isomorphism.*

Proof. I. The canonical mapping of T into $\mathbf{P}(P)$ is bijective and order-preserving and therefore an isomorphism. – II. Now let $\{\mathscr{U}_a \mid a \in A\}$ be a non-void family in T and let $\mathscr{V}_a$ be the collection of all finite square elements of $\mathscr{U}_a$ for each a. Thus $\mathscr{V}_a$ is a base for $\mathscr{U}_a$. The union $\mathscr{V}$ of $\mathscr{V}_a$ is a sub-base for $\inf\{\mathscr{U}_a\}$. Since $\mathscr{V}$ consists of finite square vicinities, the smallest filter base containing $\mathscr{V}$ also consists of finite square vicinities and therefore, by 38 B.11, $\inf\{\mathscr{U}_a\}$ is proximally coarse. Thus T is completely meet-stable in $\mathbf{U}(P)$. An alternate proof follows from 25 B.20.

Remark. The supremum in $\mathbf{P}(P)$ of proximally coarse semi-uniformities need not be a proximally coarse semi-uniformity (see 25 ex. 9). On the other hand, if the supremum of a family of proximally coarse semi-uniformities is a uniformity $\mathscr{U}$, then $\mathscr{U}$ is proximally coarse because $\mathscr{U}$ is totally bounded and a totally bounded uniformity is proximally coarse.

38 B.13. Remark. The mapping $\gamma : \mathbf{U} \to \mathbf{P}$ is not meet-preserving, i.e., if $\mathscr{U}$ and $\mathscr{V}$ are semi-uniformities for a set Q and p and q are proximities induced by $\mathscr{U}$ and $\mathscr{V}$ respectively, then $\inf(\mathscr{U}, \mathscr{V})$ need not induce $\inf(p, q)$ (even if $p = q$). This was shown by Example 25 B.10 where we proved that if P is an infinite set, $\mathscr{W}_1$ is the uniformly finest uniformity for P, $\mathscr{W}_2$ is the uniformly finest proximally coarse uniformity for P, then $\mathscr{W}_1 \times \mathscr{W}_2$ and $\mathscr{W}_2 \times \mathscr{W}_1$ are proximally equivalent and $\inf(\mathscr{W}_1 \times \mathscr{W}_2, \mathscr{W}_2 \times \mathscr{W}_1)$ is the uniformly finest uniformity for $P \times P$ which induces the proximally finest proximity for $P \times P$; but $\mathscr{W}_2 \times \mathscr{W}_1$ as well as $\mathscr{W}_1 \times \mathscr{W}_2$ induces a proximity which is not the proximally finest proximity for $P \times P$.

On the other hand, the following proposition holds.

38 B.14. *If at least one of the semi-uniformities $\mathscr{U}$ or $\mathscr{V}$ is proximally coarse, then the proximity induced by* $\inf(\mathscr{U}, \mathscr{V})$ *coincides with the infimum of the proximities induced by $\mathscr{U}$ and $\mathscr{V}$.*

Proof. Suppose that $\mathscr{U}$ is proximally coarse and $\mathscr{U}'$ is the set of all finite square elements of $\mathscr{U}$. The set $\mathscr{W}$ of all $U \cap V$, $U \in \mathscr{U}'$, $V \in \mathscr{V}$, is a base for $\inf(\mathscr{U}, \mathscr{V})$. If p is induced by $\mathscr{U}$, q is induced by $\mathscr{V}$ and $r = \inf(p, q)$, then each element of $\mathscr{U}$ as well as each element of $\mathscr{V}$ is an r-proximal vicinity. Since each element of $\mathscr{U}'$ is finite square, by 25 B.6 each element of $\mathscr{W}$ is an r-proximal vicinity. Consequently $\mathscr{W}$ is a base for a proximally continuous semi-uniformity for $\langle P, r\rangle$. It is evident that $\inf(\mathscr{U}, \mathscr{V})$ always induces a proximity proximally finer than r, and therefore $\inf(\mathscr{U}, \mathscr{V})$ induces r.

Corollary. *Let P be a set and let $\{\mathscr{U}_a\}$ be a non-void family in* $\mathbf{U}(P)$. *If all the $\mathscr{U}_a$, excepting at most one, are proximally coarse, then* $\inf\{\mathscr{U}_a\}$ *induces* $\inf\{p_a\}$, *where p_a is induced by $\mathscr{U}_a$ for each a.* — 38 B.12, 38 B.14.

38 B.15. Definition. A semi-uniformity $\mathscr{U}$ is said to be *proximally fine* if $\mathscr{U}$ is the uniformly finest semi-uniformity inducing the proximity induced by $\mathscr{U}$.

Evidently any fine semi-uniformity is proximally fine. It follows that if p is a fine proximity then there exists a proximally fine semi-uniformity which induces p; it coincides with the fine semi-uniformity which induces the closure induced by p. In general a proximity need not be induced by a proximally fine semi-uniformity; e.g. in 38 B.13 the proximity induced by $\mathscr{W}_1 \times \mathscr{W}_2$ is induced by no proximally fine semi-uniformity.

38 B.16. *A semi-uniformity $\mathscr{U}$ for a set P is proximally fine if and only if $\mathscr{U}$ contains each vicinity V of the diagonal of $P \times P$ which has the following property: For each $X \subset P$ there exists a U in $\mathscr{U}$ such that $U[X] \subset V[X]$.*

Proof. Let p be the proximity induced by $\mathscr{U}$. The property of V stated is equivalent to the fact that V is a p-proximal vicinity. It follows from 38 B.14 that $\mathscr{U}$ is proximally fine only if $\mathscr{U}$ consists of all p-proximal vicinities. On the other hand "if" is evident.

C. UNIFORMIZABLE PROXIMITIES

Recall that a uniformizable proximity is a proximity induced by a uniformity. We begin with a review of some earlier results.

38 C.1. (a) *Each of the following conditions is necessary and sufficient for a proximity p for a set P to be uniformizable:*

(α) *every two non-proximal subsets of $\langle P, p\rangle$ have disjoint proximal neighborhoods* (25 B.2);

(β) $p = \mathbf{E}\{\langle X, Y\rangle \mid X \subset P,\ Y \subset P,\ f \in \mathbf{P}(\langle P, p\rangle, \mathsf{R}) \Rightarrow \operatorname{dist}(f[X], f[Y]) = 0\}$ (25 C.5);

(γ) *the coarse semi-uniformity inducing p is a uniformity* (25 B.9).

(b) *If p is a proximity then there exists a proximally finest uniformizable proximity proximally coarser than p (the so-called uniformizable modification of p)* (25 C.2).

(c) *If $\mathscr{V}$ is the proximally coarse semi-uniformity inducing the same proximity as a semi-uniformity $\mathscr{U}$, then the uniform modification of $\mathscr{V}$ induces the same proximity as the uniform modification of $\mathscr{U}$* (25 C.2).

Recall that the symbol $\mathfrak{v}\mathbf{C}$ ($\mathfrak{v}\mathbf{U}$) denotes the ordered subclass of $\mathbf{C}$ ($\mathbf{U}$) consisting of all uniformizable closures (uniformities). The symbol $\mathfrak{v}_{\mathbf{U}}$ has been used to denote the uniform modification, that is, the single-valued relation which assigns to each semi-uniformity $\mathscr{U}$ the uniformly finest uniformity uniformly coarser than $\mathscr{U}$; the symbol $\mathfrak{v}_{\mathbf{C}}$ denotes the uniformizable modification for closures, that is, the single-valued relation which assigns to each closure u the finest uniformizable closure coarser than u. It should be noted that as $\mathfrak{v}_{\mathbf{U}}$ as well as $\mathfrak{v}_{\mathbf{C}}$ are sometimes abbreviated to $\mathfrak{v}$.

38 C.2. Remark. $\mathfrak{v}\mathbf{P}$ denotes the ordered class of all uniformizable proximities; if P is a set then $\mathfrak{v}\mathbf{P}(P)$ denotes the ordered subset of $\mathbf{P}(P)$ consisting of all uniformizable proximities, and $\mathfrak{v}\mathbf{P}(P, u)$, u being a closure for P, denotes the ordered set of all uniformizable proximities inducing the closure u. Finally, if p is a proximity for a set P, then $\mathfrak{v}\mathbf{U}(P, p)$ denotes the ordered set of all uniformities inducing the proximity p. The uniformizable modification for proximities, denoted by $\mathfrak{v}_{\mathbf{P}}$ or simply $\mathfrak{v}$, is the single-valued relation which assigns to each proximity p the proximally finest uniformizable proximity proximally coarser than p, i.e. $\mathfrak{v}p$ is the upper modification of p in $\mathfrak{v}\mathbf{P}$. The symbol $\mathfrak{v}_{\mathbf{P}}$ also denotes $\{\langle P, p\rangle \to \langle P, \mathfrak{v}_{\mathbf{P}}p\rangle \mid \langle P, p\rangle \in \mathbf{P}\}$.

38 C.3. Theorem. *Let P be a set. The ordered set $\mathfrak{v}\mathbf{P}(P)$ is completely meet-stable and completely meet-preserving in $\mathbf{P}(P)$ and the proximally coarsest and proximally* *finest proximities for P are uniformizable, and hence they are the proximally coarsest and proximally finest uniformizable proximities for P. Furthermore, $\mathfrak{v}\mathbf{P}(P)$ is order-complete and $\mathfrak{v}\mathbf{P}(P) = \mathfrak{v}[\mathbf{P}(P)]$. The mapping $\mathfrak{v} : \mathbf{P}(P) \to \mathfrak{v}\mathbf{P}(P)$ is completely lattice-preserving, $\mathfrak{v} \circ \mathfrak{v} = \mathfrak{v}$ and $\mathfrak{v}p$ is proximally coarser than p for each p in $\mathbf{P}(P)$. If $\{p_a\}$ is any family in $\mathbf{P}(P)$ then*

$$\mathfrak{v} \sup \{p_a\} = \mathfrak{v} \sup \{\mathfrak{v}p_a\} = \sup \{\mathfrak{v}p_a\}$$

where the last supremum is taken in $\mathfrak{v}\mathbf{P}(P)$.

Corollary. *The class $\mathfrak{v}\mathbf{P}$ is completely meet-stable and completely meet-preserving in $\mathbf{P}$, and $\mathfrak{v}\mathbf{P} = \mathbf{E}\mathfrak{v}$. The class $\mathfrak{v}\mathbf{P}$ is boundedly order-complete and contains each proximally accrete or proximally discrete proximity. We have $\mathfrak{v} \circ \mathfrak{v} = \mathfrak{v}$, $\mathfrak{v}p$ is proximally coarser than p for each p in $\mathbf{P}$, and the mapping $\mathfrak{v} : \mathbf{P} \to \mathfrak{v}\mathbf{P}$ is surjective and completely lattice-preserving.*

Proof. By 38 C.1 (b) each element of $\mathbf{P}(P)$ has an upper modification in $\mathfrak{v}\mathbf{P}(P)$, by 38 A.1 the set $\mathbf{P}(P)$ is order-complete. Lemma 31 B.2 applies, and we obtain all the statements except that each proximally discrete proximity is uniformizable; however, this follows readily from 38 C.1 (a) (α) and the description of proximally discrete proximities (38 A.1).

It has already been shown (36 B.7) that $\gamma_{\mathbf{CU}} \circ \mathfrak{v}_{\mathbf{U}} \neq \mathfrak{v}_{\mathbf{C}} \circ \gamma_{\mathbf{CU}}$, but $(\gamma_{\mathbf{CU}} \circ \mathfrak{v}_{\mathbf{U}}) \mathscr{U}$ is always coarser than $(\mathfrak{v}_{\mathbf{C}} \circ \gamma_{\mathbf{CU}}) \mathscr{U}$.

38 C.4. Theorem. *The uniform modification of a semi-uniformity $\mathscr{U}$ induces the the uniformizable modification of the proximity induced by $\mathscr{U}$, in symbols,*

$$\gamma_{\mathbf{PU}} \circ \mathfrak{v}_{\mathbf{U}} = \mathfrak{v}_{\mathbf{P}} \circ \gamma_{\mathbf{PU}} .$$

Proof. Let $\mathscr{U}$ be a semi-uniformity for a set P and let $\mathscr{V}$ be the coarse semi-uniformity inducing the same proximity as $\mathscr{U}$. It is enough to show that the uniform modification $\mathscr{V}_1$ of $\mathscr{V}$ induces the same proximity as the uniform modification $\mathscr{U}_1$ of $\mathscr{U}$; however, this was recalled in 38 C.1.

38 C.5. Remark. (a) $\gamma_{\mathbf{CP}} \circ \mathfrak{v}_{\mathbf{P}} \neq \mathfrak{v}_{\mathbf{C}} \circ \gamma_{\mathbf{CP}}$. In point of fact, assuming $\gamma_{\mathbf{CP}} \circ \mathfrak{v}_{\mathbf{P}} = \mathfrak{v}_{\mathbf{C}} \circ \gamma_{\mathbf{CP}}$, we immediately obtain from 38 C.4 that $\gamma_{\mathbf{CU}} \circ \mathfrak{v}_{\mathbf{U}} = \mathfrak{v}_{\mathbf{C}} \circ \gamma_{\mathbf{CU}}$; but this is not true as noted above.

(b) Since $\mathfrak{v}\mathbf{P}(P) = \mathfrak{v}_{\mathbf{P}}[\mathbf{P}(P)]$ for each set P, it follows from 38 C.4 that $\mathfrak{v}\mathbf{U}(P, p) = \mathfrak{v}_{\mathbf{U}}[\mathbf{U}(P, p)]$ provided that p is a uniformizable proximity. On the other hand it is not true that $\mathfrak{v}\mathbf{P}(P, u) = \mathfrak{v}_{\mathbf{P}}[\mathbf{P}(P, u)]$ even if u is a uniformizable closure (by (a)).

38 C.6. Theorem. *Let P be a set. The canonical mapping of $\mathfrak{v}\mathbf{U}(P)$ into $\mathfrak{v}\mathbf{P}(P)$ (i.e. the mapping $\gamma_{\mathbf{PU}} : \mathfrak{v}\mathbf{U}(P) \to \mathfrak{v}\mathbf{P}(P)$) is completely join-preserving, and*

$$\gamma_{\mathbf{PU}} \sup \{\mathscr{U}_a \mid a \in A\} = \sup \{\gamma_{\mathbf{PU}}\mathscr{U}_a \mid a \in A\}$$

for each family $\{\mathscr{U}_a\}$ in $\mathfrak{v}\mathbf{U}(P)$, where the supremum on the left side is taken in $\mathfrak{v}\mathbf{U}(P)$ and that on the right side in $\mathfrak{v}\mathbf{P}(P)$. The mapping $\gamma : \mathfrak{v}\mathbf{U} \to \mathfrak{v}\mathbf{P}$ is completely join-preserving.

Proof. Denote by $\mathcal{U}$ or $\mathcal{U}_1$ the supremum of $\{\mathcal{U}_a\}$ taken in $\mathfrak{v}\mathbf{U}(P)$ or $\mathbf{U}(P)$, respectively, and by p or p_1 the supremum of $\{\gamma_{\mathbf{PU}}\mathcal{U}_a\}$ taken in $\mathfrak{v}\mathbf{P}(P)$ or $\mathbf{P}(P)$ respectively. By 38 B.4 we have $\gamma_{\mathbf{PU}}\mathcal{U}_1 = p_1$. Since $\mathcal{U} = \mathfrak{v}_{\mathbf{U}}\mathcal{U}_1$ and $p = \mathfrak{v}_{\mathbf{P}}p_1$, we obtain from 38 C.4 that $\gamma_{\mathbf{PU}}\mathcal{U} = p$.

Remark. In general the canonical mapping $\gamma : \mathfrak{v}\mathbf{U}(P) \to \mathfrak{v}\mathbf{P}(P)$ is not meet-preserving. By 38 B.13 there exist two proximally equivalent uniformities whose infimum induces a strictly proximally finer proximity.

38 C.7. Corollary. *For each proximity p the ordered set $\mathfrak{v}\mathbf{U}(P, p)$ is boundedly order-complete, and $\mathfrak{v}\mathbf{U}(P, p)$ is completely join-stable and completely join-preserving in $\mathfrak{v}\mathbf{U}(P)$.*

38 C.8. Theorem. *Let P be a set. The canonical mapping $\gamma : \mathfrak{v}\mathbf{P}(P) \to \mathfrak{v}\mathbf{C}(P)$ is completely meet-preserving, and*

$$(*) \qquad \gamma \inf \{p_a\} = \inf \{\gamma p_a\}$$

for each family $\{p_a\}$ (not necessarily non-void) in $\mathfrak{v}\mathbf{P}(P)$, where the infimum on the left side is taken in $\mathfrak{v}\mathbf{P}(P)$ and that on the right side in $\mathfrak{v}\mathbf{C}(P)$.

Corollary. *The canonical mapping $\gamma : \mathfrak{v}\mathbf{P} \to \mathfrak{v}\mathbf{C}$ is completely meet-preserving.*

Proof. By 38 C.3 (31.B.4) the greatest lower bounds in $\mathfrak{v}\mathbf{P}(P)$ ($\mathfrak{v}\mathbf{C}(P)$) coincide with those in $\mathbf{P}(P)$ ($\mathbf{C}(P)$, respectively). On the other hand, by 38 B.3 the relation $(*)$ holds if the greatest lower bounds are taken in $\mathbf{P}(P)$ and $\mathbf{C}(P)$ respectively.

38 C.9. Corollary. *For each closure u for a set P the ordered set $\mathfrak{v}\mathbf{P}(P, u)$ is boundedly order-complete and the set $\mathfrak{v}\mathbf{P}(P, u)$ is completely meet-stable and completely meet-preserving in $\mathfrak{v}\mathbf{P}(P)$.*

Remark. The set $\mathfrak{v}\mathbf{P}(P, u)$ is non-void if and only if u is uniformizable. If $\mathfrak{v}\mathbf{P}(P, u)$ is non-void, then (by 38 C.9) there exists a finest element of $\mathfrak{v}\mathbf{P}(P, u)$, the so-called Čech proximity of $\langle P, u\rangle$ (see 28 A.1). On the other hand, the coarsest element need not exist, e.g. if $\langle P, u\rangle = \mathbf{Q}$ (see 41 D.6).

38 C.10. Definition. A *proximally fine* (*coarse*) *uniformity* is a uniformity $\mathcal{U}$ such that $\mathcal{U}$ is the uniformly finest (coarsest) uniformity inducing the proximity $\gamma\mathcal{U}$. A *fine* (*coarse*) *uniformizable proximity* is a uniformizable proximity p such that p is the proximally finest (coarsest) uniformizable proximity inducing the closure γp. Thus the term fine uniformizable proximity is an alternate name for the Čech proximity.

38 C.11. (a). *A uniformity $\mathcal{U}$ is a proximally coarse semi-uniformity if and only if $\mathcal{U}$ is a proximally coarse uniformity.* (b) *If a proximally fine semi-uniformity $\mathcal{U}$ is a uniformity, then $\mathcal{U}$ is a proximally fine uniformity (but a proximally fine uniformity need not be a proximally fine semi-uniformity).* (c) *If a fine (coarse) proximity p is uniformizable, then p is a fine (coarse) uniformizable proximity; but a fine (coarse) uniformizable proximity need not be a fine (coarse) proximity.*

Proof. I. If a uniformity $\mathscr{U}$ is a proximally coarse semi-uniformity, then obviously $\mathscr{U}$ is a proximally coarse uniformity. If $\mathscr{U}$ is a proximally coarse uniformity, then $\mathscr{U}$ is a proximally coarse semi-uniformity, by 38 C.1 (c).

II. The first statements of (b) and (c) are evident.

III. Let $\langle P, u \rangle$ be a subspace of R, where P is the set consisting of zero and all points of the form $1/n$, $n = 1, 2, \ldots$. It is easily seen that the fine proximity of $\langle P, u \rangle$ differs from the fine uniformizable proximity of $\langle P, u \rangle$ (i.e., the Čech proximity of $\langle P, u \rangle$).

IV. An example of a coarse uniformizable proximity which is not a coarse proximity can be obtained as follows: take a coarse uniformity $\mathscr{U}$ which is not a coarse semi-uniformity; clearly the proximity induced by $\mathscr{U}$ is a coarse uniformizable proximity but not a coarse proximity. An example of a proximally fine uniformity which is not a proximally fine semi-uniformity can be obtained as follows: take a pseudometrizable uniform space $\langle P, \mathscr{U} \rangle$ which is not uniformly quasi-discrete; $\mathscr{U}$ is a proximally fine uniformity but not a proximally fine semi-uniformity, see ex. 4.

38 C.12. *A uniformity $\mathscr{U}$ is a proximally coarse uniformity if and only if $\mathscr{U}$ is totally bounded.* Indeed, a uniformity $\mathscr{U}$ is a proximally coarse uniformity if and only if $\mathscr{U}$ is a proximally coarse semi-uniformity (by 38 C.11 (a)), and a uniformity $\mathscr{U}$ is a proximally coarse semi-uniformity if and only if $\mathscr{U}$ is totally bounded (by 25 B.12).

38 C.13. A description of proximally fine uniformities was given in 25 B.22. Observe that the uniform modification of a proximally fine semi-uniformity is a proximally fine uniformity.

38 C.14. *A proximity p is a proximally fine uniformizable proximity (i.e., a Čech proximity) if and only if p is the uniformizable modification of a fine proximity.*

Proximally coarse uniformizable proximities will be described in 41 D.

39. PROJECTIVE AND INDUCTIVE GENERATION FOR PROXIMITY SPACES

The projective and inductive generation for closure spaces or semi-uniform spaces was studied in 32, 33 and 37. The present section concerns projective and inductive generation for proximity spaces. Whereas the proofs and examples in 32, 33 and 37 were given with all the details, and some theorems were proved twice, the proofs in this section are rather short and, possibly, rather concise. Particular attention is given to interrelations between the generation of closure spaces, semi-uniform spaces and proximity spaces (for a summary see 39 C.6). The reader familiar with the fundamentals of the theory of categories is invited to carry over generations and the interrelations between the generation for various kind of spaces to the theory of categories (e.g. with given functors into the category of sets); the proofs persist.

A. GENERALITIES

39 A.1. Definition. A *proximity* p for a set P is said to be *projectively generated by a family of mappings* $\{f_a \mid a \in A\}$ if $\{f_a\}$ is a projective family of mappings for proximity spaces with a common domain carrier P or $\langle P, p\rangle$ and p is the proximally coarsest proximity for P such that all the mappings $f_a : \langle P, p\rangle \to \mathbf{E}^* f_a$ are continuous; in this case the family $\{f_a\}$ is said to be a *projective generating family for* $\langle P, p\rangle$. A *proximity space* $\langle P, p\rangle$ is said to be *projectively generated by a family of mappings* $\{f_a\}$ if $\{f_a\}$ is a projective generating family for $\langle P, p\rangle$ and $\langle P, p\rangle$ is the common domain carrier of all f_a. A *proximity* p for a set P is said to be *inductively generated by a family* $\{f_a\}$ if $\{f_a\}$ is an inductive family of mappings with a common range carrier P or $\langle P, p\rangle$ and p is the proximally finest proximity such that all the mappings $f_a : \mathbf{D}^* f_a \to \langle P, p\rangle$ are continuous; in this case $\{f_a\}$ is said to be an *inductive generating family for* $\langle P, p\rangle$. A *proximity space* is said to be *inductively generated by* $\{f_a\}$ if $\{f_a\}$ is an inductive generating family for $\langle P, p\rangle$ and $\langle P, p\rangle$ is the common range carrier of all the f_a. The definitions just stated are carried over to collections of mappings and single mappings in such a way that a collection $\mathscr{F}$ has a property $\mathfrak{P}$ if and only if the family $\{f \mid f \in \mathscr{F}\}$ has the property $\mathfrak{P}$, and a mapping f has a property $\mathfrak{P}$ if and only if the singleton (f) has the property $\mathfrak{P}$.

39 A.2. Examples. (a) The empty set $\emptyset$ is a projective generating family for each proximally accrete space and an inductive generating family for each proximally discrete space. (b) A proximity space $\langle P, p\rangle$ inductively generated by a family of constant mappings is a proximally discrete space, and a proximity space projectively generated by a family of constant mappings is a proximally accrete space (remember that every constant mapping for proximity spaces is proximally continuous). (c) A proximity space projectively generated by a family of mappings into proximally accrete spaces is a proximally accrete space, and a proximity space inductively generated by a family of mappings of proximally discrete spaces is a proximally discrete space (remember that a mapping f for proximity spaces is proximally continuous whenever $\mathbf{E}^*f$ is proximally accrete or $\mathbf{D}^*f$ is proximally discrete). (d) A proximal homeomorphism is both a projective generating mapping and an inductive generating mapping. (e) If $\{p_a\}$ is family in $\mathbf{P}(P)$, then inf $\{p_a\}$ is projectively generated by the family $\{\mathsf{J} : P \to \langle P, p_a\rangle\}$ and sup $\{p_a\}$ is inductively generated by the family $\{\mathsf{J} : \langle P, p_a\rangle \to \to P\}$.

39 A.3 proj. Theorem. *Every projective family of mappings for proximity spaces with a common domain carrier P which is a set, projectively generates exactly one proximity p for P. If p is projectively generated by a single mapping* $f : P \to \langle Q, q\rangle$, *then* XpY *if and only if* $X \subset P$, $Y \subset P$ *and* $f[X]\, q\, f[Y]$. *If a proximity p for a set P is projectively generated by a family* $\{f_a\}$ *and if each* p_a *is projectively generated by the mapping* f_a, *then p is the infimum of* $\{p_a\}$.

39 A.3 ind. Theorem. *Every inductive family of mappings* $\{f_a\}$ *for proximity spaces with a common range carrier P which is a set, inductively generates exactly one proximity for P. If p is inductively generated by a single mapping* $f : \langle Q, q\rangle \to P$, *then* XpY *if and only if* $X \subset P$, $Y \subset P$ *and either* $X \cap Y \neq \emptyset$ *or* $f^{-1}[X]\, q\, f^{-1}[Y]$. *If a proximity p for a set P is inductively generated by a family* $\{f_a\}$ *and* p_a *is inductively generated by the mapping* f_a *for each a, then p is the supremum of* $\{p_a\}$.

Proof of 39 A.3 proj. I. Uniqueness is clear.

II. Let f be a mapping of P into a proximity space $\langle Q, q\rangle$ and let p be the set of all $\langle X, Y\rangle$ such that $X \subset P$, $Y \subset P$ and $f[X]\, q\, f[X]$. We shall prove that p is the proximity projectively induced by f. The proof of the fact that p is a proximity for P is left to the reader because the verification of the conditions (prox i) is straightforward. If XpY, then $f[X]\, q\, f[Y]$ (by the definition of p) and hence the mapping $f : \langle P, p\rangle \to \langle Q, q\rangle$ is proximally continuous. If r is a proximity such that the mapping $f : \langle P, r\rangle \to \langle Q, q\rangle$ is proximally continuous, then XrY implies $f[X]\, q\, f[Y]$ and hence XpY, which shows that p is the proximally coarsest proximity which renders f continuous.

III. Let $\{f_a\}$ be a projective family for proximity spaces with a common domain carrier P, which is a set, and let p_a be the proximity projectively generated by f_a. We shall prove that $p = \inf \{p_a\}$ is projectively generated by $\{p_a\}$. The mapping

$g_a = f_a : \langle P, p \rangle \to \mathbf{E}^* f_a$ is proximally continuous for each a because g_a is the composite of two proximally continuous mappings, namely $\mathrm{J} : \langle P, p \rangle \to \langle P, p_a \rangle$ and $f_a : \langle P, p_a \rangle \to \mathbf{E}^* f_a$. On the other hand, if r is a proximity such that each $f_a : \langle P, r \rangle \to \mathbf{E}^* f_a$ is proximally continuous, then r is proximally finer than each p_a, and hence r is proximally finer than $\inf \{p_a\} = p$.

Proof of 39 A.3 ind. We shall only prove that if f is a mapping of a proximity space $\langle Q, q \rangle$ into a set P then the set p of all $\langle X, Y \rangle$ such that $X \subset P$, $Y \subset P$ and either $X \cap Y \neq \emptyset$ or $f^{-1}[X] \, q \, f^{-1}[Y]$, is the proximity inductively generated by f. It is easily seen that p is a proximity for P. If $X_1 q Y_1$, then $f[X_1] \, p \, f[Y_1]$ because $f^{-1}[f[X_1]] \supset X_1$, $f^{-1}[f[Y_1]] \supset Y_1$, and hence $X_1 q Y_1$ implies $f^{-1}[f[X_1]] \, q \, f^{-1}[f[Y_1]]$. Thus the mapping $f : \langle Q, q \rangle \to \langle P, p \rangle$ is proximally continuous. If r is a proximity for P such that $f : \langle Q, q \rangle \to \langle P, r \rangle$ is proximally continuous, and if XpY, then either $X \cap Y \neq \emptyset$, in which case XrY because r is a proximity, or $f^{-1}[X] \, q \, f^{-1}[Y]$, in which case $f[f^{-1}[X]] \, r \, f[f^{-1}[X]]$ and hence also XrY because $X \supset f[f^{-1}[X]]$ and $Y \supset f[f^{-1}[Y]]$. Thus XpY implies XrY, and hence r is proximally coarser than p.

39 A.4 proj. Theorem. *Let $\{f_a\}$ be a projective family of mappings for proximity spaces with a common domain carrier $\mathcal{P}$ which is a space. Then $\{f_a\}$ is a projective generating family for proximity spaces if and only if the following condition is fulfilled:*

A mapping f of a proximity space into $\mathcal{P}$ is proximally continuous if and only if all the mappings $f_a \circ f$ are proximally continuous.

39 A.4 ind. Theorem. *Let $\{f_a\}$ be an inductive family of mappings for proximity spaces with a common range carrier $\mathcal{P}$ which is a space. Then $\{f_a\}$ is an inductive generating family for proximity spaces if and only if the following condition is fulfilled:*

A mapping f of $\mathcal{P}$ into a proximity space is proximally continuous if and only if all the mappings $f \circ f_a$ are proximally continuous.

Proof of 39 A.4 proj. I. Suppose that $\mathcal{P}$ is projectively generated by $\{f_a\}$ and f is a mapping of a proximity space $\mathcal{Q}$ into $\mathcal{P}$. If f is proximally continuous, then all the $f_a \circ f$ are proximally continuous as composites of proximally continuous mappings. Conversely, suppose that all the $f_a \circ f$ are proximally continuous. Write $\mathcal{Q} = \langle Q, q \rangle$, $\mathcal{P} = \langle P, p \rangle$. Consider the proximity r inductively generated by $f : \langle Q, q \rangle \to P$. To prove that f is continuous it is sufficient to show that r is proximally finer than p, and to prove that r is proximally finer than p it is sufficient to show that each mapping $f_a : \langle P, r \rangle \to \mathbf{E}^* f_a$ is proximally continuous. Suppose XrY; we must show that the sets $f_a[X]$ and $f_a[Y]$ are proximal in $\mathbf{E}^* f_a$. If $X \cap Y \neq \emptyset$ then $f_a[X] \cap f_a[Y] \neq \emptyset$ and therefore the sets $f_a[X]$ and $f_a[Y]$ are proximal in $\mathbf{E}^* f$. If $X \cap Y = \emptyset$, then $f^{-1}[X] \, q \, f^{-1}[Y]$ (by 39 A.3 ind) and hence, $f_a \circ f$ being proximally continuous, the images under $f_a \circ f$ of the sets $X_1 = f^{-1}[X]$ and $Y_1 = f^{-1}[Y]$ must be proximal in $\mathbf{E}^* f_a$. Since evidently $f_a[X] \supset f_a[f[X_1]]$ and $f_a[Y] \supset f_a[f[Y_1]]$, we obtain that

$f_a[X]$ and $f_a[Y]$ are also proximal in $\mathbf{E}^*f$; this establishes the proximal continuity of $f_a : \langle P, r\rangle \to \mathbf{E}^*f_a$.

II. Suppose that the condition is fulfilled. Taking $f = \mathrm{J} : \mathscr{P} \to \mathscr{P}$ we see that all the mappings $f_a = f_a \circ f$ are proximally continuous. Write $\mathscr{P} = \langle P, p\rangle$. If r is a proximity for P such that all the mappings $f_a : \langle P, r\rangle \to \mathbf{E}^*f_a$ are proximally continuous, then $\mathrm{J} : \langle P, r\rangle \to \langle P, p\rangle$ is proximally continuous because $f_a : \langle P, r\rangle \to \mathbf{E}^*f_a = f_a \circ (\mathrm{J} : \langle P, r\rangle \to \langle P, p\rangle)$, and therefore r is proximally finer than p. Thus $\mathscr{P}$ is projectively generated by the family $\{f_a\}$.

Proof of 39 A.4 ind. I. Suppose that $\mathscr{P}$ is inductively generated by $\{f_a\}$ and f is a mapping of $\mathscr{P}$ into a proximity space $\mathscr{Q}$. If f is proximally continuous then all the $f \circ f_a$ are proximally continuous as composites of proximally continuous mappings. Conversely, assume that all the mappings $f \circ f_a$ are proximally continuous. We must show that f is proximally continuous. Write $\mathscr{P} = \langle P, p\rangle$, $\mathscr{Q} = \langle Q, q\rangle$ and let r be the proximity projectively generated by $f : P \to \langle Q, q\rangle$. It is sufficient to prove that p is proximally finer than r, and hence, that each mapping $f_a : \mathbf{D}^*f_a \to \langle P, r\rangle$ is proximally continuous.

Assume that X and Y are proximal in $\mathbf{D}^*f_a$. We must show $f_a[X]\; r\, f_a[Y]$. Since $f \circ f_a$ is proximally continuous we have $f[f_a[X]]\; q\, f[f_a[Y]]$ and therefore, r being projectively generated by $f : P \to \langle Q, q\rangle$, $f_a[X]\; r\, f_a[Y]$.

II. Suppose that the condition is fulfilled. Taking $f = \mathrm{J} : \mathscr{P} \to \mathscr{P}$ we obtain that each f_a is proximally continuous. If r is a proximity for $|\mathscr{P}|$ such that all $f_a : \mathbf{D}^*f_a \to \langle |\mathscr{P}|, r\rangle$ are proximally continuous, then $\mathrm{J} : \mathscr{P} \to \langle |\mathscr{P}|, r\rangle$ is proximally continuous (by the condition) and hence $\mathscr{P}$ is inductively generated by $\{f_a\}$.

From theorem 39 A.4 we immediately obtain the following important theorem on the associativity of projective and inductive constructions.

39 A.5 proj. Associativity theorem. *Let $\{f_a \mid a \in A\}$ be a projective family for proximity spaces and let the range carrier $\mathbf{E}^*f_a$ be projectively generated by a family $\{g_{ab} \mid b \in B_a\}$ for each a in A. Then $\{f_a\}$ is a projective generating family for proximity spaces if and only if the family $\{g_{ab} \circ f_a \mid a \in A,\ b \in B_a\}$ is a projective generating family for proximity spaces.*

39 A.5 ind. Associativity theorem. *Let $\{f_a \mid a \in A\}$ be an inductive family for proximity spaces and let the domain carrier $\mathbf{D}^*f_a$ be inductively generated by a family $\{g_{ab} \mid b \in B_a\}$ for each a in A. Then $\{f_a\}$ is an inductive generating family for proximity spaces if and only if the family $\{f_a \circ g_{ab} \mid a \in A,\ b \in B_a\}$ is an inductive generating family for proximity spaces.*

Proof of 39 A.5 proj. Applying 39 A.4 proj. to each $\mathbf{D}^*g_{ab}$ we find that $f_a \circ f$ is proximally continuous for each a if and only if the mapping $g_{ab} \circ f_a \circ f$ is proximally continuous for each $a \in A$ and $b \in B_a$. Again by 39 A.4, $\{f_a\}$ is a projective generating family if and only if $\{g_{ab} \circ f_a\}$ is a projective generating family.

Proof of 39 A.5 ind. Applying 39 A.4 ind. to each $\mathbf{D}^*f_a$ we see that, given a mapping f of the common range carrier of all f_a into a proximity space, then all the

$f \circ f_a$ are proximally continuous if and only if all the $f \circ f_a \circ g_{ab}$ are proximally continuous. Now the conclusion follows from 39 A.4.

39 A.6. Examples. (a) *Every surjective projective generating mapping is inductive generating.* In fact, if $f : \langle Q, q \rangle \to \langle P, p \rangle$ is a surjective projective generating mapping, XpY and $X_1 = f^{-1}[X]$, $Y_1 = f^{-1}[Y]$, then $f[X_1] = X$, $f[Y_1] = Y$, because f is surjective, and hence $X_1 q Y_1$ because f is a projective generating mapping. Thus XpY implies $f^{-1}[X] \, q \, f^{-1}[Y]$, which means that f is an inductive generating mapping.

(b) *Every injective inductive generating mapping is a projective generating mapping.* Use the fact that $f^{-1}[f[X]] = X$ if f is injective.

(c) *If $\langle P, p \rangle$ is a proximity space, then the relativization q of p to a subset Q of P is projectively generated by $\mathsf{J} : Q \to \langle P, p \rangle$*; stated in other words, if $Q \subset P$ then $\mathsf{J} : \langle Q, q \rangle \to \langle P, p \rangle$ is a projective generating mapping if and only if $\langle Q, q \rangle$ is a subspace of $\langle P, p \rangle$. Remember that the relativization of p to a subset Q of P is is defined to be $p \cap (\exp Q \times \exp Q)$, i.e. XqY if and only if $X \subset Q$, $Y \subset Q$ and XpY.

(d) *If f is a proximally continuous mapping of $\langle P, p \rangle$ into $\langle Q, q \rangle$ and if there exists a proximally continuous mapping $g : \langle Q, q \rangle \to \langle P, p \rangle$ such that $f \circ g = \mathsf{J} :$ $: \langle Q, q \rangle \to \langle Q, q \rangle$, then f is a surjective inductive generating mapping.*

(e) *If f is a proximally continuous mapping of $\langle P, p \rangle$ into $\langle Q, q \rangle$ and if there exists a proximally continuous mapping g of a subspace of $\langle Q, q \rangle$ into $\langle P, p \rangle$ such that $g \circ f = \mathsf{J} : \langle P, p \rangle \to \langle P, p \rangle$, then f is an injective projective generating mapping, i.e. a proximal embedding (first notice that f is injective).*

(f) *If f and g are proximally continuous mappings and if $f \circ g$ is a proximal homeomorphism (in particular, $\mathbf{D}^*f = \mathbf{E}^*g$), then f is a surjective inductive generating mapping and g is an injective projective generating mapping (i.e. a proximal embedding).*

39 A.7 proj. Commutativity. *If $\{f_a\}$ is a projective generating family for a proximity space $\langle P, p \rangle$ and $\langle Q, q \rangle$ is a subspace of $\langle P, p \rangle$, then $\{g_a\}$ is a projective generating family for $\langle Q, q \rangle$ where $g_a = f_a : \langle Q, q \rangle \to \mathbf{E}^*f_a$ for each a.*

39 A.7 ind. Partial commutativity. *If $\langle Q, q \rangle$ is a subspace of a proximity space $\langle P, p \rangle$ and $\{f_a\}$ is an inductive generating family for $\langle P, p \rangle$, then $\{g_a\}$ is an inductive generating family for $\langle Q, q \rangle$ where g_a is the restriction of f to a mapping of the subspace $f^{-1}[Q]$ of $\mathbf{D}^*f_a$ into $\langle Q, q \rangle$ for each a.*

Proof of 39 A.7 proj. The identity mapping f of $\langle Q, q \rangle$ into $\langle P, p \rangle$ is a projective generating mapping (by 39 A.6 (c)) and hence $\{f_a \circ f\}$ is a projective generating family (by 39.A 5). Clearly $f_a \circ f = g_a$.

Proof of 39 A.7 ind. We have XqY if and only if $X \subset Q$, $Y \subset Q$ and XpY; and if $X \subset Q$, $Y \subset Q$ then the sets $f_a^{-1}[X]$ and $f_a^{-1}[Y]$ are proximal in $\mathbf{D}^*f_a$ if and only if the sets $g_a^{-1}[X] = f_a^{-1}[X]$ and $g_a^{-1}[X] = f_a^{-1}[X]$ are proximal in $\mathbf{D}^*g_a$. The statement follows from the following description of inductively generated proximities.

39 A.8. *In order that a proximity space $\langle P, p\rangle$ be inductively generated by a family of mappings $\{f_a : \langle Q, q_a\rangle \to \langle P, p\rangle\}$ it is necessary and sufficient that XpY if and only if $X \subset P$, $Y \subset P$ and either $X \cap Y \neq \emptyset$ or $f_a^{-1}[X]\, q_a f^{-1}[Y]$ for some a.*

Proof. If p_a is inductively generated by $f_a : \langle Q_a, q_a\rangle \to P$ then $p = \sup\{p_a\}$ is inductively generated by $\{f_a : \langle Q_a, q_a\rangle \to P\}$ (cf. 39 A.3). We have $p = \bigcup\{p_a\}$ (by 38 A.1) and Xp_aY if and only if $X \subset P$, $Y \subset P$ and either $X \cap Y \neq \emptyset$ or $f_a^{-1}[X]\ q_a\ f_a^{-1}[Y]$.

Remark. Notice that the proofs of 39 A.7 proj. and 39 A.7 ind. are entirely different. This follows from the fact that a subspace can be defined projectively (an identity mapping is a projective generating mapping, see 39 A.6 (c)) but not inductively.

39 A.9. Definition. Let $\{\langle P_a, p_a\rangle\}$ be a family of proximity spaces. The *product of* $\{\langle P_a, p_a\rangle\}$, denoted by $\Pi\{\langle P_a, p_a\rangle\}$, is defined to be the proximity space $\langle \Pi\{P_a\}, p\rangle$ where p is the proximity projectively generated by the family $\{\mathrm{pr}_a : \Pi\{P_a\} \to \langle P_a, p_a\rangle\}$. The *sum of the family* $\{\langle P_a, p_a\rangle\}$, denoted by $\Sigma\{\langle P_a, p_a\rangle\}$, is defined to be the proximity space $\langle \Sigma\{P_a\}, p\rangle$ where p is the proximity inductively generated by the family $\{\mathrm{inj}_a : \langle P_a, p_a\rangle \to \Sigma\{P_a\}\}$.

As an immediate consequence of Definition 39 A.9 and Theorems 39 A.4 and 39 A.5 we obtain the following important results.

39 A.10 proj. *Let f be a mapping of a proximity space into the product $\mathscr{P}$ of a family $\{\mathscr{P}_a\}$ of proximity spaces. Then f is continuous if and only if all $(\mathrm{pr}_a : \mathscr{P} \to \mathscr{P}_a) \circ f$ are continuous, and f is a projective generating mapping if and only if $\{(\mathrm{pr}_a : \mathscr{P} \to \mathscr{P}_a) \circ f\}$ is a projective generating family.*

39 A.10 ind. *Let f be a mapping of the sum $\mathscr{P}$ of a family $\{\mathscr{P}_a\}$ of proximity spaces into a proximity space. Then f is proximally continuous if and only if all the mappings $f \circ (\mathrm{inj}_a : \mathscr{P}_a \to \mathscr{P})$ are proximally continuous, and f is an inductive generating mapping if and only if $\{f \circ (\mathrm{inj}_a : \mathscr{P}_a \to \mathscr{P})\}$ is an inductive generating family.*

39 A.11. Definition. The *product of a family $\{f_a\}$ of mappings for proximity spaces* is the mapping of $\Pi\{\mathbf{D}^*f_a\}$ into $\Pi\{\mathbf{E}^*f_a\}$ whose graph is the relational product of the family $\{\mathrm{gr}\, f_a\}$. The *reduced product of a projective family $\{f_a\}$ of mappings for proximity spaces* with common domain carrier $\mathscr{P} = \langle P, p\rangle$ is the mapping of $\mathscr{P}$ into $\Pi\{\mathbf{E}^*f_a\}$ whose graph is the reduced relational product of $\{\mathrm{gr}\, f_a\}$, i.e. $fx = \{f_a x\}$. The *sum of a family $\{f_a\}$ of mappings for proximity spaces* is defined to be the mapping of $\Sigma\{\mathbf{D}^*f_a\}$ into $\Sigma\{\mathbf{E}^*f_a\}$ whose graph is the relational sum of $\{\mathrm{gr}\, f_a\}$, and the *reduced sum of an inductive family $\{f_a\}$ of mappings for proximity spaces* with common range carier $\mathscr{P} = \langle P, p\rangle$ is the mapping of $\Sigma\{\mathbf{D}^*f_a\}$ into $\mathscr{P}$ whose graph is the reduced relational sum of $\{\mathrm{gr}\, f_a\}$.

Theorems 39 A.10 can be restated as follows:

39 A.12 proj. *The reduced product f of a family of mappings $\{f_a\}$ for a proximity space is proximally continuous (a projective generating mapping) if and only if all f_a are proximally continuous ($\{f_a\}$ is a projective generating family).*

Indeed, $f_a = (\mathrm{pr}_a : \mathbf{E}^*f \to \mathbf{E}^*f_a) \circ f$ for each a.

39 A.12 ind. *The reduced sum f of a family of mappings $\{f_a\}$ for proximity spaces is proximally continuous (an inductive generating mapping) if and only if all the f_a are proximally continuous ($\{f_a\}$ is an inductive generating family).*

Indeed, $f \circ (\mathrm{inj}_a : \mathbf{D}^*f_a \to \mathbf{D}^*f) = f_a$ for each a.

For an examination of products and sums of mappings we shall need the following result.

39 A.13 proj. *If $\mathscr{P}$ is the product of a family $\{\mathscr{P}_a\}$ of proximity spaces and* $\mathrm{pr}_\alpha : \mathscr{P} \to \mathscr{P}_\alpha$ *is surjective (in particular if $|\mathscr{P}| \neq \emptyset$), then* $\mathrm{pr}_\alpha : \mathscr{P} \to \mathscr{P}_\alpha$ *is an inductive generating mapping.*

39 A.13 ind. *If $\mathscr{P}$ is the sum of a family $\{\mathscr{P}_a\}$ of proximity spaces, then each* $\mathrm{inj}_\alpha : \mathscr{P}_\alpha \to \mathscr{P}$ *is a projective generating mapping (and hence an embedding).*

Proof of 39 A.13 proj. Suppose that $f = \mathrm{pr}_\alpha : \mathscr{P} \to \mathscr{P}_\alpha$ is surjective. If $|\mathscr{P}| = \emptyset$, then $|\mathscr{P}_a| = \emptyset$ and the statement is trivial. Assuming $|\mathscr{P}| \neq \emptyset$ choose an x in $|\mathscr{P}|$ and consider the mapping g of $\mathscr{P}_\alpha$ into $\mathscr{P}$ which assigns to each $y \in \mathscr{P}_\alpha$ the point gy whose α-th coordinate is y and the other coordinates coincide with those of x. Thus $f \circ g$ is the identity mapping and $\mathrm{pr}_a \circ g : \mathscr{P} \to \mathscr{P}_a$ is constant for each $a \neq \alpha$. Since all $(\mathrm{pr}_a : \mathscr{P} \to \mathscr{P}_a) \circ g$ are proximally continuous, the mapping g is proximally continuous. By 39 A.6 (d) the mapping f is an inductive generating mapping.

Proof of 39 A.13 ind. Let g be the mapping $\langle \alpha, x \rangle \to x$ of the subspace $\mathrm{inj}_\alpha |\mathscr{P}_\alpha|$ of $\mathscr{P}$ onto $\mathscr{P}_\alpha$. By 39 A.4 and 39 A.7 ind. the mapping g is proximally continuous because clearly $g \circ (\mathrm{inj}_\alpha : \mathscr{P}_\alpha \to \mathscr{P}) = \mathrm{J} : \mathscr{P}_\alpha \to \mathscr{P}_\alpha$. By 39 A.6 (e) the mapping $\mathrm{inj}_\alpha : : \mathscr{P}_\alpha \to \mathscr{P}$ is a proximal embedding.

39 A.14 proj. Theorem. *Let f be the product of a family $\{f_a\}$ of mappings for proximity spaces. If all the f_a are proximally continuous, then f is proximally continuous. Conversely, if $\mathbf{D}f \neq \emptyset$ and f is proximally continuous, then all the f_a are proximally continuous.*

39 A.14 ind. Theorem. *Let f be the sum of a family $\{f_a\}$ of mappings for proximity spaces. The mapping f is proximally continuous if and only if all the mappings f_a are proximally continuous.*

Proof of 39 A.14 proj. If $\mathbf{D}f = \emptyset$ then f is proximally continuous. It remains to show that, if $\mathbf{D}f \neq \emptyset$, then f is proximally continuous if and only if all the mappings f_a are proximally continuous, and this follows from the following obvious equality:

$$(\mathrm{pr}_a : \mathbf{E}^*f \to \mathbf{E}^*f_a) \circ f = f_a \circ (\mathrm{pr}_a : \mathbf{D}^*f \to \mathbf{D}^*f_a) .$$

Indeed, if f is proximally continuous, then the left side is proximally continuous and hence f_a is continuous because $(\mathrm{pr}_a : \mathbf{D}^*f \to \mathbf{D}^*f_a)$ is an inductive generating mapping by 39 A.13 proj. If each f_a is proximally continuous, then f is continuous by 39 A.4 proj. because $\{\mathrm{pr}_a : \mathbf{E}^*f \to \mathbf{E}^*f_a\}$ is a projective generating family.

Proof of 39 A.14 ind. Notice that

$$f \circ (\mathrm{inj}_a : \mathbf{D}^*f_a \to \mathbf{D}^*f) = (\mathrm{inj}_a : \mathbf{E}^*f_a \to \mathbf{E}^*f) \circ f_a .$$

If each f_a is proximally continuous then the right side is proximally continuous, and therefore f is proximally continuous by 39 A.4 because $\{\mathrm{inj}_a : \mathbf{D}^*f_a \to \mathbf{D}^*f\}$ is an inductive generating family. If f is proximally continuous then the right side is proximally continuous for each a, and therefore f_a is continuous because $(\mathrm{inj}_a : \mathbf{E}^*f_a \to \mathbf{E}^*f)$ is a projective generating mapping (by 39 A.13 ind.).

39 A.15 proj. *If f is the product of a family $\{f_a\}$ of mappings for proximity spaces and each f_a is a projective generating mapping, then f is a projective generating mapping.*

39 A.15 ind. *If f is the sum of a family $\{f_a\}$ of mappings for proximity spaces and each f_a is an inductive generating mapping, then f is an inductive generating mapping.*

Proof of 39 A.15 proj. If $\mathbf{D}f = \emptyset$, then $\mathbf{E}f = \emptyset$ and f is a projective generating mapping. If $\mathbf{D}f \neq \emptyset$ then $\mathbf{E}f \neq \emptyset$ and the formula of the proof of 39 A.14 proj. holds. Since $\{\mathrm{pr}_a : \mathbf{D}^*f \to \mathbf{D}^*f_a\}$ is a projective generating family and each f_a projectively generates $\mathbf{D}^*f_a$, $\{f_a \circ (\mathrm{pr}_a : \mathbf{D}^*f \to \mathbf{D}^*f_a)\}$ is a projective generating family (by 39 A.5) and by the formula, $\{(\mathrm{pr}_a : \mathbf{E}^*f \to \mathbf{E}^*f_a) \circ f\}$ is a projective generating family, and $\{\mathrm{pr}_a : \mathbf{E}^*f \to \mathbf{E}^*f_a\}$ being a projective generating family, f is a projective generating family by 39 A.5.

Proof of 39 A.15 ind. is left to the reader.

Remark. One can show that if $\mathbf{D}f \neq \emptyset$ and f is a generating mapping, then each f_a is a generating mapping.

In conclusion we shall state the factorization theorems, the proofs of which are left to the reader.

39 A.16. Projective factorization. *If $\{f_a\}$ is a non-void projective family of mappings for proximity spaces, then there exists a unique projective generating family $\{g_a\}$ for proximity spaces and an identity mapping h such that $f_a = g_a \circ h$ for each a, i.e.*

$$\{f_a\} = [\{g_a\}] \circ h .$$

The mapping h is proximally continuous if and only if all the mappings f_a are proximally continuous, and h is a proximal homeomorphism if and only if $\{f_a\}$ is a projective generating family.

Inductive factorization. *If* $\{f_a\}$ *is a non-void inductive family of mappings for proximity spaces, then there exists a unique inductive generating family* $\{g_a\}$ *and an identity mapping h such that* $h \circ g_a = f_a$ *for each a, i.e.*

$$\{f_a\} = h \circ [\{g_a\}] .$$

The mapping h is proximally continuous if and only if all the mappings f_a *are proximally continuous, and h is a proximal homeomorphism if and only if* $\{f_a\}$ *is an inductive generating mapping.*

B. PROJECTIVE GENERATION

We begin with two descriptions of projectively generated proximities.

39 B.1. Theorem. *If a proximity space* $\langle P, p\rangle$ *is projectively generated by a non-void family of mappings* $\{f_a : \langle P, p\rangle \to \langle Q_a, q_a\rangle\}$ *then*

(a) XpY *for* $X \subset P$, $Y \subset P$ *if and only if, for each a and any finite decompositions* $\{X_i\}$ *and* $\{Y_j\}$ *of X and Y respectively, then* $f_a[X_i]\ q_a\ f_a[Y_j]$ *for some i and j;*

(b) *U is a proximal neighborhood of X in* $\langle P, p\rangle$ *if and only if it contains a finite intersection of sets of the form* $\bigcup\{f_{a_i}^{-1}[U_i]\}$ *with* U_i *proximal neighborhoods of* $f_{a_i}[X_i]$ *in* $\langle Q_{a_i}, q_{a_i}\rangle$ *and* $\{X_i\}$ *a finite cover of X.*

Proof. Combine 39 A.3 with the descriptions 38 A.5 and 38 A.8 of infima in $\mathbf{P}(P)$.

39 B.2. Theorem. *If* $\{f_a : \langle P, p\rangle \to \langle Q_a, q_a\rangle\}$ *is a projective generating family for proximity spaces, then* $\{f_a : \langle P, \gamma_{\mathbf{CP}} p\rangle \to \langle Q_a, \gamma_{\mathbf{CP}} q_a\rangle\}$ *is a projective generating family; stated in other words, if* $\{f_a\}$ *is a projective generating family for proximity spaces, then* $\{g_a\}$ *is a projective generating family for closure spaces where each* g_a *is the transpose of* f_a *to a mapping for closure spaces.*

Proof. A direct proof can be obtained from description 39 B.1 (b). Indeed, if $X = (x)$, then U is a neighborhood of x if and only if U is a proximal neighborhood of x, and the statement is obtained by combining 39 B.1 (b) with the description 32 A.6 of projectively generated closures. Another proof may be in order. Since the canonical mapping of $\mathbf{P}(P)$ into $\mathbf{C}(P)$ is completely meet-preserving, it is sufficient to prove the statement for a family consisting of one member, i.e. for projective generating mappings. Suppose that $f : \langle P, p\rangle \to \langle Q, q\rangle$ is a projective generating mapping and let u be the closure induced by p and v be the closure induced by q. Thus $x \in uX$ if and only if $(x)\ p\ X$, $(x)\ p\ X$ if and only if $(fx)\ q\ f[X]$ (by 39 A.3), and $(fx)\ q\ f[X]$ if and only if $fx \in vf[X]$. Thus $x \in uX$ if and only if $fx \in vf[X]$, which shows that u is projectively generated by $f : P \to \langle Q, v\rangle$.

39 B.3. *If* $\{f_a\}$ *is a projective generating family for semi-uniform spaces, then the transposed family* $\{g_a\} = \{f_a : \gamma_{\mathbf{PU}}\mathbf{D}^* f_a \to \gamma_{\mathbf{PU}}\mathbf{E}^* f_a\}$ *need not be a projective generating family for proximity spaces (even if the index set consists of two elements and all the semi-uniform spaces in question are uniform spaces).* This follows from the fact that the canonical mappings $\gamma : \mathbf{U} \to \mathbf{P}$ and also $\gamma : \upsilon\mathbf{U} \to \upsilon\mathbf{P}$ are not meet-preserving. We shall prove a general result which shows that if $\{f_a\}$ is a projective generating family for semi-uniform spaces then $\{g_a\}$ is usually not a projective generating family for proximity spaces.

39 B.4. Theorem. *The mapping*

$$\{\mathscr{U} \to \gamma_{\mathbf{PU}}(\mathscr{U} \times \mathscr{U})\} : \upsilon\mathbf{U} \to \mathbf{P}$$

is an order-embedding. In particular,

$$\gamma_{\mathbf{PU}}(\mathscr{U} \times \mathscr{U}) \neq \gamma_{\mathbf{PU}}\mathscr{U} \times \gamma_{\mathbf{PU}}\mathscr{U}$$

provided that $\mathscr{U}$ is a uniformity which is not proximally coarse.

Proof. Clearly it is sufficient to show that the mapping in question is injective, and this follows from the following simple result.

39 B.5. *If $\mathscr{U}$ and $\mathscr{V}$ are semi-uniformities for a set P, then the filter of all proximal neighborhoods of a set X in $\langle P \times P, \mathscr{U} \times \mathscr{V}\rangle$ has the following sets for a base: $U \circ X \circ V$ with U in $\mathscr{U}$ and V in $\mathscr{V}$. In particular, the sets $U \circ V$, $U \in \mathscr{U}$, $V \in \mathscr{V}$, form a base of the filter of proximal neighborhoods of the diagonal of $P \times P$.*

Proof. By definition, the collection of all

$$U \times_{\mathrm{rel}} V = \mathbf{E}\{\langle\langle x_1, x_2\rangle, \langle y_1, y_2\rangle\rangle \mid \langle x_1, y_1\rangle \in U,\ \langle x_2, y_2\rangle \in V\},$$

$U \in \mathscr{U}$, $V \in \mathscr{V}$, is a base for the product semi-uniformity $\mathscr{U} \times \mathscr{V}$. It is easily seen that

$$(U \times_{\mathrm{rel}} V)[X] = U \circ X \circ Y$$

for each symmetric U and V, and each $X \subset P \times P$.

Remark. Notice that it follows from 39 B.4 that the theory of uniform spaces can be reduced to the theory of uniformizable proximities.

39 B.6. Theorem. *The transpose of a projective generating mapping for semi-uniform spaces to a mapping for proximity spaces is a projective generating mapping for proximity spaces.*

Proof. Let $f : \langle P, \mathscr{U}\rangle \to \langle Q, \mathscr{V}\rangle$ be a projective generating mapping for semi-uniform spaces and let p and q be the proximities induced by $\mathscr{U}$ and $\mathscr{V}$. By our assumption the set $\mathscr{U}'$ of all $(f \times f)^{-1}[V]$, $V \in \mathscr{V}$, is a base for $\mathscr{U}$ and we must show that XpY if and only if $f[X]\, q\, f[Y]$. By the definition of induced proximities we have $f[X]\, q\, f[Y]$ if and only if $V[f[X]] \cap f[Y] \neq \emptyset$ for each V in $\mathscr{V}$, and XpY if and only if $U[X] \cap Y \neq \emptyset$ for each U in $\mathscr{U}$, and hence, $\mathscr{U}'$ being a base for $\mathscr{U}$, XpY if and only if $U[X] \cap Y \neq \emptyset$ for each $U \in \mathscr{U}'$. Since evidently $V[f[X]] \cap f[Y] \neq \emptyset$ if and only if $(f \times f)^{-1}[V][X] \cap Y \neq \emptyset$, we obtain XpY if and only if $f[X]\, q\, f[Y]$.

39 B.7. Theorem. *Let $\{f_a\}$ be a projective generating family for semi-uniform spaces such that all $\mathbf{E}^*f_a$ except for at most one index a, are proximally coarse. If g_a is the transpose of f_a to a mapping for proximity spaces, then $\{g_a\}$ is a projective generating family of mappings for proximity spaces.*

Proof. Let $\langle P, \mathscr{U}\rangle$ be the common domain carrier of all f_a, $\mathscr{U}_a$ be the semi-uniformity projectively generated by $f_a : P \to \mathbf{E}^* f_a$, and p_a be the proximity induced by $\mathscr{U}_a$. By 37 A.2 $\mathscr{U} = \inf\{\mathscr{U}_a\}$ and, by 39 B.6, p_a is projectively generated by the mapping $g_a : P \to \mathbf{E}^* g_a$. By 39 A.3 the proximity generated by the family $\{g_a : P \to \mathbf{E}^* g_a\}$ is $\inf\{p_a\}$, and we need only prove that $\inf\{p_a\}$ is induced by $\mathscr{U} = \inf\{\mathscr{U}_a\}$. But this follows from the corollary to 38 B.14 because each $\mathscr{U}_a$, excepting at most one, is proximally coarse (by Theorem 37 B.3 which asserts that a semi-uniformity projectively generated by a mapping into a proximally coarse semi-uniform space is proximally coarse).

39 B.8. Corollary. *If $\langle P, \mathscr{U}\rangle$ is the product of a family of semi-uniform spaces $\{\langle P_a, \mathscr{U}_a\rangle\}$ and if p_a is the proximity induced by $\mathscr{U}_a$ and p is the proximity induced by $\mathscr{U}$, then $\langle P, p\rangle$ is the product of the family $\{\langle P_a, p_a\rangle\}$ whenever all the $\mathscr{U}_a$, excepting at most one, are proximally coarse.*

Combining 39 B.4 with 39 B.8 we obtain the following interesting characterization of proximally coarse semi-uniformities.

39 B.9. Theorem. *In order that a uniformity $\mathscr{U}$ be proximally coarse (i.e., totally bounded) it is necessary and sufficient that*

$$\gamma_{\mathbf{PU}}(\mathscr{U} \times \mathscr{U}) = \gamma_{\mathbf{PU}}\mathscr{U} \times \gamma_{\mathbf{PU}}\mathscr{U} .$$

Now we proceed to a description of classes of semi-uniform spaces which are stable under projective constructions.

39 B.10. Definition. A *class K of proximity spaces* is said to be *projective-stable* if every proximity space projectively generated by a family of mappings with range carriers in K belongs to K. A *class L of proximities* is said to be *projective-stable* if the class K consisting of all proximity spaces whose proximity structures belong to L is projective-stable.

39 B.11. Theorem. *Let K be a class of proximity spaces and L be the class consisting of proximity structures of spaces from K. Then K is projective-stable if and only if the following two conditions are fulfilled:*

(a) *L is completely meet-stable in* **P** *and contains all proximally accrete proximities.*

(b) *If f is a projective generating mapping for proximity spaces and $\mathbf{E}^* f \in K$, then $\mathbf{D}^* f \in K$.*

Proof. Apply 39 A.3 (compare with the corresponding results for projective construction for closure spaces and semi-uniform spaces).

39 B.12. Remark. Notice that condition (a) is equivalent to the statement that every proximity has an upper modification in L.

39 B.13. Theorem. *The class $\mathfrak{v}\mathbf{P}$ of all uniformizable proximity spaces is projective-stable.*

Proof. We shall verify conditions (a) and (b) of 39 B.11. Condition (a) is fulfilled by 38 C.3. Let f be a projective generating mapping of a proximity space $\langle P, p\rangle$ into a uniformizable proximity space $\langle Q, q\rangle$; we shall prove that $\langle P, p\rangle$ fulfils condition (prox 5). If X non p Y, then $f[X]$ non $p\, f[Y]$ and, $\langle Q, q\rangle$ being uniformizable, $\langle Q, q\rangle$ fulfils (prox 5), and therefore we can choose proximal neighborhoods U of $f[X]$ and V of $f[Y]$ such that $U \cap V = \emptyset$. Since f is proximally continuous, $f^{-1}[U]$ is a proximal neighborhood of X in $\langle P, p\rangle$ and $f^{-1}[V]$ is a proximal neighborhood of Y in $\langle P, p\rangle$.

39 B.14. Definition. The *projective progeny of a class K of proximity spaces*, denoted by $\text{proj}_{\mathbf{P}} K$ or simply proj K, is the class consisting of all proximity spaces projectively generated by families of mappings with range carriers in K. The *projective progeny of a class L of proximities* is the class consisting of proximity structures of spaces of proj K where K is the class of all proximity spaces whose proximity structures belong to L.

39 B.15. Theorem. *For any class K of proximity spaces*

$$\text{proj proj } K = \text{proj } K ,$$

that is to say, the projective progeny of any class K of proximity spaces is projective-stable.

Proof: 39 A.5 proj.

39 B.16. Theorem. *Let K be any class of proximity spaces and let K_1 be the class of all proximally accrete spaces. A proximity space $\mathscr{P}$ belongs to* proj K *if and only if $\mathscr{P}$ is homeomorph to a subspace of the product of a family of spaces from $K \cup K_1$.*

Corollary. *A class K of proximity spaces is projective-stable if and only if K contains all proximally accrete spaces, K is hereditary and completely productive, and, of course, K contains the proximal homeomorphs of all of its elements.*

Proof. Let K_2 be the class of all spaces satisfying the condition. Clearly $K_2 \subset \subset$ proj K. Suppose that $\mathscr{P} \in$ proj K and $\{f_a \mid a \in A\}$ is a projective generating family for $\mathscr{P}$ such that $\mathbf{E}^* f_a \in K$ for each a. If A is empty then $\mathscr{P}$ is a proximally accrete space and hence $\mathscr{P} \in K_2$. Assuming $A \neq \emptyset$ consider the reduced product f of $\{f_a\}$; by 39 A.12, f is a projective generating mapping. Let $\mathscr{Q}$ be the set $|\mathscr{P}|$ endowed with the proximally accrete proximity and let g be the reduced product of $\mathsf{J} : \mathscr{P} \to \mathscr{Q}$ and f. Clearly g is a projective generating mapping for $\mathscr{P}$, and $\mathbf{E}^* g \in K_2$. Since g is injective, g is a proximal embedding and hence $\mathscr{P} \in K_2$.

39 B.17. *Let K be a class of proximity spaces. The projective progeny of $\gamma_{\mathbf{CP}}[K]$ (in $\mathbf{C}$) coincides with $\gamma_{\mathbf{CP}}[\text{proj}_{\mathbf{P}} K]$ (i.e. the class of all spaces induced by spaces of the projective progeny of K).*

Corollary. *If a class K of proximity spaces is projective-stable, then the class* $\gamma_{\mathbf{CP}}[K]$ *is projective-stable.*

Proof. Use 39 B.2.

39 B.18. Example. Since the class of all uniformizable proximities is projective-stable and a closure space is uniformizable if and only if it is induced by a uniformizable proximity, we obtain from 39 B.16 a new proof of the fact that the class of all uniformizable spaces is projective-stable.

C. INDUCTIVE GENERATION

This subsection is concerned with the development of the properties of the inductive generation. We begin with a description of inductively generated proximities. Then we shall show that the transpose of an inductive generating family for semi-uniform spaces to a family of mapping for proximity spaces is an inductive generating family of mappings for proximity spaces (39 C.2), but the transpose of an inductive generating family of mappings for proximity spaces to a family of mappings for closure spaces need not be an inductive generating family (39 C.5). In 39 C.6 we shall summarize all earlier results which concern transposed families inheriting the properties of being an inductive generating family and of being a projective generating family. Then we shall give a characterization of inductive-stable classes of proximity spaces. We shall show that the class of all uniformizable proximity spaces is not inductive-stable and therefore, in the exercises, we shall introduce the notion of an inductive generating family for uniformizable proximity spaces.

39 C.1. Theorem. *Let a proximity space* $\langle P, p\rangle$ *be inductively generated by a non-void family of mappings* $\{f_a : \langle Q_a, q_a\rangle \to \langle P, p\rangle\}$. *Then*

(a) XpY *if and only if* $X \subset P$, $Y \subset P$ *and either* $X \cap Y \neq \emptyset$ *or* $f_a^{-1}[X]\, q_a f_a^{-1}[Y]$ *for some* a;

(b) *a subset* U *of* P *is a proximal neighborhood of a set* $X \subset P$ *if and only if* $U \supset X$ *and* $f_a^{-1}[U]$ *is a proximal neighborhood of* $f_a^{-1}[X]$ *in* $\langle Q_a, q_a\rangle$ *for each* a.

Proof. Combine 39 A.3 with the descriptions 38 A.1 and 38 A.6 of suprema in $\mathbf{P}(P)$.

39 C.2. Theorem. *If* $\{f_a\}$ *is an inductive generating family for semi-uniform spaces, then* $\{\gamma_{\mathbf{PU}} f_a\}$ *is an inductive generating family for proximity spaces (where* $\gamma_{\mathbf{PU}} f_a$ *denotes the transpose of* f_a *to a mapping for proximity spaces). Stated in other words, if a semi-uniformity* $\mathcal{U}$ *is inductively generated by a family of mapings* $\{f_a : \langle Q_a, \mathcal{V}_a\rangle \to P\}$, *then the proximity* p *induced by* $\mathcal{U}$ *is inductively generated by the family* $\{f_a : \langle Q_a, q_a\rangle \to P\}$ *where* q_a *is the proximity induced by* $\mathcal{V}_a$.

Proof. Every uniformly continuous mappings is proximally continuous, and therefore the proximity p' inductively generated by the family $\{f_a : \langle Q_a, q_a\rangle \to P\}$

is proximally finer than the proximity p induced by $\mathcal{U}$. To prove that p' is proximally coarser than p we must use the direct descriptions of $\mathcal{U}$, p and p'. Assuming XpY we must show that $Xp'Y$. Since p is induced by $\mathcal{U}$ we have $U[X] \cap Y \neq \emptyset$ for each $U \in \mathcal{U}$. Consider the families $\{X_a\}$ and $\{Y_a\}$, where $X_a = f_a^{-1}[X]$ and $Y_a = f_a^{-1}[Y]$ for each a. By 39 C.1, to prove $Xp'Y$ it is sufficient to show that $X \cap Y \neq \emptyset$ or $X_a\, q_a Y_a$ for some a. Thus the proof will be completed by showing that the assumption "$X \cap Y = \emptyset$ and X_a non $q_a Y_a$ for each a" leads to a contradiction. Since q_a is induced by $\mathcal{V}_a$ we can choose a family $\{V_a \mid a \in A\}$ such that $V_a \in \mathcal{V}_a$ and $V_a[X_a] \cap Y_a = \emptyset$ for each a. Put

$$U = J_P \cup \bigcup\{(f_a \times f_a)[V_a] \mid a \in A\}.$$

Since $\mathcal{U}$ is inductively generated by the family $\{f_a : \langle Q_a, \mathcal{V}_a\rangle \to P\}$, by 37 A.2 ind., the set U belongs to $\mathcal{U}$, and hence, by our assumption, $U[X] \cap Y \neq \emptyset$. On the other hand, it is easily seen that

$$f_a[V_a[X_a]] = ((f_a \times f_a)[V_a])[X]$$

and hence

$$U[X] \cap Y = (X \cap Y) \cup \bigcup\{((f_a \times f_a)[V_a])[X]) \cap Y \mid a \in A\} = \emptyset.$$

39 C.3. Theorem. *Every proximity space is inductively generated by a surjective mapping whose domain carrier is a discrete uniformizable proximity space.*

Proof. Let $\langle P, p\rangle$ be a proximity space. Choose a semi-uniformity $\mathcal{U}$ inducing p. By 37 A.8 the semi-uniform space $\langle P, \mathcal{U}\rangle$ is inductively generated by a surjective mapping $f : \langle Q, \mathcal{V}\rangle \to P$ where $\langle Q, \mathcal{V}\rangle$ is a discrete uniform space. If q is the proximity induced by $\mathcal{V}$, then p is inductively generated by the mapping $f : \langle Q, q\rangle \to P$ (39 C.2). Evidently $\langle Q, q\rangle$ is uniformizable and discrete.

From 39 C.3 we obtain the following two results:

39 C.4. *A proximity space inductively generated by a mapping of a uniformizable proximity space need not be uniformizable.*

Proof. Let $\mathcal{P}$ be a proximity space which is not uniformizable. By 39 C.3 the space $\mathcal{P}$ is inductively generated by a (surjective) mapping f such that $\mathbf{D}^*f$ is a uniformizable proximity space.

39 C.5. *If f is an inductive generating mapping for proximity spaces, then the transpose $g = \gamma_{\mathbf{CP}} f$ of f to a mapping for closure spaces need not be an inductive generating mapping for closure spaces.*

Proof. Let $\langle P, p\rangle$ be a non-discrete proximity space. By 39 C.3 there exists an inductive generating mapping f of a discrete uniformizable space $\langle Q, q\rangle$ into $\langle P, p\rangle$. If v is the closure induced by q, then v is discrete, and hence the closure u inductively generated by $f : \langle Q, v\rangle \to P$ is also discrete. Since p is not discrete, u is not induced by p.

39 C.6. For convenience we recall earlier results concerning transposed mappings. If $\{f_a\}$ is an inductive generating family for semi-uniform spaces, then $\{\gamma_{\mathbf{PU}} f_a\}$

is an inductive generating family for proximity spaces (39 C.2), and $\{\gamma_{\mathbf{CU}} f_a\}$ need not be an inductive generating family for closure spaces (e.g. by 37 A.8). If $\{f_a\}$ is a projective generating family for semi-uniform spaces, then $\{\gamma_{\mathbf{PU}} f_a\}$ need not be a projective generating family for proximity spaces (by 39 B.3), but $\{\gamma_{\mathbf{CU}} f_a\}$ is a projective generating family for closure spaces (by 37 B.6). If $\{f_a\}$ is an inductive generating family for proximity spaces, then $\{\gamma_{\mathbf{CP}} f_a\}$ need not be an inductive generating family for closure spaces (by 39 C.5). On the other hand, if $\{f_a\}$ is a projective generating family for proximity spaces, then $\{\gamma_{\mathbf{CP}} f_a\}$ is a projective generating family for closure spaces (by 39 B.2).

Now we proceed to inductive-stable classes of proximity spaces.

39 C.7. Definition. A *class K of proximity spaces* is said to be *inductive-stable* if every proximity space inductively generated by a family of mappings with domain carriers in K belongs to K. A *class L of proximities* is said to be *inductive-stable* if the class K consisting of all proximity space whose proximity structures belong to L is inductive-stable.

39 C.8. Theorem. *Let K be a class of proximity spaces and let L be the class consisting of proximity structures of spaces from K. Then K is inductive-stable if and only if the following two conditions are fulfilled:*

(a) *L is completely join-stable in* **P** *and contains all the proximally discrete proximities.*

(b) *If f is an inductive generating mapping for proximity spaces and* $\mathbf{D}^* f$ *belongs to K, then* $\mathbf{E}^* f$ *also belongs to K.*

Proof. Apply 39 A.3.

39 C.9. Remark. Notice that condition (a) is equivalent to the statement that every proximity has a lower modification in L.

39 C.10. Definition. The *inductive progeny of a class K of proximity spaces*, denoted by $\operatorname{ind}_{\mathbf{P}} K$ or simply $\operatorname{ind} K$, is the class of all spaces inductively generated by a family of mappings with the domain carriers in K.

Thus K is inductive-stable if and only if $\operatorname{ind} K = K$

39 C.11. *For any class K of proximity spaces*

$$\operatorname{ind} \operatorname{ind} K = \operatorname{ind} K .$$

that is to say, $\operatorname{ind} K$ *is inductive-stable.*

Proof: 39 A.5 ind.

D. QUOTIENTS

We have introduced the concepts of a quotient of a closure space under a mapping or an equivalence (33 C.1), a quotient mapping of a closure space into another one (33 C.1), a quotient of a semi-uniform space under a mapping or an equivalence

(37 D.1), and a quotient in the uniform sense (37 D.1). Now we shall introduce the corresponding concepts for proximity spaces.

39 D.1. Definition. Let $\mathscr{P}$ be a proximity space. If f is any mapping such that $\mathbf{D}\operatorname{gr} f$ contains the underlying class of $\mathscr{P}$, then the *proximal quotient of* $\mathscr{P}$ *under* f, denoted by $\mathscr{P}/f$, is defined to be the set $f[|\mathscr{P}|]$ endowed with the proximity inductively generated by the mapping $f : \mathscr{P} \to f[|\mathscr{P}|]$. A mapping of $\mathscr{P}$ into a proximity space $\mathscr{Q}$ is said to be a *proximal quotient mapping* or a *quotient mapping for proximity spaces* if $\mathscr{P}/f$ is a subspace of $\mathscr{Q}$. Finally, if ϱ is an equivalence on $\mathscr{P}$, then the *proximal quotient of* $\mathscr{P}$ *under* ϱ is defined to be the proximal quotient space $\mathscr{P}/f$, where f is the mapping $\{x \to \varrho[x]\}$ of $\mathscr{P}$ onto $|\mathscr{P}|/\varrho$ $(= \mathbf{E}\{\varrho[x] \mid x \in |\mathscr{P}|\})$.

39 D.2. Any inductive generating mapping for proximity spaces is a quotient mapping for proximity spaces. On the other hand a quotient mapping for proximity spaces f need not be an inductive generating mapping unless f is surjective. Indeed, if f is an inductive generating mapping for proximity spaces and X and Y are proximal and contained in $|\mathbf{E}^*f| - \mathbf{E}f$, then $X \cap Y \neq \emptyset$. On the other hand an embedding of a proximity space into another one is a quotient mapping. Thus e.g. $\mathsf{J} :]\!]\, 0, 1\, [\![\to \mathsf{R}$ is a quotient mapping for proximity spaces but not an inductive generating mapping for proximity spaces.

39 D.3. Theorem. *If f is a quotient mapping for semi-uniform spaces, then the transpose $\gamma_{\mathsf{PU}} f$ of f to a mapping for proximity spaces is a quotient mapping for proximity spaces; loosely speaking, a uniform quotient mapping is a proximal quotient mapping. If ϱ is an equivalence on a semi-uniform space $\mathscr{P}$, then*

$$\gamma_{\mathsf{PU}}(\mathscr{P}/\varrho) = (\gamma_{\mathsf{PU}}\mathscr{P})/\varrho\,.$$

Finally, if $\mathscr{P}$ is a semi-uniform space and f is a mapping such that $\mathbf{D}\operatorname{gr} f \supset |\mathscr{P}|$, then $\gamma_{\mathsf{PU}}(\mathscr{P}/f) = (\gamma_{\mathsf{PU}}\mathscr{P})/f$.

Proof. Evidently it is sufficient to prove the last statement under the additional assumption that f is a surjective generating mapping for semi-uniform spaces and $\mathscr{P}$ is the domain carrier of f, i.e. $\mathbf{E}^*f = \mathscr{P}/f$. By 39 C.2 the transpose g of f to a mapping for proximity spaces is an inductive generating mapping for proximity spaces. On the other hand, $\mathbf{D}^*g = \gamma_{\mathsf{PU}}\mathscr{P}$ and hence, f being surjective, $\mathbf{E}^*g = (\gamma_{\mathsf{PU}}\mathscr{P})/f$. Since $\mathbf{E}^*g = \gamma_{\mathsf{PU}}\mathbf{E}^*f = \gamma_{\mathsf{PU}}(\mathscr{P}/f)$, the formula is proved.

On the other hand a proximal quotient mapping need not be a quotient mapping (although a proximal continuous mapping is continuous). Indeed, Theorem 39 C.3 can be restated as follows.

39 D.4. *Every proximity space is a proximal quotient of a discrete uniformizable proximity space.*

It is important to know some sufficient condition for the formula $\gamma_{\mathsf{CP}}(\mathscr{P}/\varrho) = (\gamma_{\mathsf{CP}}\mathscr{P})/\varrho$ to be true. For convenience we shall give a direct description of the proximity structure of the proximal quotient $\mathscr{P}/f$ by means of the proximity structure of $\mathscr{P}$.

39 D.5. *Let f be a proximal quotient mapping of a proximity space $\mathscr{P} = \langle P, p\rangle$ onto a proximity space $\mathscr{Q} = \langle Q, q\rangle$; thus $\mathscr{Q} = \mathscr{P}/f$. Then*

$$XqY \Leftrightarrow X \subset Q,\ Y \subset Q,\ f^{-1}[X]pf^{-1}[Y],$$

and $U \subset Q$ is a proximal neighborhood of $X \subset Q$ in $\mathscr{Q}$ if and only if the set $f^{-1}[U]$ is a proximal neighborhood of $f^{-1}[X]$ in $\mathscr{P}$.

Proof. Since f is surjective, f is an inductive generating mapping for proximity spaces, and both statements then follow from 39 A.3.

39 D.6. Definition. A *proximity* p for a set P is said to be *fine around a subset X of P* if each neighborhood of X in $\langle P, \gamma_{\mathsf{CP}}p\rangle$ is a proximal neighborhood of X in $\langle P, p\rangle$. A *semi-uniformity* $\mathscr{U}$ for a set P is said to be *fine around a subset X of P* if the proximity induced by $\mathscr{U}$ is fine around X.

Notice that any proximity for a set P is fine around each singleton (x), $x \in P$. A simple characterization 39 D.7 is followed by the main result.

39 D.7. Theorem. *Let X be a non-void subset of a set P and let $\varrho = \mathsf{J}_P \cup (X \times X)$. A proximity p for P is fine around X if and only if*

$$(*) \qquad \gamma_{\mathsf{CP}}(\langle P, p\rangle/\varrho) = (\gamma_{\mathsf{CP}}\langle P, p\rangle)/\varrho\,.$$

A semi-uniformity $\mathscr{U}$ for P is fine around X if and only if

$$\gamma_{\mathsf{CU}}(\langle P, \mathscr{U}\rangle/\varrho) = (\gamma_{\mathsf{CU}}\langle P, \mathscr{U}\rangle)/\varrho\,.$$

Proof. It is sufficient to prove the first statement. Write $Q = P/\varrho$, $f = \{x \to \varrho[x] \mid x \in P\}$. Let u be the closure induced by p, q be the proximity inductively generated by $f : \langle P, p\rangle \to Q$ and v be the closure inductively generated by $f : \langle P, u\rangle \to Q$. Relation $(*)$ can then be written $\gamma_{\mathsf{CP}}q = v$. First assume that the equality holds. If U is a neighborhood of the set X in $\langle P, u\rangle$, then $V = f[U]$ is a neighborhood of the point X in $\langle Q, v\rangle$ because evidently $U = f^{-1}[V]$. According to the equality, the set V is a neighborhood of the point X in $\langle Q, \gamma_{\mathsf{CP}}q\rangle$ and hence (X) non $q\,(Q - V)$, and thus by 39 D.5, the sets $f^{-1}[(X)] = X$ and $f^{-1}[Q - V] = P - U$ are distant in $\langle P, p\rangle$; this shows that U is a proximal neighborhood of the set X in $\langle P, p\rangle$. Conversely, assuming that each neighborhood of X in $\langle P, u\rangle$ is a proximal neighborhood in $\langle P, p\rangle$ we must show that $\gamma_{\mathsf{CP}}q = v$. This follows, however, from the following theorem because each proximity is fine around each singleton.

39 D.8. Theorem. *If f is a quotient mapping for proximity spaces (semi-uniform spaces) and if the proximity structure (semi-uniform structure) of* $\mathbf{D}^*f$ *is fine around each inverse fibre of f, then the transpose of f to a mapping for closure spaces is a quotient mapping. If $\mathscr{P}$ is a proximity (semi-uniform) space whose structure is fine around each fibre of a given equivalence ϱ on $\mathscr{P}$, then the space $\gamma_{\mathsf{CP}}(\mathscr{P}/\varrho)$ or $\gamma_{\mathsf{CU}}(\mathscr{P}/\varrho)$ coincides with the space $(\gamma_{\mathsf{CP}}\mathscr{P})/\varrho$ or $(\gamma_{\mathsf{CU}}\mathscr{P})/\varrho$ respectively.*

Proof. It is sufficient to prove the statement concerning the quotients of proximity spaces under equivalences. Let ϱ be an equivalence on a proximity space $\langle P, p\rangle$,

$Q = P/\varrho$, $f = \{x \to \varrho[x] \mid x \in P\}$, q be the proximity inductively generated by $f : \langle P, p\rangle \to Q$, u the closure induced by p and v the closure inductively generated by $f : \langle P, u\rangle \to Q$. Assuming that p is fine around each fibre of ϱ (i.e. around inverse fibres of f), we must show that $v = \gamma_{\mathbf{CP}}q$. Now v is finer than $\gamma_{\mathbf{CP}}q$, and hence it is enough to prove that $\gamma_{\mathbf{CP}}q$ is finer than v, i.e. that if $(X) \in Q$ and V is a neighborhood of (X) in $\langle Q, v\rangle$, then V is a neighborhood of (X) in $\langle Q, \gamma_{\mathbf{CP}}q\rangle$. Let V be a neighborhood of (X) in $\langle Q, v\rangle$. Since the mapping $f : \langle P, u\rangle \to \langle Q, v\rangle$ is continuous, the set $U = f^{-1}[V]$ is a neighborhood of $X = f^{-1}[((X))]$ in $\langle P, u\rangle$. Since p is fine around X, the set U is proximal neighborhood of X in $\langle P, p\rangle$ and hence, by 39 D.5, V is a proximal neighborhood of the one-point set $((X))$ in $\langle Q, q\rangle$ because $X = f^{-1}[((X))]$, $U = f^{-1}[V]$.

39 D.9. Examples. Let $\mathscr{P} = \langle P, p\rangle$ be a proximity space and let u be the closure induced by p. (a) The proximity p is fine around each $X \subset P$ if and only if $X \cap uY = \emptyset$ implies $uX \cap Y = \emptyset$ and p is a fine proximity (i.e. $XpY \Leftrightarrow (uX \cap Y) \cup (X \cap uY) \neq \emptyset$). (b) The proximity p is fine around an open set X if and only if X is closed and distant to $P - X$. (c) If p is fine around each $X \subset P$, then u is a quasi-discrete closure. (d) If p is the Wallman proximity of $\langle P, u\rangle$ and u is topological, then p is fine around each closed set.

39 D.10. *A uniformizable closure space $\mathscr{P}$ is normal if and only if the Čech proximity of $\mathscr{P}$ is fine around each closed subset of $\mathscr{P}$.* — 39 D.9 (d).

We know that the projective progeny of a class K of proximity spaces consists of all homeomorphs of subspaces of arbitrary products of proximally accrete spaces or spaces from K. It has already been noted that to projective concepts (subspace, accrete, product) there correspond inductive concepts (quotient, discrete, sum). E.g. we shall establish the following description of the inductive progeny of a class of proximity spaces.

39 D.11. Theorem. *Let K be a class of proximity spaces and let K_1 be the class of all proximally discrete spaces. The inductive progeny of K consists of all quotients of sums of spaces from $K \cup K_1$.*

Corollary. *A class K is inductive-stable if and only if quotients of spaces of K belong to K, sums of families of spaces of K belong to K, and K contains all proximally discrete spaces.*

Proof. Consider the class K_2 of all spaces satisfying the condition. Evidently $K_2 \subset \operatorname{ind} K$. We shall prove that K_2 contains $\operatorname{ind} K$. Assuming that a space $\mathscr{P}$ is inductively generated by a family $\{f_a\}$ with $\mathbf{D}^*f_a$ in $K \cup K_1$ for each a, let us consider the reduced sum f of $\{f_a\}$; by 39 A.12 f is an inductive generating mapping for $\mathscr{P}$. If f is surjective then $\mathscr{P} = \mathbf{D}^*f/f$ and hence $\mathscr{P} \in K_2$ (because $\mathbf{D}^*f$ is a sum of spaces of $K \cup K_1$). If f is not surjective then consider the proximally discrete space $\mathscr{Q}$ such that $|\mathscr{P}| = |\mathscr{Q}|$. The reduced sum g of $\mathrm{J} : \mathscr{Q} \to \mathscr{P}$ and f is a surjective inductive generating mapping and hence $\mathscr{P} = \mathbf{D}^*g/g$. Clearly $\mathbf{D}^*g$ is the sum of spaces from $K \cup K_1$.

40. PRESHEAVES

If $\{P_a \mid a \in A\}$ is a family of sets and $\{f_a\}$ is a family, each f_a being a mapping of a set P into P_a, then there exists a unique mapping f of P into $\Pi\{P_a\}$ such that $f_\alpha = (\mathrm{pr}_\alpha : \Pi\{P_a\} \to P) \circ f$ for each α in A. The mapping f is the reduced product of $\{f_a\}$. If P_a are closure spaces, proximity spaces or uniform spaces, and all f_a are continuous, proximally continuous or uniformly continuous respectively, then the same is true with f continuous, proximally continuous or uniformly continuous respectively. Let a family of sets $\{P_a \mid a \in A\}$ be given, and for each pair $\langle a, b\rangle \in R$, where R is a given subset of $A \times A$, let f_{ab} be a mapping of P_a into P_b. It is then natural to inquire whether there exists a set Q and mappings $g_a : Q \to P_a$ with the following property: if $\{f_a\}$ is any family of mappings $f_a : P \to P_a$ with $f_b = f_{ab} \circ f_a$ whenever $\langle a, b\rangle \in R$, then there exists a unique $f : P \to Q$ with $f_a = g_a \circ f$ for each a. The answer is in the affirmative under the assumption that $f_{bc} \circ f_{ab} = f_{ac}$ for each $\langle a, b\rangle$, $\langle b, c\rangle$ and $\langle a, c\rangle$ in R; we shall be concerned with the case where R is a quasi-order for A; then $\mathscr{S} = \langle\{P_a\}, \{f_{ab} \mid \langle a, b\rangle \in R\}\rangle$ will be called a presheaf of sets over $\langle A, R\rangle$ and a certain Q will be called the projective limit of $\mathscr{S}$ (denoted by $\varprojlim \mathscr{S}$); the product $\Pi\{P_a\}$ will be a particular case of $\varprojlim \mathscr{S}$. The same problem and definitions apply to presheaves of closure spaces (P_a are spaces, f_{ab} are continuous mappings, f, f_a, g_a are required to be continuous), proximity spaces or semi-uniform spaces. A "dual" reasoning leads to inductive limits. Roughly speaking, the projective limits are related to the products as the inductive limits to the sums.

Subsection A concerns limits of presheaves of sets. We shall show in subsection B that any projective presheaf of sets over the ordered set of open subsets of a closure space $\mathscr{P}$ is isomorphic to the sheaf of continuous sections of a covering fibration over $\mathscr{P}$. Subsection C concerns limits of presheaves of spaces.

A. PRESHEAVES OF SETS AND THEIR LIMITS

For convenience we shall begin with a review of terminology concerning quasi-ordered sets. A quasi-ordered set will be a $\langle A, \leqq\rangle$ where A is a set and $\leqq$ is a reflexive and transitive relation on A. A subset B of a $\langle A, \leqq\rangle$ is left-cofinal or left-saturated if, respectively, each $a \in A$ follows some $b \in B$, or each $a \in A$ preceding some element of B belongs to B. The right-cofinal and right-saturated subsets are defined similarly. Finally, $\langle A, \leqq\rangle$ is left-directed (right-directed) if $A \neq \emptyset$ and every two elements of A are preceded (followed) by an element of A. A subset B of a quasi-ordered set $\langle A, \leqq\rangle$ will usually be considered as a quasi-ordered subset of $\langle A, \leqq\rangle$ and usually the relativized quasi-order will also be denoted by $\leqq$. A left-cofinal (right-cofinal) subset of a left-directed (right-directed) set is left-directed (right-directed).

40 A.1. Definition. A *presheaf of sets over a quasi-ordered set* $\langle A, \leqq\rangle$ is a pair $\mathcal{S} = \langle\{P_a \mid a \in A\}, \{f_{ab} \mid a \leqq b\}\rangle$ such that $\{P_a\}$ is a family of sets, each f_{ab} is a mapping of P_a into P_b (i.e. $P_a = \mathbf{D}^* f_{ab}$, $P_b = \mathbf{E}^* f_{ab}$) and the following two conditions are fulfilled:

(a) f_{aa} is the identity mapping of P_a;

(b) if $a \leqq b$ and $b \leqq c$, then $f_{ac} = f_{bc} \circ f_{ab}$.

If B is an ordered subset of $\langle A, \leqq\rangle$, then the presheaf $\langle\{P_a \mid a \in B\}, \{f_{ab} \mid a \leqq b, a \in B, b \in B\}\rangle$ will be called the *restriction of* $\mathcal{S}$ *to* B and will be denoted by $\mathcal{S}_B$. The mappings f_{ab} are called *connecting mappings of* $\mathcal{S}$ and the quasi-ordered set $\langle A, \leqq\rangle$ is called the *base of* $\mathcal{S}$.

It is to be noted that a presheaf is uniquely determined by the family $\{f_{ab}\}$; indeed, P_a is the domain of f_{aa}. Nevertheless, in the most important examples we shall only be interested in the sets P_a; A will be a collection of sets ordered by inverse inclusion $\supset$, each P_a will be a collection of mappings with domain a and f_{ab} will be a mapping assigning to each $x \in P_a$ the restriction of x to b; thus f_{ab} will be uniquely determined by a, b, P_a and P_b.

40 A.2. Presheaves of continuous mappings. Let P and Q be closure spaces. For each $X \subset P$ let C_X be the set of all continuous mappings of the subspace X into Q (i.e. $C_X = C(X, Q)$), and for $X \supset Y$ let f_{XY} be the mapping of C_X into C_Y which assigns to each $g \in C_X$ the domain-restriction $g \mid Y$ of g to Y (we must know that the restriction of a continuous mapping is a continuous mapping). Clearly $\langle\{C_X\}, \{f_{XY}\}\rangle$ is a presheaf over $\langle \exp P, \supset\rangle$. This presheaf will be called the *presheaf of continuous mappings of* P *into* Q and the mappings f_{XY} are usually called *restriction mappings*. It should be noted that the term restriction mapping is sometimes used for mappings f_{ab} in any presheaf. Now if P and Q are semi-uniform spaces then $\langle\{C_X\}, \{f_{XY}\}\rangle$ will be a presheaf over $\langle \exp P, \supset\rangle$ if C_X is the collection of all uniformly continuous mappings of the subspace X of P into Q and if f_{XY} are the corresponding restriction mappings. This presheaf will be called the *presheaf of uniformly continuous mappings of* P *into* Q. In a similar way we define the *presheaf of proximally continuous mappings of a proximity space* P *into another one* Q. In what follows, if $\langle\{P_a\}, \{f_{ab}\}\rangle$ is a presheaf over a collection of sets A such that P_a is a set of mappings with domain a and f_{ab} is the restriction mapping, then this presheaf will be denoted simply by $\{P_a\}$.

From the definition of presheaves we shall derive the following simple but useful result.

40 A.3. *If $\langle\{P_a\}, \{f_{ab}\}\rangle$ is a presheaf over $\langle A, \leqq\rangle$ and $a \leqq b$, $b \leqq a$, then $f_{ab} = f_{ba}^{-1}$; in particular, f_{ab} is bijective.*

Proof. Indeed, from Definition 40 A.1, condition (b), we obtain $f_{aa} = f_{ba} \circ f_{ab}$, $f_{bb} = f_{ab} \circ f_{ba}$. Since by condition (a) of the definition the mappings f_{aa} and f_{bb} are identity mappings of P_a and P_b respectively, the first equality implies that f_{ab} is injective and f_{ba} is surjective and the second one implies f_{ba} is injective and f_{ab} is surjective. Thus both mappings f_{ba} and f_{ab} are simultaneously injective and surjective, that is bijective. Now, f_{ab} is the inverse of f_{ba}, e.g. by the first equality.

With every presheaf of sets there are associated two sets — the projective limit and the inductive limit. We begin with the former.

40 A.4. Definition. Let $\mathscr{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ be a presheaf of sets over a quasi-ordered set $\langle A, \leqq\rangle$. The *projective limit of* $\mathscr{S}$ (in another terminology, the *inverse limit of* $\mathscr{S}$) is defined to be the set of all elements $x = \{x_a\}$ of the product of the family $\{P_a \mid a \in A\}$ such that

$$a \leqq b \quad \text{implies} \quad f_{ab}x_a = x_b\,.$$

The projective limit of $\mathscr{S}$ will be denoted by $\varprojlim \mathscr{S}$ $(= \varprojlim \langle\{P_a\}, \{f_{ab}\}\rangle)$, and this notation will sometimes be abbreviated to $\varprojlim \{P_a\}$. For each a in A the mapping $\mathrm{pr}_a : \varprojlim \mathscr{S} \to P_a$ will called the *projection of* $\mathscr{S}$ *into* P_a or the a-th *projection of* $\varprojlim \mathscr{S}$, and will usually be denoted by f_a.

Obviously the projections of a projective limit are restrictions of corresponding projections of the corresponding product set. The projective limit is derived from the notion of the product set. On the other hand the product set is a special case of the projective limit. Indeed, if $\{P_a \mid a \in A\}$ is a family of sets and $a \leqq b$ if and only if $a = b \in A$, then $\mathscr{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$, where f_{ab} is the identity mapping of P_a onto P_b (since $a = b$), is a presheaf over $\langle A, \leqq\rangle$ and its projective limit coincides with the product of the family $\{P_a\}$.

40 A.5. Definition. A *projective family* $\{g_a \mid a \in A\}$ *of mappings is* said to be *compatible for a presheaf* $\mathscr{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ *over* $\langle A, \leqq\rangle$ if $P_a = \mathbf{E}^*f_a$ for each a in A and $a \leqq b$ implies $g_b = f_{ab} \circ g_a$.

40 A.6. The family of all projections of the projective limit of a presheaf $\mathscr{S}$ is compatible for $\mathscr{S}$.

Proof. With the usual notation, let $a \leqq b$ and $x = \{x_c \mid c \in A\}$ be any point of $\varprojlim \mathscr{S}$. By definition 40 A.4 we have $f_{ab}x_a = x_b$, $f_a x = x_a$ and $f_b x = x_b$, which yields $(f_{ab} \circ f_a)\, x = f_b x$.

40 A.7. Theorem. *If $\mathscr{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ is a presheaf over $\langle A, \leqq\rangle$ and $\{g_a\}$ is a family of mappings of a set Q and $\{g_a\}$ is compatible for $\mathscr{S}$, then the relation $\{x \to \{g_a x \mid a \in A\} \mid x \in Q\}$, which is the relational reduced product of the family $\{\mathrm{gr}\, g_a\}$, ranges in the projective limit of $\mathscr{S}$.*

Proof. Fix an x in Q. If $a \leqq b$, then $g_b = f_{ab} \circ g_a$ and hence $g_b x = (f_{ab} \circ g_a)\, x = f_{ab}(g_a x)$.

Corollary (a). *There exists exactly one mapping g of Q into $\varprojlim \mathcal{S}$ such that*

(*) $$g_a = (\mathrm{pr}_a : \varprojlim \mathcal{S} \to P_a) \circ g \textit{ for each } a.$$

The graph of g is the relational reduced product of $\{g_a\}$, that is, $gx = \{g_a x \mid a \in A\}$ for each $x \in Q$.

Proof. If g fulfils (*) then necessarily $g_a x = \mathrm{pr}_a\, gx$ and hence there exists at most one such mapping. By the theorem there exists at least one such mapping, namely that from the second statement of the corollary.

Corollary (b). *Suppose that $\mathcal{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ is a presheaf over $\langle A, \leqq\rangle$, R is a set, $\{h_a\}$ is a family of mappings compatible for $\mathcal{S}$ such that $\mathbf{D}h_a = R$ for each a in A, and finally assume that, if Q is a set and $\{g_a \mid a \in A\}$ is a family of mappings compatible for $\mathcal{S}$ such that $\mathbf{D}g_a = Q$ for each a, then there exists exactly one mapping g of Q into R such that $g_a = h_a \circ g$ for each a in A.*

Then there exists an injective mapping k of R onto the projective limit of $\mathcal{S}$ such that $h_a = (\mathrm{pr}_a : \varprojlim \mathcal{S} \to P_a) \circ k$ for each a in A.

40 A.8. Definition. Let $\mathcal{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ be a presheaf over $\{A, \leqq\rangle$ and let α be a lower bound of a subset B of A. By the corollary of 40 A.7 there exists exactly one mapping g of P_α into $\varprojlim \mathcal{S}_B$ such that $f_{\alpha b} = (\mathrm{pr}_b : \varprojlim \mathcal{S}_B \to P_b) \circ g$ for each $b \in B$. This mapping will be called the *canonical mapping of P_α into $\varprojlim \mathcal{S}_B$*.

40 A.9. *Under the assumptions of* 40 A.8, *if $\alpha \in B$ (thus $\alpha = \inf B$) then the canonical mapping g of P_α into $\varprojlim \mathcal{S}_B$ is bijective.*

Proof. If $gx = gy$, then $\mathrm{pr}_b\, gx = \mathrm{pr}_b\, gy$ for each $b \in B$; substituting $b = \alpha$ we obtain $x = y$ because always $\mathrm{pr}_\alpha \circ g = f_{\alpha\alpha}$. Thus g is injective. If $\{x_b \mid b \in B\}$ is any point of $\varprojlim \mathcal{S}_B$, then $x_b = f_{\alpha b} x_\alpha$ for each $b \in B$ because $\alpha \leqq b$ for each $b \in B$. Now clearly $\{x_b \mid b \in B\} = gx_\alpha$.

Corollary. *If $\mathcal{S}$ is a presheaf over $\langle A, \leqq\rangle$ and α is a least element of A, then $\{x \to \{f_{\alpha a} x \mid a \in A\}\}$ is a one-to-one relation on P_α onto $\varprojlim \mathcal{S}$.*

Remark. Let $\mathcal{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ be any presheaf over $\langle A, \leqq\rangle$. Let us consider the set A' consisting of all points of A and one further point, say α. Define a quasi-order $\prec$ on A' so that $\langle A, \leqq\rangle$ becomes a quasi-ordered subset of A' and α is the least element of A'. Finally put $P_\alpha = \varprojlim \mathcal{S}$ and $f_{\alpha a} = \mathrm{pr}_a : P_\alpha \to P_a$ for a in A. Then $\mathcal{S}' = \langle\{P_a \mid a \in A'\}, \{f_{ab}\}\rangle$ is a presheaf over $\langle A', \prec\rangle$, and the canonical mapping of P_α into $\varprojlim \mathcal{S}'_A = \varprojlim \mathcal{S}$ (because $\mathcal{S}_A = \mathcal{S}$) is bijective. Thus any presheaf $\mathcal{S}$ over A is the restriction of a presheaf $\mathcal{S}' = \langle\{P_a\}, \{f_{ab}\}\rangle$ over a base with a least element α such that the canonical mapping of P_α into $\varprojlim \mathcal{S}$ corresponding to $\{f_{\alpha a} \mid a \in A\}$ is bijective.

It should be noted that, in general, a canonical mapping g of P_α into $\varprojlim \mathcal{S}_B$, where α is a lower bound of B, is neither injective nor surjective. For example, if $\alpha \leqq \beta$, and $B = (\beta)$, then $\varprojlim \mathcal{S}_B = \Pi\{P_a \mid a \in (\beta)\} = (\beta) \times P_\beta$ and the mapping g assigns to each $x \in P_\alpha$ the point $\langle \beta, f_{\alpha\beta} x\rangle$; it follows that g is injective or surjective if and only if the mapping $f_{\alpha\beta}$ has the corresponding property. But $f_{\alpha\beta}$ need not be injective nor surjective. It seems to be appropriate to examine injectivity and surjectivity of canonical mappings g for each α and B in a suitable presheaf.

40 A.10. Example. Let us consider the presheaf $\mathcal{S} = \langle\{P_X\}, \{r_{XY}\}\rangle$ over $\langle \exp R, \supset\rangle$, where R is a set, $P_X = Q^X$ (i.e. the set of all single-valued relations on X into Q) where Q is a given set independent on X, and r_{XY} is the restriction mapping of P_X into P_Y, that is, r_{XY} assigns to each $\sigma \in P_X$ the restriction $\sigma|Y$ of σ to Y (notice that the mappings r_{XY} are surjective). Choose a non-void ordered subset $\mathcal{X}$ of $\langle \exp R, \supset\rangle$ and a lower bound U of $\mathcal{X}$, i.e. $U \supset \bigcup\mathcal{X}$, and consider the canonical mapping g of P_U into $\varprojlim \mathcal{S}_{\mathcal{X}}$; thus g assigns to each $\sigma \in P_U$ the family $\{\sigma|x \mid x \in \mathcal{X}\}$. Now if Q possesses at most one point, then each P_X consists of the constant relation, and clearly g is bijective. In what follows let us suppose that Q has at least two points.

(α) The mapping g is injective if and only if $U = \bigcup\mathcal{X}$ (that is, $U = \inf \mathcal{X}$).

Proof. First suppose that $U = \bigcup\mathcal{X}$ and $\varrho, \sigma \in P_U$, $\varrho \neq \sigma$. There exists a point x of U such that $\varrho x \neq \sigma x$. If we choose an X in $\mathcal{X}$ containing the point x, then $\varrho|X \neq \sigma|X$, but $\varrho|X$ and $\sigma|X$ are, respectively, the X-th coordinate of $g\varrho$ and $g\sigma$. Thus $g\varrho \neq g\sigma$. Conversely, suppose that there exists an x in $U - \bigcup\mathcal{X}$. Since Q has at least two elements, there exist ϱ and σ in P_U such that $\varrho x \neq \sigma x$ but $\varrho y = \sigma y$ for each $y \neq x$. It follows that $\varrho|X = \sigma|X$ for each $X \in \mathcal{X}$, and consequently $g\varrho = g\sigma$.

(β) In order that the mapping g be surjective it is sufficient that $X_1 \cap X_2 \in \in \mathcal{X}$ provided that both X_1 and X_2 belong to $\mathcal{X}$; in particular, if $\mathcal{X}$ is right saturated then g is surjective.

Proof. Suppose that the condition is fulfilled and let us consider any element $\{\varrho_X \mid X \in \mathcal{X}\}$ of $\varprojlim \mathcal{S}_{\mathcal{X}}$. It is easily seen that $x \in X_1 \cap X_2, X_\iota \in \mathcal{X}$ implies $\varrho_{X_1}x = \varrho_{X_2}x$. Indeed, $X = X_1 \cap X_2$ belongs to $\mathcal{X}$ and follows both X_1 and X_2; thus $\varrho_X = \varrho_{X_1} \mid X = \varrho_{X_2} \mid X$, in particular, $\varrho_{X_1}x = \varrho_{X_2}x$. It follows that we can define a relation ϱ on U into Q so that $\varrho x = \varrho_X x$ if $x \in X \in \mathcal{X}$ (if $x \in U - \bigcup\mathcal{X}$ then ϱx can be any element of Q). Clearly $\varrho|X = \varrho_X$ for each $X \in \mathcal{X}$.

A necessary and sufficient condition for g to be surjective is rather complicated.

40 A.11. Definition. A *presheaf* $\mathcal{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ over $\langle A, \leqq\rangle$ will be called *projective at an index* $\alpha \in A$ if the following two conditions are fulfilled:

(a) if $\alpha = \inf B$, then the canonical mapping $\{x \to \{f_{\alpha b}x \mid b \in B\}\}$ of P_α into $\varprojlim \mathcal{S}_B$ is injective;

(b) if $\alpha = \inf B$ and B is right-saturated, then the canonical mapping $\{x \to \to \{f_{\alpha b}x \mid b \in B\}\}$ of P_α into $\varprojlim \mathcal{S}_B$ is surjective.

A *presheaf* is said to be *projective* if it is projective at each index. Instead of projective presheaf we will often use the term *sheaf*.

40 A.12. Examples. (a) The presheaf in example 40 A.10 is projective. (b) On the other hand the presheaves of continuous mappings usually are not projective.

For example let $\{C_X\}$ be the presheaf of continuous functions on a space P (see example 40 A.2) and consider the collection $\mathscr{X}$ consisting of all one-point sets and the void set. Clearly each C_X, $X \in \mathscr{X}$, consists of all functions on the subspace X of P. If f is a function on P which is not continuous, and if f_X is the domain-restriction of f to X, then $f_X \in C_X$ for each $X \in \mathscr{X}$ and the family $\{f_X \mid X \in \mathscr{X}\}$ belongs to the projective limit of the presheaf $\{C_X \mid X \in \mathscr{X}\}$; nevertheless, there exists no continuous function h on P such that $f_X = h|X$ for each $X \in \mathscr{X}$; indeed, if $h|X = f_X$ for each $X \in \mathscr{X}$, then necessarily $h = f$; but f was chosen not continuous. It follows that if P is not discrete, then the presheaf of continuous functions on P is not projective at P; in particular, it is not projective.

(c) Consider again the presheaf $\{C_X\}$ of continuous mappings of a space $\mathscr{P}$ into a space $\mathscr{Q}$. Let U be a subset of $\mathscr{P}$ and let $\mathscr{X}$ be a multiplicative collection of subsets of U which interiorly covers U. It is easily seen that the canonical mapping of C_U into the projective limit of $\{C_X \mid X \in \mathscr{X}\}$ is bijective. Indeed, let $\{f_X \mid X \in \mathscr{X}\}$ be an element of $\varprojlim \{C_X \mid X \in \mathscr{X}\}$. As in 40 A.10 one cand find a mapping f of U into $\mathscr{Q}$ such that $f_X = f|X$ for each $X \in \mathscr{X}$. Since $\mathscr{X}$ interiorly covers U, the mapping f is continuous (roughly speaking, since f is locally continuous, f is continuous, see 17 A.19).

(d) It follows from (c) that the restriction of the presheaf of continuous mappings of $\mathscr{P}$ into $\mathscr{Q}$ to the ordered subset $\mathscr{U}$ of all open subsets of $\mathscr{P}$ is a projective presheaf.

40 A.13. Definition. A *sheaf* is a projective presheaf $\mathscr{S}$ whose base is the set of all open subsets of a closure space $\mathscr{P}$ ordered by inclusion; we shall say that $\mathscr{S}$ is a *sheaf over* $\mathscr{P}$. More generally, if the collection of all open subsets (occasionally, all subsets) of a closure space $\mathscr{P}$ is a base of a presheaf $\mathscr{S}$, then we shall say that $\mathscr{S}$ is *over* $\mathscr{P}$.

Let $\mathscr{U}$ be the collection of all open subsets of a space $\mathscr{P}$ and let $\mathscr{Q}$ be a space. The restriction to $\mathscr{U}$ of the presheaf of continuous mappings of $\mathscr{P}$ into $\mathscr{Q}$ is a sheaf by 40 A.12 (d) which will be called the *sheaf of continuous mappings of* $\mathscr{P}$ *into* $\mathscr{Q}$.

40 A.14. Remarks to Definition 40 A.11. (α) It is easily seen that the condition (a) can be weakened by requiring the set B to be right saturated. Indeed, assume the weaker condition and let $\alpha = \inf B$. Consider the smallest right saturated set B_1 containing B; clearly B_1 consists of all $b_1 \in A$ following some $b \in B$. Clearly $g = f \circ g_1$, where g and g_1 are canonical mappings of P_α into the projective limit of $\mathscr{S}_B$ and $\mathscr{S}_{B_1}$ respectively, and f is the canonical mapping of $\varprojlim \mathscr{S}_{B_1}$ into $\varprojlim \mathscr{S}_B$, i.e. $f(\{x_b \mid b \in B_1\}) = \{x_b \mid b \in B\}$. The mapping g_1 is injective by virtue of the weakened condition (a), and the mapping f is injective because each $b_1 \in B_1$ is preceded by some $b \in B$. It follows that g is injective. Now the conditions (a) and (b) can be replaced by the following condition:

If $\alpha = \inf B$ and B is right-saturated, then the canonical mapping $\{x \to \{f_{\alpha b}x \mid b \in B\}\}$ of P_α into $\varprojlim \mathscr{S}_B$ is bijective.

(β) If each finite family in $\langle A, \leqq\rangle$ possesses a supremum, then the condition (b) can be strengthened by replacing the requirement "B is right-saturated" by the following weaker requirement:

(∗) If $b_1, b_2 \in B$ then $\sup(b_1, b_2) \in B$.

For example, if $\mathscr{S}$ is a presheaf over $\langle \exp R, \supset\rangle$ then (∗) requires B to be multiplicative. The proof is very simple. Suppose that $\mathscr{S}$ is projective at α and B is a set satisfying (∗). We must show that the canonical mapping of P_α onto $\varprojlim \mathscr{S}_B$ is surjective, i.e. that $\mathbf{E}\{x \to \{f_{\alpha b}x \mid b \in B\} \mid x \in P_\alpha\} = \varprojlim \mathscr{S}_B$.

Consider the smallest right saturated set B' containing B. Since $\mathscr{S}$ is projective at α, we have

$$\mathbf{E}\{x \to \{f_{\alpha b}x \mid b \in B'\} \mid x \in P_\alpha\} = \varprojlim \mathscr{S}_{B'} .$$

It follows that it is enough to show that each family $\{x_b \mid b \in B\} \in \varprojlim \mathscr{S}_B$ can be extended to a family $\{x_b \mid b \in B'\} \in \varprojlim \mathscr{S}_{B'}$. If $b \in B' - B$ and $b_1 \leqq b$, $b_2 \leqq b$, then also $b_3 = \sup(b_1, b_2) \in B$ and hence $f_{b_1b}x_{b_1} = f_{b_3b}x_{b_3} = f_{b_2b}x_{b_2}$, and consequently the value $f_{b_1b}x_{b_1}$ does not depend on the choice of $b_1 \leqq b$ in B. Thus we can define x_b for b in $B' - B$ as f_{b_1b} with a $b_1 \leqq b$ (such a b_1 exists).

(γ) If $\mathscr{S}$ is projective at the greatest element α, then P_α is a one-point set. Indeed, clearly $B = \emptyset$ is a right-saturated set in $\langle A, \leqq\rangle$ and $\alpha = \inf B$. Since $\varprojlim \mathscr{S}_B$ contains exactly one element (namely $\emptyset$), the set P_α contains at most one element by (a) and at least one element by (b).

We shall return to an examination of projective presheaves in subsection B. Now we present the definition of the inductive limit of a presheaf.

40 A.15. Definition. Let us suppose that $\mathscr{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ is a presheaf of sets over a quasi-ordered set $\langle A, \leqq\rangle$. Let ϱ be the smallest equivalence on the sum P of the family $\{P_a \mid a \in A\}$ such that $\langle a, x\rangle \;\varrho\; \langle b, y\rangle$ provided that $f_{ac}x = f_{bc}y$ for some c (which necessarily follows both a and b). The quotient of P under ϱ will be termed the *inductive limit* of $\mathscr{S}$, and denoted by $\varinjlim \mathscr{S}$ (i.e., $\varinjlim \langle\{P_a\}, \{f_{ab}\}\rangle$) or merely $\varinjlim \{P_a\}$. For each a the mapping $\{x \to \varrho[\langle a, x\rangle]\}$ of P_a into $\varinjlim \mathscr{S}$ will be termed the *canonical mapping of* P_a *into* $\varinjlim \mathscr{S}$ and will often be denoted by ${}^af$.

40 A.16. Remarks. (α) As it stands, the notion of inductive limit is derived from the notion of sum. On the other hand, the sum is "almost" a special case of inductive limits. Indeed, if $\{P_a \mid a \in A\}$ is a family of sets and if $a \leqq b$ if and only if $a = b \in A$, then $\mathscr{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$, where f_{ab} is the identity mapping of P_a onto P_b (since $a = b$), is a presheaf over $\langle A, \leqq\rangle$ and its inductive limit is the quotient set of the sum $\Sigma\{P_a\}$ under the identity equivalence on the sum, that is,

$$\varinjlim \mathscr{S} = \mathbf{E}\{(x) \mid x \in \Sigma\{P_a\}\} .$$

(β) Evidently $\sigma = \{\langle a, x\rangle \to \langle b, y\rangle \mid f_{ac}x = f_{bc}y \text{ for some } c \text{ in } A\}$ is a reflexive and symmetric relation on $\Sigma\{P_a\}$ which need not be transitive. Nevertheless, if $\langle A, \leqq\rangle$ is right directed then σ is transitive, and hence σ is an equivalence on P and therefore $\sigma = \varrho$. The proof of this fact is a matter of a simple calculation. Suppose $\langle a, x\rangle \sigma \langle b, y\rangle$, $\langle b, y\rangle \sigma \langle d, z\rangle$. By definition of σ, there exist c and e such that $f_{ac}x = f_{bc}y$, $f_{be}y = f_{de}z$ (thus $a, b \leqq c$ and $b, d \leqq e$). Since $\langle A, \leqq\rangle$ is right directed, we can choose an l in A following both c and e and hence also a, b, d. Now clearly

$$f_{al}x = f_{cl}f_{ac}x = f_{cl}f_{bc}y = f_{bl}y = f_{el}f_{be}y = f_{el}f_{de}z = f_{dl}z\ ,$$

hence $f_{al}x = f_{dl}z$, which proves $\langle\langle a, x\rangle, \langle d, z\rangle\rangle \in \sigma$ and establishes the transitivity.

(γ) As we noticed in (β), the relation σ generating ϱ is reflexive and symmetric but need not be transitive. Sometimes it is convenient to define ϱ as the smallest equivalence containing the following relation σ_1, which is reflexive and transitive, but in general not symmetric:

$$\sigma_1 = \{\langle a, x\rangle \to \langle b, y\rangle \mid a \leqq b \quad\text{and}\quad f_{ab}x = y\}\ .$$

Obviously $\sigma_1 \subset \sigma$. As we know (3 F.4) the relation ϱ can de described in terms of σ as follows (this description obtains under the assumption that σ is symmetric and reflexive): $\langle \xi, \eta\rangle \in \varrho$ if and only if there exists a finite sequence $\{\xi_i \mid i \leqq n\}$ such that $\xi_0 = \xi$, $\xi_n = \eta$ and $\langle \zeta_i, \zeta_{i+1}\rangle \in \sigma$ for each $i = 0, 1, \ldots, n - 1$. It is easily seen that the relation ϱ can be described in terms of σ_1 as follows: $\langle \xi, \eta\rangle \in \varrho$ if and only if there exists a finite sequence $\{\xi_i \mid i \leqq n\}$ such that $\xi_0 = \xi$, $\xi_n = \eta$, $\langle \xi_i, \xi_{i+1}\rangle \in \sigma_1 \cup \sigma_1^{-1}$ for each $i \leqq n - 1$.

(δ) The canonical mapping of P_a into the inductive limit is the composite of the canonical embedding $\{x \to \langle a, x\rangle\}$ of P_a into the corresponding sum followed by the canonical mapping $\{\langle b, y\rangle - \varrho[\langle b, y\rangle]\}$ of the sum onto the inductive limit.

(ε) If $A = \emptyset$, then $\Sigma\{P_a \mid a \in A\} = \emptyset$ and hence $\varinjlim \mathscr{S} = \emptyset$ as well.

Before presenting the general theory we give two examples. The first of them shows that the notion of the inductive limit implicitly occurs in many theorems concerning the local behavior of functions. The second very simple example serves as an introduction to further rather general theorems.

40 A.17. Germs of a presheaf over subsets of a closure space. (a) Let $\langle\{C_X\}, \{f_{XY}\}\rangle$ be a presheaf over a closure space $\mathscr{P}$, and for each x in $\mathscr{P}$ let $\mathscr{U}_x$ be the neighborhood system at x. The elements of $\varinjlim \{C_X \mid X \in \mathscr{U}_x\}$ are called the *germs of the presheaf* $\{C_X\}$ *at the point* x. Since $\mathscr{U}_x$ is right directed, the relation $\sigma = \{\langle X, s\rangle \to \langle Y, r\rangle \mid f_{XZ}s = f_{YZ}r$ for some $Z \in \mathscr{U}_x$, $Z \subset X \cap Y\}$ is an equivalence, and consequently $\varinjlim \{C_X \mid X \in \mathscr{U}_x\}$ is the quotient of $\Sigma\{C_X \mid X \in \mathscr{U}_x\}$ under σ.

(b) For example, let $\{C_X\}$ be the presheaf of continuous mappings of $\mathscr{P}$ into $\mathscr{Q}$. The germs of $\{C_X\}$ are called the *germs of continuous mappings of* $\mathscr{P}$ *into* $\mathscr{Q}$. The elements of $\Sigma\{C_X \mid X \in \mathscr{U}_x\}$ are usually called *continuous elements at* x (relative to $\mathscr{Q}$). Thus a continuous element at x is a pair $\langle X, f\rangle$, where X is a neighborhood of x and f is continuous mapping of the subspace X of $\mathscr{P}$ into $\mathscr{Q}$, the germs at x of continuous mappings into $\mathscr{Q}$ are classes of equivalent continuous elements at x and two continuous elements at x, say $\langle X, f\rangle$ and $\langle Y, g\rangle$, are equivalent if and only if there exists a neighborhood Z of x such that $f \mid Z = g \mid Z$ (and hence $Z \subset X \cap Y$). Stated in other words, two continuous elements at x determine the same germ if and only if the corresponding mappings agree on a neighborhood of x.

40 A.18. Let $\mathscr{P}$ be a space. For each $X \subset |\mathscr{P}|$ let F_X be the set of all functions on X. Clearly $\{F_X\}$ with restrictions as connecting mappings is a presheaf over $\langle \exp |\mathscr{P}|, \supset \rangle$. If ξ is a germ of $\{F_X\}$ at a point x and there exists a $\langle U, f\rangle \in \xi$ such that f is continuous at x, then for each $\langle V, g\rangle \in \xi$, g is continuous at x. Similarly, if there is defined the notion of differentiability, e.g. if $\mathscr{P} = \mathrm{R}$ and there exists a $\langle U, f\rangle$ in ζ such that the derivative of f at x exists, then for each $\langle V, g\rangle \in \zeta$ the derivative of g at x exists and equals that of f. Thus we can define the notion of a "continuous" germ and of the derivative of a germ. It seems that the statement "if a germ ξ has a derivative, then the germ ξ is continuous" expresses the well-known fact more suggestively than the usual formulation "if a function f has a derivative at a point x, then f is continuous at x". One can find many examples of local properties of functions at points which are in fact properties of germs.

40 A.19. Example. Suppose that $\langle A, \leqq \rangle$ is monotone, $\{P_a \mid a \in A\}$ is an order-preserving family of sets (that is, $a \leqq b$ implies $P_a \subset P_b$), and if $a \leqq b$ then f_{ab} is the identity mapping of P_a into P_b. Clearly $\mathscr{S} = \langle \{P_a\}, \{f_{ab}\}\rangle$ is a presheaf over $\langle A, \leqq \rangle$.

(a) It is easily seen that $\varprojlim \mathscr{S}$ consists of all constant families $\{x \mid a \in A\}$ such that $x \in \bigcap\{P_b \mid b \in A\}$; stated in other words, $\{x \to \{x \mid a \in A\} \mid x \in \bigcap\{P_b \mid b \in A\}\}$ is a one-to-one relation the domain of which is $\bigcap\{P_b \mid b \in A\}$ and the range of which is $\varprojlim \mathscr{S}$. Next, $\varinjlim \mathscr{S}$ consists of all sets of the form $X_x = \mathbf{E}\{\langle a, x\rangle \mid x \in P_a\}$, $x \in \bigcup\{P_b \mid b \in A\}$, in other words, the relation $\{x \to X_x \mid x \in \bigcup\{P_a \mid a \in A\}\}$ ranges on $\varinjlim \mathscr{S}$; clearly this relation is one-to-one.

(b) For each $a \in A$ let g_a be the identity mapping of $\bigcap\{P_b \mid b \in A\}$ into P_a, i.e. $g_a x = x$ for each $x \in \mathbf{D} g_a$. Clearly the family $\{g_a\}$ is compatible for $\mathscr{S}$, i.e. $g_b = f_{ab} \circ g_a$ for each $a \leqq b$. By 40 A.7 there exists exactly one mapping g such that $g_a = (\mathrm{pr}_a : \varprojlim \mathscr{S} \to P_a) \circ g$ for each a and $gx = \{g_a x \mid a \in A\}$. But clearly the relation $\operatorname{gr} g$ is the relation considered in (a). By (a) the mapping g is bijective. Thus g is a one-to-one mapping of $\bigcap\{P_a \mid a \in A\}$ onto $\varprojlim \mathscr{S}$ and $\{g_a\} = [\{\mathrm{pr}_a : \varprojlim \mathscr{S} \to P_a\}] \circ g$. In general, if g is a bijective mapping and fulfils the last equality, then we say that $\mathbf{D} g$ is *isomorphic by* $\{g_a\}$ *with* $\varprojlim \mathscr{S}$. In our case, we can say that the intersection of $\{P_a\}$ is isomorphic (by identity mappings) with $\varprojlim \mathscr{S}$.

(c) Now, for each $a \in A$, let g_a be the identity mapping of P_a into $\bigcup\{P_b \mid b \in B\}$ and let g be the mapping of $\varinjlim \mathscr{S}$ which assigns to each $X_x \in \varinjlim \mathscr{S}$ the point x. By (a) the mapping g is bijective and $g_a = g \circ {}^a f$ for each a, where ${}^a f$ is the canonical mapping of P_a into $\varinjlim \mathscr{S}$ which can be written as $\{g_a\} = g \circ [\{{}^a f\}]$. In general, if $\{g_a\}$ fulfils the last equality with a bijective mapping g, then we shall say that the common range carrier of mappings g_a is *isomorphic by* $\{g_a\}$ *with* $\varinjlim \mathscr{S}$. In our case we can say that the union P of $\{P_a \mid a \in A\}$ is isomorphic by the identity mappings $g_a : P_a \to P$ with $\varinjlim \mathscr{S}$.

In 40 A.5 we introduced the notion of a projective family compatible for a presheaf $\mathcal{S}$; this is a projective family $\{g_a\}$ (i.e. a family with common domain carrier) such that $g_b = f_{ab} \circ g_a$ for each $a \leqq b$, where f_{ab} are connecting mappings of $\mathcal{S}$ and $\leqq$ is a quasi-order of the base of $\mathcal{S}$. Now we introduce corresponding notions for inductive families.

40 A.20. Definition. Let $\mathcal{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ be a presheaf over $\langle A, \leqq\rangle$. By an *inductive family compatible for* $\mathcal{S}$ we shall mean an inductive family of mappings $\{g_a\}$ (i.e. a family $\{g_a\}$ with a common range carrier) such that $g_a = g_b \circ f_{ab}$ for each $a \leqq b$ (thus $\mathbf{D}g_b = P_b$ for each $b \in A$).

Before stating the main result for inductive families compatible for a presheaf we summarize the results of 40 A.6 and corollaries of 40 A.7 in the theorem which follows.

40 A.21. Theorem. *Let $\mathcal{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ be a presheaf over $\langle A, \leqq\rangle$. The family $\{f_a\}$ of projections of $\varprojlim \mathcal{S}$ is compatible for $\mathcal{S}$, and if a projective family $\{g_a\}$ is compatible for $\mathcal{S}$, then there exists exactly one mapping g such that $g_a = f_a \circ g$ for each a; this can be written as*

$$(*) \qquad \{g_a\} = [\{f_a\}] \circ g.$$

If $\{h_a\}$ is a projective family compatible for $\mathcal{S}$ such that each projective family $\{g_a\}$ admits a unique decomposition $(*)$ with f_a replaced by h_a, then there exists exactly one bijective mapping k such that $h_a = f_a \circ k$ for each a.

40 A.22. Theorem. *Let $\mathcal{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ be a presheaf over $\langle A, \leqq\rangle$. The family $\{{}^af\}$ of canonical mappings into $\varinjlim \mathcal{S}$ is an inductive family compatible for $\mathcal{S}$. If $\{g_a\}$ is any inductive family compatible for $\mathcal{S}$, then there exists exactly one mapping g of $\varinjlim \mathcal{S}$ (into the common range carrier of all g_a) such that $g_a = g \circ {}^af$ for each $a \in A$; this can be written as*

$$(*) \qquad \{g_a\} = g \circ [\{{}^af\}].$$

Stated in other words, each inductive family compatible for $\mathcal{S}$ admits a unique decomposition $(*)$ (where only g depends on $\{g_a\}$).

Corollary. *If $\{h_a\}$ is an inductive family compatible for $\mathcal{S}$ such that each inductive family $\{g_a\}$ compatible for $\mathcal{S}$ admits a unique decomposition $(*)$ with ${}^af$ replaced by h_a, then there exists a bijective mapping k such that $h_a = k \circ {}^af$ for each $a \in A$.*

Proof. By definition, the inductive limit of $\mathcal{S}$ is the quotient set $(\Sigma\{P_a\})/\varrho$ where ϱ is the smallest equivalence containing the relation $\sigma = \{\langle a, x\rangle \to \langle b, y\rangle \mid f_{ac}x = f_{bc}y$ for some $c\}$. Now if $y = f_{ab}x$, then $\langle\langle a, x\rangle, \langle b, y\rangle\rangle \in \sigma \subset \varrho$ and hence the pairs $\langle a, x\rangle$ and $\langle b, y\rangle$ belong to the same equivalence class, i.e. ${}^afx = {}^bfy$. It follows that ${}^af = {}^bf \circ f_{ab}$, which shows that $\{{}^af\}$ is compatible for $\mathcal{S}$. Now let $\{g_a\}$ be any inductive family compatible for $\mathcal{S}$ and let Q be the common range carrier of all g_a. Consider an auxiliary mapping k of $\Sigma\{P_a\}$ into Q which assigns to each $\langle a, x\rangle$ the point g_ax of Q. The relation $\{\langle\zeta, \eta\rangle \mid k\zeta = k\eta\}$ is an equivalence on $\Sigma\{P_a\}$ containing the relation σ (since $\{g_a\}$ is compatible for $\mathscr{S}$) and hence also ϱ. Therefore there exists exactly one mapping g of $(\Sigma\{P_a\})/\varrho$ into Q such that $k = g \circ \{\zeta \to \varrho[\zeta]\}$ for each ζ in the sum. Now we need only notice that a mapping g fulfils the last equality if and only if $g_a = g \circ {}^a f$ for each a.

40 A.23. Definitions. Let $\mathscr{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ be a presheaf over $\langle A, \leqq\rangle$. (a) If α is an upper bound of a subset B of A, then $\{f_{b\alpha} \mid b \in B\}$ is an inductive family compatible for the restricted presheaf $\mathscr{S}_B$ and hence, by virtue of 40 A.22, there exists exactly one mapping g of $\varinjlim \mathscr{S}_B$ into P_α such that each $f_{b\alpha}$ is the composition of the canonical mapping of P_b into $\varinjlim \mathscr{S}_B$ followed by g. This mapping g will be termed the *canonical mapping of* $\mathscr{S}_B$ *into* P_α.

(b) The presheaf $\mathscr{S}$ is said to be *inductive at* $\alpha \in A$ if the following condition is fulfilled:

If $\alpha = \sup B$ and B is left-saturated, then the canonical mapping g of $\varinjlim \mathscr{S}_B$ into P_α is bijective.

(c) The presheaf $\mathscr{S}$ will be called *inductive* if $\mathscr{S}$ is inductive at each index $\alpha \in A$ excepting, possibly, the least indices.

40 A.24. Remarks to Definition 40 A.23. (α) If $\alpha = \sup B$ and $\alpha \in B$, then the canonical mapping g of $\varinjlim \mathscr{S}_B$ into P_α is bijective, and moreover g is the inverse of the canonical mapping of P_α into $\varinjlim \mathscr{S}_B$. In particular, if α is the greatest element of A, then the canonical mapping of $\varinjlim \mathscr{S}$ into P_α is the inverse of the canonical mapping of P_α into $\varinjlim \mathscr{S}$ (compare with analogous result 40 A.9 for projective limits). The proof is straightforward and may be left to the reader.

(β) If $\mathscr{S}$ is inductive at α and α is a least element of A, then $P_\alpha = \emptyset$. Indeed, $B = \emptyset$ is left saturated and $\sup B = \alpha$. Clearly $\varinjlim \mathscr{S}_B = \emptyset$, and hence $P_\alpha = \emptyset$. Conversely, if α is a least element of A and $P_\alpha = \emptyset$, then $\mathscr{S}$ is inductive at α.

(γ) Notice that a presheaf was defined to be projective if it is projective at each index (not excepting the greatest elements), but a presheaf was defined to be inductive if it is inductive at each index with the exception of the least elements. The reasons for this were explained in (β).

(δ) Very little is known about inductive presheaves. Therefore we shall restrict ourselves to examples. It should be remarked that more useful concepts are obtained by imposing some additional properties on B in 40 A.23 (b).

40 A.25. Examples. Let $\mathscr{S} = \langle\{S_a\}, \{f_{ab}\}\rangle$ be a presheaf over $\langle A, \leqq\rangle$.

(a) If $\leqq \cap (B \times B) = \mathrm{J}_B$ then $\varinjlim \mathscr{S}_B$ "almost" is $\Sigma\{S_b \mid b \in B\}$, and hence $\mathscr{S}$ need not be inductive even if all the f_{ab} are bijective and $\langle A, \leqq\rangle$ is a boundedly complete lattice.

(b) $\mathscr{S}$ is said to be filter-inductive at α if the condition in 40 A.23 (b) holds with B a right filter. If each filter is directed then $\mathscr{S}$ is filter-inductive provided that all f_{ab} are bijective.

(c) Let $\langle A, \leqq \rangle$ be the collection of all closed subsets of a topological space $\mathscr{P} = \langle P, u \rangle$. $\mathscr{S}$ is filter-inductive at α if for each open additive cover $\mathscr{U}$ of $P - \alpha$ the following condition is fulfilled: if $s \in S_a$ then $s = f_{P-U\,a} r$ with r in S_{P-U} for some U in $\mathscr{U}$ (i.e. the canonical mapping is surjective), and also if $f_{aa} s = f_{ba} r$ with $P - a \in \mathscr{U}$, $P - b \in \mathscr{U}$, then $f_{ac} s = f_{bc} r$ with $c = a \cap b$. Inductivity may be described similarly. E.g. if $\mathscr{P} = \mathrm{C}$ and S_a is the set of all rational functions holomorphic on a, then $\mathscr{S}$ is filter-inductive at any infinite a, but in general not inductive. On the other hand, $\mathscr{S}$ is inductive at any infinite a provided that S_a is the set of all rational functions holomorphic on a which do have at most one singularity.

In the following subsection we shall need the notion of an isomorphism of two presheaves over the same base. For the sake of completeness we also introduce some related notions, and show how morphisms of presheaves induce mappings of their limits.

40 A.26. Definition. Suppose that $\mathscr{S} = \langle \{P_a\}, \{f_{ab}\} \rangle$ and $\mathscr{S}' = \langle \{P'_a\}, \{f'_{ab}\} \rangle$ are presheaves over the same quasi-ordered set $\langle A, \leqq \rangle$. A *morphism of* $\mathscr{S}$ *into* $\mathscr{S}'$ is a family $\varphi = \{\varphi_a \mid a \in A\}$ such that each φ_a is a mapping of P_a into P'_a and $f'_{ab} \circ \varphi_a = \varphi_b \circ f_{ab}$ for each $a \leqq b$; that is, the diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{\;f'_{ab}\;} & \cdot \\ \varphi_a \Big\uparrow & & \Big\uparrow \varphi_b \\ \cdot & \xrightarrow[\;f_{ab}\;]{} & \cdot \end{array}$$

is commutative for each $a \leqq b$. A morphism $\varphi = \{\varphi_a\}$ is an *epimorphism* or a *monomorphism* if each mapping φ_a is surjective or injective, respectively. A morphism φ is called an *isomorphism* if φ is simultaneously an epimorphism and a monomorphism.

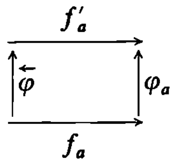
Notice that we employ terms usually used in the general theory of categories. It follows from the following theorem that all presheaves over a given quasi-ordered set with the morphisms just defined as morphisms is a category. Moreover, one can easily show that monomorphisms and epimorphisms just defined coincide with the monomorphisms and epimorphisms in this category.

40 A.27. *Let $\mathscr{S}$, $\mathscr{S}'$ and $\mathscr{S}''$ be presheaves over the same base $\langle A, \leqq \rangle$. If $\{\varphi_a\}$ and $\{\psi_a\}$ are morphisms (monomorphisms, epimorphisms, isomorphisms) of $\mathscr{S}$ into $\mathscr{S}'$ and of $\mathscr{S}'$ into $\mathscr{S}''$ respectively, then $\{\psi_a \circ \varphi_a\}$ is a morphism (monomorphism, epimorphism, isomorphism) of $\mathscr{S}$ into $\mathscr{S}''$. If $\{\varphi_a\}$ is an isomorphism then $\{\varphi_a^{-1}\}$ is an isomorphism of $\mathscr{S}'$ into $\mathscr{S}$. Let B be a quasi-ordered subset of $\langle A, \leqq \rangle$. If $\{\varphi_a \mid a \in A\}$ is a morphism, monomorphism, epimorphism or isomorphism of $\mathscr{S}$ into $\mathscr{S}'$, then $\{\varphi_a \mid a \in B\}$ is a morphism, monomorphism, epimorphism or isomorphism of $\mathscr{S}_B$ into $\mathscr{S}'_B$.*

Now we shall prove that each morphism of $\mathscr{S}$ into $\mathscr{S}'$ induces, in a natural way, a mapping of $\varprojlim \mathscr{S}$ into $\varprojlim \mathscr{S}'$ and a mapping of $\varinjlim \mathscr{S}$ into $\varinjlim \mathscr{S}'$.

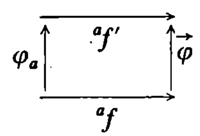
40 A.28. Theorem. *Suppose that $\mathscr{S}$ and $\mathscr{S}'$ are presheaves over the same quasi-ordered set $\langle A, \leqq\rangle$ and $\varphi = \{\varphi_a\}$ is a morphism of $\mathscr{S}$ into $\mathscr{S}'$. Then*

(a) *There exists exactly one mapping $\overleftarrow{\varphi}$ of $\varprojlim \mathscr{S}$ into $\varprojlim \mathscr{S}'$ such that the diagram*



is commutative for each a in A, where $\{f_a\}$ and $\{f'_a\}$ are the families of canonical projections of $\varprojlim \mathscr{S}$ and $\varprojlim \mathscr{S}'$ respectively. The mapping $\overleftarrow{\varphi}$ is injective or bijective provided that all φ_a possess the corresponding property, i.e., provided that φ is a monomorphism or an isomorphism.

(b) *There exists exactly one mapping $\overrightarrow{\varphi}$ of $\varinjlim \mathscr{S}$ into $\varinjlim \mathscr{S}'$ such that the diagram*



is commutative for each a in A, where $\{^af\}$ and $\{^af'\}$ are the families of canonical mappings into $\varinjlim \mathscr{S}$ and $\varinjlim \mathscr{S}'$. The mapping $\overrightarrow{\varphi}$ is surjective or bijective provided that all mappings φ_a possess the corresponding property, i.e. provided that φ is an epimorphism or an isomorphism.

Proof. The reader is requested to prove the existence and the uniqueness in both statements (a) and (b), even though both are special cases of 40 A.29. We restrict ourselves to the proof of additional properties of $\overleftarrow{\varphi}$ and $\overrightarrow{\varphi}$.

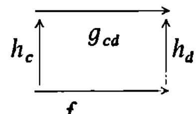
(a) It follows immediately from the diagram in (a) that $\overleftarrow{\varphi}(\{x_a\}) = \{\varphi_a x_a\}$ for each $\{x_a\}$ in $\varprojlim \mathscr{S}$. Thus clearly $\overleftarrow{\varphi}$ is injective if each φ_a is injective. If φ is an isomorphism, then both φ and $\varphi^{-1} = \{\varphi_a^{-1}\}$ are morphisms and clearly $\overleftarrow{\varphi^{-1}} \circ \overleftarrow{\varphi}$ is the identity mapping of $\varprojlim \mathscr{S}$. It follows that $\overleftarrow{\varphi}$ is bijective.

(b) Suppose that φ is an epimorphism and choose an $\eta \in \varinjlim \mathscr{S}'$. There exists an a in A and y in P'_a such that $^af'\langle a, y\rangle = \eta$. Since φ_a is surjective, we can choose an x in P_a such that $\varphi_a x = y$. Put $\xi = {}^af\langle a, x\rangle$. It follows from the diagram in (b) that $\overrightarrow{\varphi}\xi = \eta$. Thus $\overrightarrow{\varphi}$ is surjective. Now let φ be an isomorphism. Since φ^{-1} is also an isomorphism and $\overrightarrow{\varphi^{-1}} \circ \overrightarrow{\varphi}$ is the identity mapping of $\varinjlim \mathscr{S}$, $\overrightarrow{\varphi}$ is necessarily bijective.

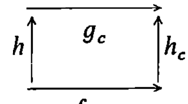
Remark. It should be noted that $\overleftarrow{\varphi}$ need not be surjective if φ is an epimorphism, and $\overrightarrow{\varphi}$ need not be injective if φ is a monomorphism.

40 A.29. Theorem. *Suppose that $\mathscr{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ is a presheaf over $\langle A, \leqq\rangle$ and $\mathscr{S}_1 = \langle\{Q_c\}, \{g_{cd}\}\rangle$ is a presheaf over $\langle C, \prec\rangle$.*

(a) *Let α be an order-preserving mapping of $\langle C, \prec \rangle$ into $\langle A, \leqq \rangle$ and $\{h_c \mid c \in C\}$ be a family of mappings, with each h_c from $P_{\alpha(c)}$ into Q_c, such that the diagram*

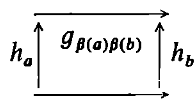


is commutative for each $c \prec d$. There exists exactly one mapping h of $\varprojlim \mathscr{S}$ into $\varprojlim \mathscr{S}_1$ such that the diagram

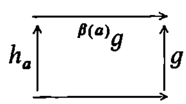


is commutative for each c in C, where $\{f_a\}$ and $\{g_c\}$ are families of all projections of $\varprojlim \mathscr{S}$ and $\varprojlim \mathscr{S}_1$, respectively.

(b) *Let β be an order-preserving mapping of $\langle A, \leqq \rangle$ into $\langle C, \prec \rangle$ and $\{h_a \mid a \in A\}$ be a family of mappings, with each h_a from P_a into $Q_{\beta(a)}$, such that the diagram*



is commutative for each $a \leqq b$. Then there exists exactly one mapping g of $\varinjlim \mathscr{S}$ into $\varinjlim \mathscr{S}_1$ such that the diagram



is commutative for each a in A, where $\{{}^a f\}$ and $\{{}^c g\}$ are families of canonical mappings into $\varinjlim \mathscr{S}$ and $\varinjlim \mathscr{S}_1$, respectively.

Proof. I. First we shall prove statement (a). There exists at most one h such that $g_c \circ h = h_c \circ f_{\alpha(c)}$ for each c in C, because if $y = hx$ then necessarily $g_c y = h_c[f_{\alpha(c)}x]$ for each c in C, and hence each $g_c y$ is uniquely determined by x; this implies that y is uniquely determined by x. On the other hand, if $x \in \varprojlim \mathscr{S}$, then the family $\{h_c(f_{\alpha(c)}x) \mid c \in C\}$ belongs to $\varprojlim \mathscr{S}_1$ because of the commutativity of the first diagram in (a). Thus $h = \langle \{x \to \{h_c(f_{\alpha(c)}x)\} \mid x \in \varprojlim \mathscr{S}\}, \varprojlim \mathscr{S}, \varprojlim \mathscr{S}_1 \rangle$ is a mapping and it clearly makes the second diagram of (a) commutative.

II. Proof of (b). The uniqueness is almost evident. Indeed, if $\zeta \in \varinjlim \mathscr{S}$, then ${}^a f\langle a, x \rangle = \zeta$ for some $a \in A$ and $x \in P_a$, and the commutativity of the second diagram in (b) yields $g\zeta = {}^{\beta(a)}g h_a x$. The existence of g can be proved as follows. By definition we have

$$\varinjlim \mathcal{S} = \Sigma\{P_a \mid a \in A\}/\varrho$$

and

$$\varinjlim \mathcal{S}_1 = \Sigma\{Q_c \mid c \in C\}/\varrho_1$$

where ϱ and ϱ_1 are the smallest equivalences containing the relations

$$\sigma = \{\langle\langle a, x\rangle, \langle b, y\rangle\rangle \mid f_{a\alpha}x = f_{b\alpha}y \text{ for some } \alpha\}$$

and

$$\sigma_1 = \{\langle\langle c, x'\rangle, \langle d, y'\rangle\rangle \mid g_{c\gamma}x' = g_{d\gamma}y' \text{ for some } \gamma\},$$

respectively. Consider an auxiliary mapping k of $\Sigma\{P_a\}$ into $\Sigma\{Q_c\}$ which assigns to each $\langle a, x\rangle$ the point $\langle\beta(a), h_a x\rangle$. It follows from the commutativity of the first diagram in (b) that $\langle\zeta, \eta\rangle \in \sigma$ implies $\langle k\zeta, k\eta\rangle \in \sigma_1$ and this implies $\langle k\zeta, k\eta\rangle \in \varrho_1$. Since ϱ_1 is an equivalence and ϱ is the smallest equivalence containing σ, we have $\langle\zeta, \eta\rangle \in \varrho \Rightarrow \langle k\zeta, k\eta\rangle \in \varrho_1$. But from the last implication one can conclude at once that there exists a mapping g of $\varinjlim \mathcal{S}$ into $\varinjlim \mathcal{S}_1$ such that

$$\{\eta \to \varrho_1[\eta]\} \circ k = g \circ \{\zeta \to \varrho[\zeta]\}.$$

Obviously this mapping g makes the second diagram in (b) commutative.

Corollaries. *Let $\mathcal{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ be a presheaf over $\langle A, \leqq\rangle$ and let $B \subset A$. Then*

(a) *There exists exactly one mapping h of $\varprojlim \mathcal{S}$ into $\varprojlim \mathcal{S}_B$ such that the composition h with the canonical projection of $\varprojlim \mathcal{S}_B$ into P_b is the canonical projection of $\varprojlim \mathcal{S}$ into P_b for each b in B.*

(b) *There exists exactly one mapping k of $\varinjlim \mathcal{S}_B$ into $\varinjlim \mathcal{S}$ such that, for each b in B, the composition of the canonical mapping of P_b into $\varinjlim \mathcal{S}_B$ followed by k is the canonical mapping of P_b into $\varinjlim \mathcal{S}$.*

Proof. Put $\alpha = \beta = \{b \to b \mid b \in B\}$ and apply the theorem (in case (a) $\mathcal{S} = \mathcal{S}$ and $\mathcal{S}_1 = \mathcal{S}_B$ and in case (b) $\mathcal{S} = \mathcal{S}_B$, $\mathcal{S}_1 = \mathcal{S}$).

40 A.30. Remark to Corollaries of 40 A.29.

(a) The mapping h is injective provided that B is left-cofinal in A, and h is bijective if B is left-cofinal in A and A is left-directed.

(b) If B is right-cofinal in A, then the mapping k is bijective.

Proof. (a) First suppose that B is left-cofinal in A and $x = \{x_a\}$ and $y = \{y_a\}$ are elements of $\varprojlim \mathcal{S}$ such that $x_b = y_b$ for each b in B (i.e. $hx = hy$). We must show that $x_a = y_a$ for each a in A. Pick an a in A. There exists a b in B preceding a. Since $f_{ba}x_b = x_a$, $f_{ba}y_b = y_a$, and $x_b = y_b$, we obtain $x_a = y_a$. Now suppose that, in addition, A is left-directed and $\{x_b \mid b \in B\}$ is any element of $\varprojlim \mathcal{S}_B$. We must find a family $\{y_a \mid a \in A\} \in \varprojlim \mathcal{S}$ such that $y_b = x_b$ for each b in B. From our assumptions we shall obtain that, given an a in A, the value $f_{ba}x_b$, $b \leqq a$ does not depend on b. This will enable us to define $y_a = f_{ba}x_b$ for some $b \leqq a$; clearly $\{y_a\} \in \varprojlim \mathscr{S}$ and $y_b = x_b$ for each b in B. Suppose that $a \in A$ and b_1 and b_2 are elements of B preceding a. Since B is left-cofinal and A is left-directed we can choose a b in B preceding both b_1 and b_2. Now $f_{b_ia}x_{b_i} = f_{b_ia}f_{bb_i}x_b = f_{ba}x_b$, $i = 1, 2$, which proves $f_{b_1a}x_{b_1} = f_{b_2a}x_{b_2}$.

(b) Suppose that B is right cofinal. If $\eta \in \varinjlim \mathscr{S}$, then ${}^af\langle a, x\rangle = \eta$ for some $a \in A$ and $x \in P_a$; now, if $b \in B$, $a \leqq b$, then also ${}^bf\langle b, f_{ab}x\rangle = \eta$. If ξ is the image of $\langle b, f_{ab}x\rangle$ under the canonical mapping of P_b into $\varinjlim \mathscr{S}_B$, then clearly $k\xi = \eta$. Thus k is surjective. It remains to show that k is injective. It is enough to show that

$$(*) \qquad \varrho_B \supset \varrho \cap (\Sigma\{P_b \mid b \in B\}) \times (\Sigma\{P_b \mid b \in B\})$$

where ϱ and ϱ_B are equivalences such that

$$\varinjlim \mathscr{S} = (\Sigma\{P_a\})/\varrho \quad \text{and} \quad \varinjlim \mathscr{S}_B = (\Sigma\{P_b\})/\varrho_B\,.$$

The proof of the inclusion $(*)$ is straightforward and is left to the reader.

Remark. The assumption "A is left directed" cannot be omitted from (a) in 40 A.30. Indeed, put $A = (1, 2, 3, 4)$ and define the order on A so that 1 and 2 are incomparable, 3 follows both 1 and 2, and 4 follows 3. The sets P_1, P_2 and P_4 are one-point and the set P_3 has at least two points. Finally, the connecting mappings f_{ij} are defined so that $f_{13}[P_1]$ does not meet $f_{23}[P_2]$. If $\mathscr{S}$ is the resulting presheaf of sets and $B = (1, 2, 4)$, then B is left cofinal in A, $\varprojlim \mathscr{S} = \emptyset$ but $\varprojlim \mathscr{S}_B \neq \emptyset$ (and hence the canonical mapping of $\varprojlim \mathscr{S}$ into $\varprojlim \mathscr{S}_B$ is not surjective).

B. SHEAVES OF SETS AND COVERING FIBRATIONS

One of the main results of this subsection asserts that every sheaf over a space $\mathscr{P}$ is isomorphic with the sheaf of continuous sections of a covering fibration over $\mathscr{P}$.

40 B.1. We shall introduce some abbreviated terminology. By 7 B.12 a fibration is a correspondence f such that f^{-1} is a mapping, i.e. f is a range-full correspondence such that the fibres are disjoint. Here we shall only consider fibrations f such that both $\mathbf{D}^*f$ and $\mathbf{E}^*f$ are closure spaces; for brevity we shall mean by a fibration a fibration such that both carriers are spaces. By 7 C.8 a section of a fibration f is a mapping such that $\mathbf{E}^*s = \mathbf{E}^*f$ and $\operatorname{gr} s \subset \operatorname{gr} f$; thus $\mathbf{D}s \subset \mathbf{D}f$ and no compatibility requirement on the structure of $\mathbf{D}^*s$ is involved; for brevity we shall always assume that $\mathbf{D}^*s$ is a subspace of $\mathbf{D}^*f$. The space $\mathbf{E}^*f$ will be called the *fibre space* of f.

Let f be a fibration over a closure space $\mathscr{P}$ and for each open subset U of $\mathscr{P}$ let S_U be the set of all continuous sections of f over U (i.e. sections over U which are continuous mappings). Since the domain-restrictions of continuous sections are continuous sections and a domain-restriction s of a continuous section s_1 which is a domain-restriction of a continuous section s_2 is a domain-restriction of s_2, the family $\{S_U \mid U \text{ open}\}$ endowed with restriction mappings as connecting mappings (see 40 A.2) is a presheaf over open subsets of $\mathscr{P}$ ordered by $\supset$.

We shall prove that $\{S_U\}$ *is a projective presheaf*, i.e. a sheaf over $\mathscr{P}$. A direct proof can be given, but we prefer to reduce the projectivity of $\{S_U\}$ to the projectivity of the sheaf $\{C_U\}$ of continuous mappings of $\mathscr{P}$ into $\mathbf{E}^*f$ (see 40 A.12). Clearly $S_U \subset C_U$ for each U, and an $s \in C_U$ belongs to S_U if and only if $\operatorname{gr} s$ is a section of $\operatorname{gr} f$, i.e. $f^{-1}sx = x$ for each x in U. Now if $U = \bigcup \mathscr{V}$ and $\{s_V \mid V \in \mathscr{V}\}$ is a family such that $s_V \in S_V$ for each V in $\mathscr{V}$ and the restrictions of s_{V_1} and s_{V_2} to $V_1 \cap V_2$ coincide for each V_1 and V_2 in $\mathscr{V}$, then $\{C_U\}$ being projective at U, there exists an s in C_U such that each s_V is a restriction of s to V. It is easily seen that $f^{-1}sx = x$ for each x in U, and hence $s \in S_U$ as we noticed above.

40 B.2. Definition. If f is a fibration over a space $\mathscr{P}$ then the sheaf over $\mathscr{P}$ just defined will be called the *sheaf of continuous sections of* f and will be denoted by $\mathscr{S}_f$.

It turns out that any sheaf over a space $\mathscr{P}$ is isomorphic with the sheaf of continuous sections of a fibration f over $\mathscr{P}$; furthermore, the fibration f can be chosen with some important additional properties, namely f may be taken to be a covering fibration.

40 B.3. Definition. A *covering fibration* is a lower semi-continuous and inversely continuous fibration f such that $\mathbf{D}f$ is open in $\mathbf{D}^*f$ and each point of the fibre space $\mathbf{E}^*f$ has an open neighborhood U such that the mapping $f^{-1} : U \to \mathbf{D}^*f$ is injective.

40 B.4. *In order that a fibration f be a covering fibration it is necessary and sufficient that each point of $\mathbf{E}^*f$ have an open neighborhood U such that the domain-restriction g of f^{-1} to the subspace U of $\mathbf{E}^*f$ is an embedding and the set $\mathbf{E}g$ $(= g[U] = f^{-1}[U])$ is open in $\mathbf{D}^*f$.*

Proof. I. Assuming that f is a covering fibration, choose a point y of $\mathbf{E}^*f$ and an open neighborhood U of y such that $g = f^{-1} : U \to \mathbf{D}^*f$ is injective. We shall prove that g is an embedding (U is considered as a subspace of $\mathbf{E}^*f$) and $\mathbf{E}g$ is open. The mapping g is continuous as the restriction of the continuous mapping, and g is inversely lower semi-continuous as the domain-restriction of an inversely lower semi-continuous mapping to an open subspace, namely of f^{-1}. Finally, $\mathbf{E}g = f^{-1}[U]$ is open in the subspace $\mathbf{E}f^{-1} = \mathbf{D}f$ because f is lower semi-continuous; since $\mathbf{D}f$ is open in $\mathbf{D}^*f$, $\mathbf{E}g$ is open in $\mathbf{D}^*f$. — II. Assume the condition. Evidently f^{-1} is injective on an open neighborhood of any point of $\mathbf{E}^*f$, and $\mathbf{E}f^{-1} = \mathbf{D}f$ is open as the union of open sets, namely the ranges of embeddings in the condition. The mapping f^{-1} is continuous because f^{-1} is continuous on a neighborhood of any point of $\mathbf{E}^*f$. To prove that f is lower semi-continuous we shall show that if V is a neighborhood of a point y of $\mathbf{E}^*f$ then $f^{-1}[V]$ is a neighborhood of $f^{-1}y$ in $\mathbf{D}^*f$. Let U be an open neighborhood of y such that $f^{-1} : U \to \mathbf{D}^*f$ is an embedding and $f^{-1}[U]$ is open. The set $U \cap V$ is a neighborhood of y in U and hence $f^{-1}[U \cap V]$ is a neighborhood of $f^{-1}y$ in $f^{-1}[U]$ and so certainly in $\mathbf{D}^*f$ because $f^{-1}[U]$ is open.

40 B.5. *In order that a fibration f be a covering fibration it is necessary and sufficient that*

(a) *ranges of continuous sections over open sets be open and cover* $\mathbf{E}^*f$ *and*

(b) *either* f^{-1} *be continuous or* $\mathbf{D}^*f$ *be topological.*

Proof. I. Assume (a) and (b). Let y be any point of $\mathbf{E}^*f$; by (a) we can choose a continuous section s over an open set V such that $U = \mathbf{E}s$ is open and contains y. The mapping s is an embedding; this is evident if f^{-1} is continuous, and if $\mathbf{D}^*f$ is topological then s is an embedding because $\mathbf{D}^*s$ is topological and s carries open sets of $\mathbf{D}^*s$ into open sets of $\mathbf{E}^*s = \mathbf{E}^*f$ (by (a)). Since s is an embedding, the mapping $g = s^{-1} : U \to \mathbf{D}^*f = f^{-1} : U \to \mathbf{D}^*f$ is also an embedding and $\mathbf{E}g = V$ is open. By 40 B.4, f is a covering fibration. — II. Assume that f is a covering fibration. Clearly (b) is fulfilled. Let s be a continuous section over an open set U. We shall prove that the set $\mathbf{E}s$ ($= s[U]$), denoted by V, is a neighborhood of each of its points. Assuming $y \in V$ choose an open neighborhood W of y such that $f^{-1}[W]$ is open and the mapping $g = f^{-1} : W \to \mathbf{D}^*f$ is an embedding (by 40 B.4). We shall prove that $W \cap V$ is open. It is sufficient to show that $g[W \cap V]$ is open. Clearly $g[W \cap V] = s^{-1}[W]$. The set $s^{-1}[W]$ is open in $\mathbf{D}^*s$ because s is continuous and W is open; since $U = \mathbf{D}s$ is open, $s^{-1}[W]$ is open in $\mathbf{D}^*f$. Finally, we must show that ranges of continuous sections over open sets cover $\mathbf{E}^*f$, and this follows from 40 B.4, because if $g = f^{-1} : : U \to \mathbf{D}^*f$ is an embedding such that $\mathbf{E}g = f^{-1}[U]$ is open, then $g^{-1} : \mathbf{E}g \to \mathbf{E}^*f$ is an embedding and so a continuous section.

Remark. If $\mathscr{P}$ is a closure space which is not topological and $f = \mathsf{J} : \mathscr{P} \to \tau\mathscr{P}$, then (a) is fulfilled but (b) is not.

For the proof of the main results we shall need the following characterization of covering fibrations.

40 B.6. Theorem. *In order that a fibration f be a covering fibration it is necessary and sufficient that there exist a family $\{s_a\}$ of continuous sections over open sets such that*

(a) *$\{s_a\}$ inductively generates the fibre space $\mathbf{E}^*f$,*

(b) *$\{\mathbf{E}s_a\}$ covers $\mathbf{E}^*f$, and*

(c) *if s_{a_1} and s_{a_2} agree at a point x (i.e. $s_{a_1}x = s_{a_2}x$) then both sections agree on an open neighborhood of x.*

Remarks. (1) Condition (c) is equivalent to the following condition:

(c′) The set $\mathbf{E}\{x \mid s_{a_1}x = s_{a_2}x\}$ is open for each a_1 and a_2.

(2) As $\{s_a\}$ we can take any family of continuous sections over open sets satisfying (b).

Proof. Evidently (c) and (c′) are equivalent. I. Assume that f is a covering fibration and $\{s_a\}$ is any family of continuous sections over open sets satisfying condition (b) (such a family exists by 40 B.5). Condition (a) is fulfilled since the s_a are embeddings and $\{\mathbf{E}s_a\}$ is an open cover of $\mathbf{E}^*f$. The set X in (c′) is the inverse image of the open set $\mathbf{E}s_{a_1} \cap \mathbf{E}s_{a_2}$ (by 40 B.5) under a continuous mapping, e.g. s_{a_1}, and therefore X is open. — II. Assuming (a)–(c) we shall prove that f is a covering fibration by showing that f fulfils the condition in 40 B.4. First notice that f^{-1} is continuous (because $f^{-1} \circ$

$\circ\, s_a$ is an identity mapping and thus certainly a continuous mapping for each a); thus f is an inversely continuous fibration. Now let y be any point of $\mathbf{E}^*f$ and let us choose an s_a such that $y \in \mathbf{E}s_a$. We shall prove that $g = f^{-1} : \mathbf{E}s_a \to \mathbf{D}^*f$ is an embedding and $\mathbf{E}s_a$ is open in $\mathbf{E}^*f$. Since f is inversely continuous and s_a is a continuous section, s_a is an embedding; since $\operatorname{gr} s_a = \operatorname{gr} g^{-1}$, g is also an embedding. To prove that $\mathbf{E}s_a$ is open it is sufficient to show that $s_a^{-1}[\mathbf{E}s_a]$ is open in $\mathbf{D}^*s_a$ for each a, and this immediately follows from condition (c′) (we have $s_a^{-1}[\mathbf{E}s_a] = \mathbf{E}\{x \mid s_a x = s_a x\}$).

Remark. Let r and s be continuous sections over an open connected set U of a covering fibration f whose fibre space is separated. Then $r = s$ or $\mathbf{E}r \cap \mathbf{E}s = \emptyset$. In fact, $X = \mathbf{E}\{x \mid sx = rx\}$ is open in U by 40 B.6, and X is closed in U because $\mathbf{E}^*f$ is separated (27 A.7). Since U is connected we have $X = U$ or $X = \emptyset$.

Now we are prepared to prove our main result. Given a sheaf $\mathscr{S}$ over a space $\mathscr{P}$, we shall construct a covering fibration f over $\mathscr{P}$ and an isomorphism of $\mathscr{S}$ onto the sheaf of continuous sections of f. We shall prove somewhat more: for any presheaf over open subsets of a space $\mathscr{P}$ we shall construct a covering fibration f over $\mathscr{P}$ and a morphism φ of $\mathscr{S}$ into $\mathscr{S}_f$ which has some important properties, e.g. φ is an isomorphism provided that $\mathscr{S}$ is a sheaf. In 40 A.18 we introduced the concept of a germ at a point of a presheaf over subsets of a closure space. We shall need germs of a presheaf over open sets of a closure space. It is to be noted that if the space is not topological then there is an essential difference between the germs of a presheaf $\mathscr{S}$ over subsets of a closure space and the germs of $\mathscr{S}$ restricted to open sets. If the space is topological then this difference is merely formal.

40 B.7. Definition. Let $\mathscr{S}$ be a presheaf over open subsets of a closure space $\mathscr{P}$ and for each x let $\mathscr{U}_x$ be the collection of all open sets containing x. The inductive limit of $\mathscr{S}$ restricted to $\mathscr{U}_x$ is called the *stalk of $\mathscr{S}$ over x*, and the elements of the stalk over x are called the *germs of $\mathscr{S}$ over x* or *at x*.

Notice that the germs of $\mathscr{S}$ over open sets of $\mathscr{P}$ and the germs of $\mathscr{S}$ over open sets of $\tau\mathscr{P}$ coincide.

40 B.8. Let $\mathscr{S} = \langle\{S_U\}, \{f_{UV}\}\rangle$ be a presheaf over the collection $\mathscr{U}$ of all open subsets of a space $\mathscr{P}$ and let $\mathscr{U}_x = \mathbf{E}\{U \mid x \in U \in \mathscr{U}\}$ for each x in $\mathscr{P}$. We shall construct a covering fibration f and the required morphism of $\mathscr{S}$ into $\mathscr{S}_f$.

(a) Denote by Q_x the stalk of $\mathscr{S}$ over x. The set

$$Q = \Sigma\{Q_x \mid x \in \mathscr{P}\} = \Sigma\{\varinjlim \mathscr{S}_{\mathscr{U}_x} \mid x \in \mathscr{P}\}$$

will be the underlying set of the fibre space of f, and the relation

$$\varrho = \Sigma\{(x) \times Q_x \mid x \in \mathscr{P}\}$$

will be the graph of f (ϱ consists of all $\langle x, \langle x, y\rangle\rangle$, $x \in \mathscr{P}$, $y \in Q_x$)). For each U in $\mathscr{U}$ and s in S_U let $\tilde{s}$ be the single-valued relation which assigns to each $x \in U$ the point $\langle x, y\rangle$ where y is the value at s of the canonical mapping of S_U into the inductive limit Q_x (40 A.15). The following assertion is obvious:

(*) *Each* $\tilde{s}$ *is a section of* ϱ, *and if* $s = f_{UV}s_1$, *then* $\tilde{s}$ *is the domain-restriction of* $\tilde{s}_1$ *to* V.

We shall also need the following:

(**) *If* $x \in U_1 \cap U_2$, $s_i \in S_{U_i}$ $(i = 1, 2)$ *and* $\tilde{s}_1 x = \tilde{s}_2 x$ *then there exists a* $U \subset U_1 \cap U_2$ *such that* $x \in U$ *and* $f_{U_1U}s_1 = f_{U_2U}s_2 = s \in S_U$; *in particular, the domain-restrictions of* $\tilde{s}_1$ *and* $\tilde{s}_2$ *to* U *coincide.*

Since $\mathscr{U}_x$ is directed and $\tilde{s}_1 x = \tilde{s}_2 x$ we can choose a U in $\mathscr{U}_x$ following both U_1 and U_2 (i.e. $U \subset U_1 \cap U_2$) such that $f_{U_1U}s_1 = f_{U_2U}s_2$ (by 40 A.16 (β)).

(b) Let u be the closure inductively generated by the family of all mappings $\tilde{s} : U \to Q$ (where U is considered as a subspace of $\mathscr{P}$) with U in $\mathscr{U}$ and s in S_U, and let $f = \varrho : \mathscr{P} \to \langle Q, u \rangle$.

We shall prove that f is a covering fibration by verifying conditions (a)–(c) in 40 B.6 for the family $\{\tilde{s} : U \to \mathbf{E}^*f \mid U \in \mathscr{U}, s \in S_U\}$. Evidently each member is a continuous section of f over an open set and the ranges of the members cover $\mathbf{E}^*f$. Condition (c) immediately follows from (**).

In what follows the symbol $\tilde{s}$ with s in S_U will denote the continuous section $\tilde{s} : U \to \mathbf{E}^*f$.

(c) Let $\{S'_U\}$ denote the sheaf of continuous sections of f. For each U in $\mathscr{U}$ let φ_U be the mapping of S_U into S'_U which assigns to each s the continuous section $\tilde{s}$. It follows from (*) that the family $\varphi = \{\varphi_U \mid U \in \mathscr{U}\}$ is a morphism of $\mathscr{S}$ into $\{S'_U\}$.

We shall prove that φ *is a monomorphism whenever* $\mathscr{S}$ *fulfils condition* (a) *of definition* 40 A.11 *of projective presheaves, and* φ *is an epimorphism whenever* $\mathscr{S}$ *is projective (i.e., a sheaf).* It will follow that φ *is an isomorphism whenever* $\mathscr{S}$ *is a sheaf.*

Take any $\mathscr{S}$ which fulfils condition (a) at $U \in \mathscr{U}$ and two elements s_1 and s_2 of S_U such that $\tilde{s}_1 = \tilde{s}_2$. We shall prove $s_1 = s_2$. By (**) we can choose families $\{U_x \mid x \in U\}$ and $\{s_x \mid x \in U\}$ such that $x \in U_x \subset U$, $s_x \in S_{U_x}$ and $f_{UU_x}s_1 = f_{UU_x}s_2$ for each x in U. Since $\bigcup\{U_x \mid x \in U\} = U$ (i.e., $\inf\{U_x\} = U$), condition (a) implies that $s_1 = s_2$.

Suppose that $\mathscr{S}$ is a sheaf, $U \in \mathscr{U}$ and $g \in S'_U$. We must find an s in S_U such that $\tilde{s} = g$. Consider the collection $\mathscr{V}$ of all $V \subset U$ such that $\tilde{r}$ is restriction of g to V for some r in S_V. Since φ is a monomorphism, as it has already been shown, for each V in $\mathscr{V}$ there exists exactly one r in S_V such that $\tilde{r}$ is a restriction of g; let s_V stand for this r. Evidently if $V_1 \subset V \in \mathscr{V}$, $V_1 \in \mathscr{U}$, then $V_1 \in \mathscr{V}$ and $f_{VV_1}s_V = s_{V_1}$, and hence $\mathscr{V}$ is a right saturated. It remains to show that $\bigcup\mathscr{V} = U$ (i.e. $U = \inf\mathscr{V}$); indeed, it will follow from condition (b) in 40 A.11 that there exists an s in S_U such that $f_{UV}s = s_V$ for each V in $\mathscr{V}$, and hence $\tilde{s}$ is a restriction of g to U. Consider any point x of U and choose a W in $\mathscr{U}$ and an r in S_W such that $\tilde{r}x = gx$. Since $\tilde{r}$ and g are continuous sections of f and f is a covering fibration, by 40 B.6 (c) the two sections agree on an open neighborhood V of x; clearly we may assume that $V \subset U$. If $t = f_{WV}r$, then $\tilde{t}x = gx$ for each x in V, and hence $V \in \mathscr{V}$. The proof is complete.

40 B.9. Definition. Let $\mathscr{S}$ be any presheaf over open sets of a space $\mathscr{P}$. The covering fibration f constructed in 40 B.8 will be called the *covering fibration associated with* $\mathscr{S}$ (and will often be denoted by $f_{\mathscr{S}}$); the fibre space of f will be called the *fibre space of* $\mathscr{S}$, the sheaf of continuous sections of f will be termed the *sheaf associated with* $\mathscr{S}$ and denoted by $\tilde{\mathscr{S}}$. Finally the morphism φ of $\mathscr{S}$ into $\tilde{\mathscr{S}}$ constructed in 40 B.8 will be called the *canonical morphism of* $\mathscr{S}$ *into* $\tilde{\mathscr{S}}$.

The main result of 40 B.8 can be restated in terms of 40 B.9 as follows:

40 B.10. Theorem. *The canonical morphism of a presheaf $\mathscr{S}$ over open sets of a space $\mathscr{P}$ into the associated sheaf $\tilde{\mathscr{S}}$ of continuous sections of the covering fibration associated with $\mathscr{S}$ is an isomorphism if and only if $\mathscr{S}$ is a sheaf. In particular, every sheaf over a space $\mathscr{P}$ is isomorphic with the sheaf of continuous sections of a covering fibration over $\mathscr{P}$.*

40 B.11. Examples. Let $\mathscr{S} = \{S_U\}$ be a sheaf of holomorphic functions over the space C of complex numbers. The fibre space associated with $\mathscr{S}$ is clearly the underlying closure space of the Riemann surface of $\mathscr{S}$. E.g., for each open set U let S_U be the set of all holomorphic functions on U whose derivative is the function $\{z \to 1/z\}$. It is easily seen that $\mathscr{S}$ is a sheaf; this sheaf is called the logarithm. It is well-known that $S_{\mathbf{C}-(0)} = \emptyset$ and S_U is infinite for each sphere in $\mathbf{C} - (0)$. If G is an open subset of C, then the restriction $\mathscr{S}_G$ of $\mathscr{S}$ to open subsets of G is the restriction of the logarithm to G. The covering fibration associated with $\mathscr{S}_{\mathbf{C}-(0)}$ is upon $\mathbf{C} - (0)$ and the fibres are infinite.

The foregoing theorem shows that there is a close connection between sheaves and covering fibrations, and in fact every notion based on sheaves can be described in terms of covering fibrations and conversely. As an example we shall find the description of morphisms of sheaves.

40 B.12. *Let f_1 and f_2 be two fibrations over a space $\mathscr{P}$ and let $\mathscr{S}_i = \{S_{iU}\}$, $i = 1, 2$, be the sheaf of continuous sections of f_i.* (a) *Let g be a continuous mapping of $\mathbf{E}^*f_1$ into $\mathbf{E}^*f_2$ such that $f_1^{-1} = f_2^{-1} \circ g$ (or equivalently, $f_1 = g^{-1} \circ f_2$). If s is a continuous section over U of f_1 then $g \circ s$ is a continuous section over U of f_2 (we have $\mathrm{gr}\,(g \circ f_1) \subset \mathrm{gr}\, f_2$). Put $\varphi_U = \{s \to g \circ s\} : S_{1U} \to S_{2U}$. It is easily seen that $\varphi = \{\varphi_U\}$ is a morphism of $\mathscr{S}_1$ into $\mathscr{S}_2$. This morphism is called the* morphism associated with g.

(b) *Conversely, given a morphism $\varphi = \{\varphi_U\}$ of $\mathscr{S}_1$ into $\mathscr{S}_2$ we want to define a continuous mapping g of $\mathbf{E}^*f_1$ into $\mathbf{E}^*f_2$ such that φ is the morphism associated with g.* Clearly gy must be the common value at $f_2 f_1^{-1} y$ of all $\varphi_U s$ such that $f_1^{-1} y \in U$ and $s f_1^{-1} y = y$. On the other hand, a given point y may be the value of no continuous section over an open set, and if s_1 and s_2 are continuous sections over an open set U and $s_1 x = s_2 x$ for some x in U, then the points $\varphi_U s_1 x$ and $\varphi_U s_2 x$ need not coincide, moreover, the sets $\mathbf{E}\varphi_U s_1$ and $\mathbf{E}\varphi_U s_2$ may be disjoint. It follows that g need not exist. *If f_1 is a covering fibration then g exists.* In fact, for any y in $\mathbf{E}^*f$ there exists a continuous section s over an open set with $y \in \mathbf{E}s$, and if s_1 and s_2 are two continuous sections over open sets U_1 and U_2 such that $y \in \mathbf{E}s_1 \cap \mathbf{E}s_2$, then s_1 and s_2 agree on an open neighborhood U of $x = f_1^{-1}y$, i.e. restrictions of s_1 and s_2 to U coincide, and hence $\varphi_{U_1}s_1x = \varphi_U sx = \varphi_{U_2}s_2x$ where s is the domain-restriction of both s_1 and s_2 to U. Thus the mapping g is well-defined, and clearly $f_1^{-1} = f_2^{-1} \circ g$. It is easily seen that $\varphi_U = \{s \to g \circ s\} : S_{1U} \to S_{2U}$. Since $\mathbf{E}^*f_1$ is inductively generated by continuous sections over open sets (40 B.6) and each composite $g \circ s$ is continuous, g is continuous by 33 A.5.

(c) *Let g be a continuous mapping of $\mathbf{E}^*f_1$ into $\mathbf{E}^*f_2$ satisfying $f_1^{-1} = f_2^{-1} \circ g$ and let φ be the morphism associated with g. If g is injective then φ is a monomorphism.* In fact, if φ is not a monomorphism then $g \circ s_1 = g \circ s_2$ for some $s_i \in S_{1U}$ $s_1 \neq s_2$; since $s_1 \neq s_2$, we have $s_1x \neq s_2x$ for some x in U and hence $gs_1x = (g \circ s_1)\, x = (g \circ s_2)\, x = gs_2x$, which shows that g is not injective. *If φ is a monomorphism and f_i are covering fibrations then g is injective.* Indeed, assuming that g is not injective, we can choose distinct y_1 and y_2 in $\mathbf{E}^*f$ such that $gy_1 = gy_2$, and continuous sections s_1 and s_2 over open sets U_1 and U_2 with $y_i \in \mathbf{E}s_i$; since $f_1^{-1} = f_2^{-1} \circ g$, the points y_i lie over the same point relative to f_1, say x, and $(g \circ s_1)\, x = (g \circ s_2)x$. Since f_2 is a covering fibration, the continuous sections $g \circ s_1$ and $g \circ s_2$ agree on an open neighborhood U of x. Let s_i', $i = 1, 2$, be the domain-restrictions of s_i to U; evidently $s_1' \neq s_2'$ but $g \circ s_1' = g \circ s_2'$.

(d) Under the assumptions of (c) it is evident that g is surjective provided that φ is an epimorphism and the ranges of continuous sections of f_2 over open sets cover $\mathbf{E}^*f_2$ (this condition is fulfilled if f_2 is a covering fibration). If g is surjective then φ need not be an epimorphism even if both f_1 and f_2 are covering fibrations. E.g. take a covering fibration f_1 over a space $\mathscr{P}$ such that $|\mathscr{P}| = \mathbf{D}f$ (i.e. a covering fibration upon $\mathscr{P}$) with the property that S_{1U} is empty for some U (e.g. see 40 B.11) and let $f_2 = \mathsf{J} : \mathscr{P} \to \mathscr{P}$; then there exists a unique g, namely f_1^{-1}, g is surjective but φ is not an epimorphism because $\mathbf{E}\varphi_U = \emptyset$ and $S_{2U} \neq \emptyset$. On the other hand, we shall prove that

If f_i are covering fibrations and either φ is a monomorphism or $\{\mathbf{E}\varphi_U\}$ is a sheaf (subsheaf of $\mathscr{S}_2$), then g is surjective if φ is an epimorphism.

First we shall prove that if s is any continuous section of f_2 over an open set U, then for each x in U there exists an open neighborhood U_x of x such that that the restriction s_x of s to U_x belongs to $\mathbf{E}\varphi_{U_x}$, i.e. $s_x = g \circ r_x$ for some continuous section r_x for f_1 over U_x. If $x \in U$ then $gy = sx$ for some y in $\mathbf{E}^*f_1$ and we can choose a continuous section t_x of f_1 over an open set V_x such that $t_x x = y$; we have $(g \circ t_x)\, x = sx$, and hence $g \circ t_x$ and s agree on an open neighborhood U_x of x. Let r_x be the restriction of t_x to U_x; evidently $g \circ r_x$ is the restriction of s to U_x. Now if $\{\mathbf{E}\varphi_U\}$ is projective then clearly $s \in \mathbf{E}\varphi_U$. If φ is a monomorphism, then the family $\{r_x\}$ is uniquely determined, and r_{x_1} and r_{x_2} coincide in $U_{x_1} \cap U_{x_2}$ for each x_1 and x_2. It follows that there exists a continuous section r over U such that each r_x is a restriction of r ($\mathscr{S}_1$ is a sheaf). Clearly $\varphi_U r = s$.

(e) Let f_i, $i = 1, 2, 3$, be fibrations over a space $\mathscr{P}$ and $\mathscr{S}_i$ be corresponding sheaves of continuous sections. Let g_1 and g_2 be continuous mappings, g_1 of $\mathbf{E}^*f_1$

into $\mathbf{E}^*f_2$ and g_2 of $\mathbf{E}^*f_2$ into $\mathbf{E}^*f_3$, such that $f_1^{-1} = f_2^{-1} \circ g_1, f_2^{-1} = f_3^{-1} \circ g_2$. Consider the composite $g = g_2 \circ g_1$. It is clear that $f_1^{-1} = f_3^{-1} \circ g$. Let φ_1 and φ_2 be the morphisms associated with g_1 and g_2. It is easily seen that $\varphi_2 \circ \varphi_1$ is associated with g.

40 B.13. Theorem. *Let $\mathscr{P}$ be a space and let Φ be the class of all covering fibrations over $\mathscr{P}$. For each morphism φ of $\mathscr{S}_{f_1}$ into $\mathscr{S}_{f_2}$ with f_i in Φ there exists a unique mapping g of $\mathbf{E}^*f_1$ into $\mathbf{E}^*f_2$ such that $\varphi_U s = g \circ s$ for each open set U and each continuous section s of f_1 over U; this mapping is called the* mapping associated with *φ and denoted by g_φ. The relation $\{\varphi \to g_\varphi\}$ is one-to-one and ranges on the class of all continuous mappings $g : \mathbf{E}^*f_1 \to \mathbf{E}^*f_2$ with f_i in Φ such that $f_1^{-1} = f_2^{-1} \circ g$. If $\varphi = \varphi_2 \circ \varphi_1$ then $g_\varphi = g_{\varphi_2} \circ g_{\varphi_1}$. A morphism φ is a monomorphism or isomorphism if and only if g_φ is, respectively, injective or bijective. If φ is an epimorphism then g_φ is surjective, but the converse assertion need not be true.*

Proof: 40 B.12.

The mappings g_φ are continuous. Now we shall prove that the fibrations g_φ^{-1} are covering fibrations.

40 B.14. Theorem. *Let f_1 and f_2 be covering fibrations over a space $\mathscr{P}$. If h is an inversely continuous fibration such that $f_1 = h \circ f_2$ (hence h is a fibration over $\mathbf{E}^*f_2$ and $\mathbf{E}^*f_1 = \mathbf{E}^*h$), then h is a covering fibration.*

Proof. We shall use 40 B.4. Let y be any point of $\mathbf{E}^*h$ and choose a continuous sections s of f_1 over an open set U and an x such that $sx = y$. The mapping $r = h^{-1} \circ s$ is a continuous section of f_2 over U. The sets $\mathbf{E}s$ and $\mathbf{E}r$ are open because the f_i are covering fibrations. It is easily seen that $t = h^{-1} : \mathbf{E}s \to \mathbf{E}^*f_2$ is an injective mapping, $\mathbf{E}t = \mathbf{E}r$, and $r = t \circ (s : U \to \mathbf{E}s)$. The mappings r and $s : U \to \mathbf{E}s$ are embeddings, and so is certainly t. By 40 B.4 h is a covering fibration.

Remark. It is easily seen that if f_i, $i = 1, 2$, are covering fibrations and $f_2 \circ f_1$ is defined, then $f_2 \circ f_1$ is a covering fibration.

A given sheaf $\mathscr{S}$ over a space $\mathscr{P}$ may be isomorphic with the sheaf of continuous sections of various fibrations over $\mathscr{P}$, e.g. if $\mathscr{S}$ is the sheaf of continuous sections of a fibration which is not a covering fibration. If $\mathscr{S}$ is isomorphic with the sheaf of continuous sections of a fibration f_1 and if f_2 is any covering fibration such that $\mathscr{S}_{f_2}$ is isomorphic with $\mathscr{S}$, then $\mathscr{S}_{f_2}$ and $\mathscr{S}_{f_1}$ are isomorphic, say under φ, and there exists a mapping g associated with φ (40 B.12 (b)) which is a mapping of $\mathbf{E}^*f_2$ into $\mathbf{E}^*f_1$. If f_1 is a covering fibration then g is a bijective mapping (40 B.12 (c), (d)), and furthermore by 40 B.14 g^{-1} is a covering fibration, and hence a homeomorphism. Conversely if g is a homeomorphism then φ is an isomorphism. Thus we have proved

40 B.15. *Let f_i, $i = 1, 2$, be covering fibrations over a space $\mathscr{P}$. The sheaves $\mathscr{S}_{f_1}$ and $\mathscr{S}_{f_2}$ are isomorphic if and only if there exists a homeomorphism g of $\mathbf{E}^*f_1$ onto $\mathbf{E}^*f_2$ such that $f_1^{-1} = f_2^{-1} \circ g$ (or equivalently, $f_1 = g^{-1} \circ f_2$).*

C. PRESHEAVES OF SPACES

40 C.1. Definition. A *presheaf of closure spaces over a quasi-ordered set* $\langle A, \leqq \rangle$ is a pair $\mathscr{S} = \langle \{\mathscr{P}_a\}, \{f_{ab}\} \rangle$ such that $\langle \{|\mathscr{P}_a|\}, \{|f_{ab}|\} \rangle$ is a presheaf of sets over $\langle A, \leqq \rangle$, which will be called the *underlying presheaf of sets* and will be denoted by $|\mathscr{S}|$, and such that each mapping f_{ab} is continuous. The set $\langle A, \leqq \rangle$ is called the *base of* $\mathscr{S}$ and the mappings f_{ab} are called *connecting mappings of* $\mathscr{S}$. The *projective limit of* $\mathscr{S}$, denoted by $\varprojlim \mathscr{S}$, is defined to be the projective limit of $|\mathscr{S}|$ endowed with the closure projectively generated by the family of all mappings $\mathrm{pr}_a : \varprojlim |\mathscr{S}| \to \mathscr{P}_a$. The mappings $\mathrm{pr}_a : \varprojlim \mathscr{S} \to \mathscr{P}_a$ will be termed the *canonical mappings* or *canonical projections* of $\varprojlim \mathscr{S}$. The *inductive limit of* $\mathscr{S}$, denoted by $\varinjlim \mathscr{S}$, is the inductive limit of $|\mathscr{S}|$ endowed with the closure inductively generated by the family of mappings $i_a : \mathscr{P}_a \to \varinjlim |\mathscr{S}|$ where i_a is the canonical mapping of $|\mathscr{P}_a|$ into $\varinjlim |\mathscr{S}|$; the mapping $i_a : \mathscr{P}_a \to \varinjlim \mathscr{S}$ will be called the *canonical mapping of* $\mathscr{P}_a$ *into* $\varinjlim \mathscr{S}$.

Remark. If the expressions "closure space", "closure" and "continuous mapping" are replaced by "semi-uniform space", "semi-uniformity" and "uniformly continuous mapping" or by "proximity space", "proximity" and "proximally continuous mapping" we obtain the definitions of presheaves, and of their inductive and projective limits, of semi-uniform spaces and of proximity spaces. It should be noted that also all the following theorems with their proofs (with the exception of 40 C.20, which deals with local bases at points) remain true for semi-uniform spaces and proximity spaces, if the terminology for closure spaces is replaced by the corresponding terminology for semi-uniform spaces or proximity spaces. The reader is requested to modify all definitions and results which follow for semi-uniform spaces and proximity spaces.

Most of results which follow extend the theorems already proved for presheaves of sets to presheaves of spaces. The corresponding results for presheaves of sets will be applied to the underlying presheaves of sets and the remainder of the proof will consist of a verification that a certain mapping is continuous. This verification of continuity always depends on general results on continuity from Sections 32 and 33.

40 C.2. Theorem. *If* $\mathscr{S} = \langle \{\mathscr{P}_a\}, \{f_{ab}\} \rangle$ *is a presheaf of closure spaces over* $\langle A, \leqq \rangle$*, then the projective limit of* $\mathscr{S}$ *is a subspace of the product space* $\Pi\{\mathscr{P}_a \mid a \in A\}$*, and the inductive limit is the quotient space of the sum space* $\Sigma\{\mathscr{P}_a \mid a \in A\}$ *under the smallest equivalence* ϱ *containing the relation* $\sigma = \{\langle\langle a, x\rangle, \langle b, y\rangle\rangle \mid f_{ac}x = f_{bc}y$ *for some* $c \in A\}$*, i.e.*

$$\varinjlim \mathscr{S} = \Sigma\{\mathscr{P}_a \mid a \in A\}/\varrho .$$

Proof. By definition the space $\varprojlim \mathscr{S}$ is projectively generated by the family $\{\mathrm{pr}_a : \varprojlim |\mathscr{S}| \to \mathscr{P}_a \mid a \in A\}$ which is the restriction to $\varprojlim |\mathscr{S}|$ of the family $\{\mathrm{pr}_a : \Pi\{|\mathscr{P}_b| \mid b \in A\} \to \mathscr{P}_a \mid a \in A\}$. By definition, the former family projectively generates the closure of $\varprojlim \mathscr{S}$, and the latter family projectively generates the closure of $\Pi\{\mathscr{P}_a\}$. Now by 32 A.13 $\varprojlim \mathscr{S}$ is a subspace of $\Pi\{\mathscr{P}_a\}$. To prove the second assertion, denote by $\{{}^a f\}$ the family of canonical mappings of the spaces $\mathscr{P}_a$ into $\underrightarrow{\lim}\, \mathscr{S}$, by $\{i_a\}$ the family of all canonical mappings of the spaces $\mathscr{P}_a$ into the sum $\Sigma\{\mathscr{P}_b \mid b \in A\}$, and by f the canonical mapping of $\Sigma\{\mathscr{P}_b \mid b \in A\}$ onto $\underrightarrow{\lim}\, \mathscr{S}$ (thus $f\langle b, x\rangle = \varrho[(\langle b, x\rangle)])$. Since $\{{}^a f\} = f \circ [\{i_a\}]$ and the familes $\{{}^a f\}$ and $\{i_a\}$ are inductive generating families for closure spaces, the former by definition and the latter by 33 A.3, f is an inductive generating mapping by 33 A.6, i.e. $\underleftarrow{\lim}\, \mathscr{S} = (\Sigma\{\mathscr{P}_a\})/f = (\Sigma\{\mathscr{P}_a\})/\varrho$.

40 C.3. Example. Let $\{\mathscr{P}_a \mid a \in A\}$ be a family of closure spaces. Let $\leqq$ be the identity relation on A and consider the presheaf $\mathscr{S} = \langle\{\mathscr{P}_a\}, \{f_{ab}\}\rangle$ over $\langle A, \leqq\rangle$, where f_{aa} is the identity mapping of $\mathscr{P}_a$ onto $\mathscr{P}_a$. Clearly $\underleftarrow{\lim}\, \mathscr{S} = \Pi\{\mathscr{P}_a \mid a \in A\}$ and the mapping $\{x \to (x)\}$ of $\Sigma\{\mathscr{P}_a \mid a \in A\}$ onto $\underrightarrow{\lim}\, \mathscr{S}$ is a homeomorphism (compare with the corresponding results 40 A.4 and 40 A.16 for presheaves of sets).

40 C.4. Definition. Suppose that $\mathscr{S} = \langle\{\mathscr{P}_a\}, \{f_{ab}\}\rangle$ is a presheaf over $\langle A, \leqq\rangle$. A *projective (inductive) family of continuous mappings* $\{g_a \mid a \in A\}$ is said to be *compatible for* $\mathscr{S}$ if $\{|g_a| \mid a \in A\}$ is compatible for the underlying presheaf $|\mathscr{S}|$ (see 40 A.5, 40 A.20).

40 C.5. Theorem. *Let* $\mathscr{S} = \langle\{\mathscr{P}_a\}, \{f_{ab}\}\rangle$ *be a presheaf of spaces over* $\langle A, \leqq\rangle$.

(a) *The family* $\{f_a\}$ *of all projections of* $\underrightarrow{\lim}\, \mathscr{S}$ *is compatible for* $\mathscr{S}$, *and if* $\{g_a\}$ *is any projective family of continuous mappings compatible for* $\mathscr{S}$ *then there exists exactly one mapping* g *such that* $g_a = f_a \circ g$ *for each* a; *this can be written as follows*

$$(*) \qquad \{g_a\} = [\{f_a\}] \circ g;$$

the mapping g *is continuous, and if* $\{g_a\}$ *is a projective generating family, then* g *is a projective generating mapping.*

(b) *The family* $\{{}^a f\}$ *of canonical mappings into* $\underrightarrow{\lim}\, \mathscr{S}$ *is an inductive family compatible for* $\mathscr{S}$, *and if* $\{g_a\}$ *is any inductive family of continuous mappings compatible for* $\mathscr{S}$ *then there exists exactly one mapping* g *of* $\underrightarrow{\lim}\, \mathscr{S}$ *(into the common range carrier of all* g_a*) such that* $g_a = g \circ {}^a f$ *for each* a *in* A; *this can be written as*

$$(**) \qquad \{g_a\} = g \circ [\{{}^a f\}];$$

the mapping g *is continuous, and if* $\{g_a\}$ *is an inductive generating family, then* g *is an inductive generating mapping.*

Proof. (a) The family $\{f_a\}$ is a projective generating family for closure spaces and hence each f_a is a continuous mapping and the family $\{|f_a|\}$ is compatible for $|\mathscr{S}|$ by 40 A.6. Thus $\{f_a\}$ is compatible for $\mathscr{S}$. Now let $\{g_a\}$ be a projective family of continuous mappings compatible for $\mathscr{S}$, and consider the family $\{|g_a|\}$ which is compatible for $|\mathscr{S}|$. By 40 A.7 there exists exactly one mapping h such that $\{|g_a|\} = [\{|f_a|\}] \circ h$. Put $g = \langle \mathrm{gr}\, h, \mathbf{D}^* g_a, \underleftarrow{\lim}\, \mathscr{S}\rangle$. We have $\{g_a\} = [\{f_a\}] \circ g$ and clearly g is the only mapping satisfying this condition. Since $\{f_a\}$ is a projective generating family and each g_a is continuous, g is necessarily continuous (by 32 A.8).

(b) As in the proof of (a), by virtue of 40 A.22, there exists exactly one mapping h such that $\{|g_a|\} = h \circ [\{|{}^a f|\}]$. Put $g = \langle \text{gr } h, \varinjlim \mathcal{S}, \mathbf{E}^* g_a \rangle$. Clearly g is the only mapping satisfying $\{g_a\} = g \circ [\{{}^a f\}]$. Finally, g is continuous by virtue of 33 A.5 because $\{{}^a f\}$ is an inductive generating family for closure spaces and $\{g_a\}$ is a family of continuous mappings.

40 C.6. Corollaries. (a) *If $\{h_a\}$ is a projective family of continuous mappings compatible for $\mathcal{S}$ such that every projective family of continuous mappings compatible for $\mathcal{S}$ admits a unique factorization (∗) with f_a replaced by h_a and with g continuous then there exists exactly one homeomorphism k such that $h_a = f_a \circ k$ for each a.*

(b) *If $\{h_a\}$ is an inductive family of continuous mappings compatible for $\mathcal{S}$ such that any other such family $\{g_a\}$ admits a unique factorization (∗∗) with ${}^a f$ replaced by h_a and with g continuous, then there exists exactly one homeomorphism k such that $h_a = k \circ {}^a f$ for each a.*

40 C.7. Definition. Let $\mathcal{S} = \langle \{\mathcal{P}_a\}, \{f_{ab}\} \rangle$ be a presheaf of closure spaces over $\langle A, \leqq \rangle$.

(a) If α is a lower bound of a set B in $\langle A, \leqq \rangle$ then $\{f_{\alpha b} \mid b \in B\}$ is a projective family of continuous mappings compatible for the restricted presheaf $\mathcal{S}_B$, and hence, by virtue of 40 C.5, there exists exactly one mapping g of $\mathcal{P}_\alpha$ into $\varprojlim \mathcal{S}_B$ such that each $f_{\alpha b}$ is the composite of g followed by the canonical mapping of $\varprojlim \mathcal{S}_B$ into $\mathcal{P}_b$ and the mapping g is continuous. This mapping will be called the *canonical mapping of $\mathcal{P}_\alpha$ into $\varprojlim \mathcal{S}_B$.*

(b) The *presheaf $\mathcal{S}$* will be called *projective at $\alpha \in A$* if the following condition is fulfilled:

If $\alpha = \inf B$ and B is right saturated, then the canonical mapping g of $\mathcal{P}_\alpha$ into $\varprojlim \mathcal{S}_B$ is a homeomorphism.

(c) The *presheaf $\mathcal{S}$* will be called *projective* if $\mathcal{S}$ is projective at each index $\alpha \in A$.

(a′) If α is an upper bound of a subset B of A, then $\{f_{b\alpha} \mid b \in B\}$ is an inductive family of continuous mappings compatible for the restricted presheaf $\mathcal{S}_B$ and hence, by virtue of 40 C.5, there exists exactly one mapping g of $\varinjlim \mathcal{S}_B$ into P_α such that each mapping $f_{b\alpha}$ is the composition of the canonical mapping of P_b into $\varinjlim \mathcal{S}_B$ followed by g; this mapping g, which is continuous by 40 C.5, will be called the *canonical mapping of $\varinjlim \mathcal{S}_B$ into $\mathcal{P}_\alpha$.*

(b′) The presheaf $\mathcal{S}$ will be called *inductive at $\alpha \in A$*, if the following condition is fulfilled:

If $\alpha = \sup B$ and B is left saturated, then the canonical mapping g of $\varinjlim \mathcal{S}_B$ into $\mathcal{P}_\alpha$ is a homeomorphism.

(c′) The *presheaf $\mathcal{S}$* will be called *inductive* if $\mathcal{S}$ is inductive at each index $\alpha \in A$ with the exception of the least elements of A.

40 C.8. Theorem. *Let $\mathscr{S} = \langle\{\mathscr{P}_a\}, \{f_{ab}\}\rangle$ be a presheaf over $\langle A, \leqq\rangle$. If α is a lower (upper) bound of $B \subset A$, then a mapping g of $\mathscr{P}_\alpha$ ($\varinjlim \mathscr{S}_B$) into $\varprojlim \mathscr{S}_B$ ($\mathscr{P}_\alpha$) is a canonical mapping if and only if $|g|$ is the canonical mapping of $|\mathscr{P}_\alpha|$ ($\varinjlim |\mathscr{S}|_B$) into $\varprojlim |\mathscr{S}|_B$ ($|\mathscr{P}_\alpha|$). If $\mathscr{S}$ is projective (inductive) at an index $\alpha \in A$, then the underlying presheaf of sets $|\mathscr{S}|$ is projective (inductive) at α; in particular, if $\mathscr{S}$ is projective or inductive, then so is $|\mathscr{S}|$.*

A natural question arises: if each C_U is endowed with a closure operation, under what conditions will $\{C_U\}$ be a projective sheaf of closure spaces; in particular, for which usual closures for spaces of mappings will the presheaf of spaces $\{C_U\}$ of continuous mappings be projective. We restrict ourselves to two examples.

40 C.9. *Suppose that P is a set and $\mathscr{Q}$ is a closure space. We shall prove that $\{\mathscr{Q}^X \mid X \in \exp P\}$ with restrictions as connecting mappings is a projective presheaf over $\langle \exp P, \supset\rangle$.* Let $Y \in \exp P$ and $\mathscr{X} \subset \exp P$ be such that $Y = \bigcup \mathscr{X}$ (*i.e.* $Y = \inf \mathscr{X}$) and $\mathscr{X}$ is hereditary (i.e. right-saturated) and consider the canonical mapping g of $\mathscr{Q}^Y$ into $\varprojlim \{\mathscr{Q}^X \mid X \in \mathscr{X}\}$.

It has already been shown that the underlying presheaf of sets $\{|\mathscr{Q}^X| \mid X \in \mathscr{X}\}$ is projective (see 40 A.10). It follows that the canonical mapping g is bijective. As always, g is continuous (40 C.5). Thus to show that g is a homeomorphism it is enough to prove that if a net N in $\varprojlim \{\mathscr{Q}^X \mid X \in \mathscr{X}\}$ converges to ϱ, then $g^{-1} \circ N$ converges to $g^{-1}\varrho$. However, this is almost self-evident. Indeed, if N converges to ϱ in $\varprojlim \{\mathscr{Q}^X \mid X \in \mathscr{X}\}$, then $\mathrm{pr}_X \circ N$ converges to $\mathrm{pr}_X\varrho$ in $\mathscr{Q}^X$ for each $X \in \mathscr{X}$, in particular, for each $X = (x)$, $x \in Y$. Now, $\mathrm{pr}_{(x)} \circ N = \mathrm{pr}_x \circ g^{-1} \circ N$ and $\mathrm{pr}_x g^{-1}\varrho = \mathrm{pr}_{(x)}\varrho$. It follows that $\mathrm{pr}_x(g^{-1} \circ N)$ converges to $\mathrm{pr}_x(g^{-1}\varrho)$ for each $x \in Y$. Since $\mathscr{Q}^Y$ is endowed with the product closure, $g^{-1} \circ N$ necessarily converges to $g^{-1}\varrho$ (because all projections onto coordinate spaces converge to corresponding coordinates of $g^{-1}\varrho$).

40 C.10. *Now let $\mathscr{P}$ and $\mathscr{Q}$ be closure spaces, and let $\{|C_U| \mid U \in \mathscr{U}\}$ be the sheaf (of sets) of continuous mappings of $\mathscr{P}$ into $\mathscr{Q}$ (see 40 A.12). Finally, let C_U be the set $|C_U|$ endowed with the closure of pointwise convergence. Clearly $\{C_U \mid U \in \mathscr{U}\}$ is a presheaf of closure spaces on $\mathscr{P}$. We shall prove that $\{C_U\}$ is projective.* Let $\mathscr{V}$ be a right-saturated collection of open sets whose infimum is W (that is, $V_1 \subset V \in \mathscr{V}$, V_1 open implies $V_1 \in \mathscr{V}$, and $W = \bigcup \mathscr{V}$), and let g be the canonical mapping of C_W into $\varprojlim \{C_V \mid V \in \mathscr{V}\}$. Since $|g|$ is the canonical mapping of $|C_W|$ into $\varprojlim \{|C_V| \mid V \in \mathscr{V}\} = |\varprojlim \{C_V \mid V \in \mathscr{V}\}|$ and $\{|C_U|\}$ is projective, the mapping $|g|$ is bijective and consequently g is also bijective. As always, g is continuous (by 40 C.5). Therefore it remains to prove that g^{-1} is continuous, and for this it is enough to show that if a net $N = \{N_a\}$ converges to a point ϱ in $\varprojlim \{C_V \mid V \in \mathscr{V}\}$ then the net $g^{-1} \circ N$ converges to the point $g^{-1}\varrho$. Suppose that N converges to ϱ. It follows that for each V in $\mathscr{V}$, the net $\mathrm{pr}_V \circ N$ converges to the mapping $\mathrm{pr}_V\, \varrho$ in C_V; but C_V is endowed with the closure of pointwise convergence, and consequently, for each x in V, the net $\{(\mathrm{pr}_V N_a)\, x\}$ converges to $(\mathrm{pr}_V\, \varrho)\, x$ in $\mathscr{Q}$ (of course $(\mathrm{pr}_V N_a)\, x$ is the value of the mapping $\mathrm{pr}_V N_a$ of V into $\mathscr{Q}$ at the point x). Since C_W is endowed with the closure of pointwise convergence, to prove that the net $g^{-1} \circ N$ converges to $g^{-1}\varrho$, it is enough to show that, for each x in W, the net $\{(g^{-1}N_a)\,x\}$ converges to the point $(g^{-1}\varrho)\,x$ in $\mathcal{Q}$ (of course, $(g^{-1}N_a)\,x$ is the value of the mapping $g^{-1}N_a$ of W into $\mathcal{Q}$ at the point x and $(g^{-1}\varrho)\,x$ is the value of the mapping $g^{-1}\varrho$ of W into $\mathcal{Q}$ at the point x). Let x be any point of W. We can choose a V in $\mathcal{V}$ containing x. For each $\varphi \in C_W$ we have $\mathrm{pr}_V(g(\varphi)) = \varphi|V$, and hence $(g^{-1}N_a) \mid V = \mathrm{pr}_V N_a$ and $(g^{-1}\varrho) \mid V = \mathrm{pr}_V \varrho$; in particular

$$(g^{-1}N_a)\,x = (\mathrm{pr}_V N_a)\,x, \quad (g^{-1}\varrho)\,x = (\mathrm{pr}_V \varrho)\,x\,.$$

However, as we noticed above, the net $\{(\mathrm{pr}_V N_a)\,x\}$ converges to $(\mathrm{pr}_V \varrho)\,x$ in $\mathcal{Q}$.

40 C.11. Definition. Suppose that $\mathcal{S}$ and $\mathcal{S}'$ are two presheaves over the same quasi-ordered set $\langle A, \leqq \rangle$ and $\mathcal{S} = \langle \{\mathcal{P}_a\}, \{f_{ab}\} \rangle$, $\mathcal{S}' = \langle \{\mathcal{P}'_a\}, \{f'_{ab}\} \rangle$. A *morphism* of $\mathcal{S}$ into $\mathcal{S}'$ is a family $\varphi = \{\varphi_a\}$ such that each φ_a is a continuous mapping of $\mathcal{P}_a$ into $\mathcal{P}'_a$ and $|\varphi| = \{|\varphi_a|\}$ is a morphism of $|\mathcal{S}|$ into $|\mathcal{S}'|$. A morphism φ is an *isomorphism* if $|\varphi|$ is an isomorphism and $\{\varphi_a^{-1}\}$ is a morphism (i.e. all φ_a are homeomorphisms). The *composite* $\varphi \,.\, \psi$ of two morphisms $\{\psi_a\}$ and $\{\varphi_a\}$ is the morphism $\{\varphi_a \circ \psi_a\}$.

It may be noted that we do not define the notions of an epimorphism or a monomorphism. The results 40 A.28 – 40 A.30 concerning presheaves of sets are transferred to presheaves of spaces as follows:

40 C.12. Theorem. *With the notation of* 40 C.11 *let* $\varphi = \{\varphi_a\}$ *be a morphism of* $\mathcal{S}$ *into* $\mathcal{S}'$.

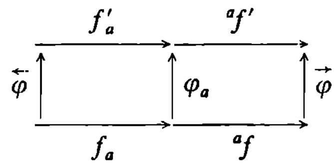
(a) *There exists exactly one mapping* $\overleftarrow{\varphi}$ *of* $\varprojlim \mathcal{S}$ *into* $\varprojlim \mathcal{S}'$ *such that* $(\mathrm{pr}_a : \varprojlim \mathcal{S}' \to \mathcal{P}'_a) \circ \overleftarrow{\varphi} = \varphi_a \circ (\mathrm{pr}_a : \varprojlim \mathcal{S} \to \mathcal{P}_a)$ *for each* a. *The mapping* $\overleftarrow{\varphi}$ *is continuous. The mapping* $\overleftarrow{\varphi}$ *is injective whenever all* φ_a *are injective, and* $\overleftarrow{\varphi}$ *is a homeomorphism whenever* φ *is an isomorphism.*

(b) *There exists exactly one mapping* $\overrightarrow{\varphi}$ *of* $\varinjlim \mathcal{S}$ *into* $\varinjlim \mathcal{S}'$ *such that* $\overrightarrow{\varphi} \circ {}^a f = {}^a f' \circ \varphi_a$ *for each* a, *where* ${}^a f$ *and* ${}^a f'$ *denote the canonical mappings into* $\varinjlim \mathcal{S}$ *and* $\varinjlim \mathcal{S}'$ *respectively. The mapping* $\overrightarrow{\varphi}$ *is continuous;* $\overrightarrow{\varphi}$ *is surjective provided that all* φ_a *are surjective, and* $\overrightarrow{\varphi}$ *is a homeomorphism provided that* φ *is an isomorphism.*

Proof. Consider the underlying presheaves $|\mathcal{S}|$ and $|\mathcal{S}'|$ and the underlying morphism $|\varphi| = \{|\varphi_a|\}$ of $|\mathcal{S}|$ into $|\mathcal{S}'|$. Obviously, if $\{f_a\}$ are the families of projections of $\varprojlim \mathcal{S}$ and $\varprojlim \mathcal{S}'$, then $\{|f_a|\}$ and $\{|f'_a|\}$ are the families of projections of $\varprojlim |\mathcal{S}|$ and $\varprojlim |\mathcal{S}'|$, respectively, and similarly for canonical mappings into inductive limits. By 40 A.28 there exists exactly one mapping $\overleftarrow{|\varphi|}$ of $\varprojlim |\mathcal{S}|$ into $\varprojlim |\mathcal{S}'|$ and exactly one mapping $\overrightarrow{|\varphi|}$ of $\varinjlim |\mathcal{S}|$ into $\varinjlim |\mathcal{S}'|$ making the following diagram commutative:

$$\begin{array}{ccccc} \cdot & \xrightarrow{\;|f'_a|\;} & \cdot & \xrightarrow{\;|{}^a f'|\;} & \cdot \\ \overleftarrow{|\varphi|}\,\Big\uparrow & & \Big\uparrow\,|\varphi_a| & & \Big\uparrow\,\overrightarrow{|\varphi|} \\ \cdot & \xrightarrow[\;|f_a|\;]{} & \cdot & \xrightarrow[\;|{}^a f|\;]{} & \cdot \end{array}$$

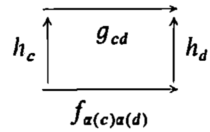
Consequently, there exists exactly one mapping $\overleftarrow{\varphi}$ of $\varprojlim \mathscr{S}$ into $\varprojlim \mathscr{S}'$ and exactly one mapping $\overrightarrow{\varphi}$ of $\varinjlim \mathscr{S}$ into $\varinjlim \mathscr{S}'$ making the diagram



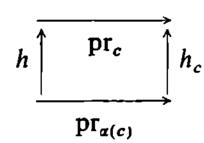
commutative. Of course, $|\overrightarrow{\varphi}| = \overrightarrow{|\varphi|}$ and $|\overleftarrow{\varphi}| = \overleftarrow{|\varphi|}$. According to Theorem 40 A.28 it is sufficient to prove that both $\overleftarrow{\varphi}$ and $\overrightarrow{\varphi}$ are continuous. The continuity of $\overleftarrow{\varphi}$ follows according to 32 A.8 from the facts that $\{f'_a\}$ is a projective generating family and $f'_a \circ \overleftarrow{\varphi}$ $(= \varphi_a \circ f_a)$ is continuous for each a. Similarly, the continuity of $\overrightarrow{\varphi}$ follows by 33 A.5 from the facts that $\{^a f\}$ is an inductive generating family and $\overrightarrow{\varphi} \circ {^a f}$ $(= {^a f'} \circ \varphi_a)$ is continuous for each a.

40 C.13. Theorem. *Suppose that* $\mathscr{S} = \langle \{\mathscr{P}_a\}, \{f_{ab}\}\rangle$ *is a presheaf of spaces over* $\langle A, \leqq \rangle$ *and* $\mathscr{S}_1 = \langle \{\mathscr{Q}_c\}, \{g_{cd}\}\rangle$ *is a presheaf of spaces over* $\langle C, \prec \rangle$.

(a) *Let* α *be an order-preserving mapping of* $\langle C, \prec \rangle$ *into* $\langle A, \leqq \rangle$ *and* $\{h_c \mid c \in C\}$ *be a family of continuous mappings, with each* h_c *a mapping of* $\mathscr{P}_{\alpha(c)}$ *into* $\mathscr{Q}_c$, *such that the diagram*

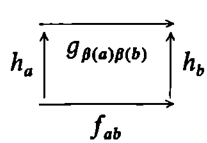


is commutative for each $c \prec d$. *There exists exactly one mapping* h *of* $\varprojlim \mathscr{S}$ *into* $\varprojlim \mathscr{S}_1$ *such that the diagram*

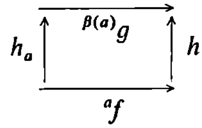


is commutative for each c *in* C, *where* pr_c *and* $\mathrm{pr}_{\alpha(c)}$ *denote the canonical projections of* $\varprojlim \mathscr{S}_1$ *into* $\mathscr{Q}_c$ *and* $\varprojlim \mathscr{S}$ *into* $\mathscr{P}_{\alpha(c)}$, *respectively. This mapping is continuous.*

(b) *Let* β *be an order-preserving mapping of* $\langle A, \leqq \rangle$ *into* $\langle C, \prec \rangle$ *and* $\{h_a\}$ *be a family, with each* h_a *a continuous mapping of* $\mathscr{P}_a$ *into* $\mathscr{Q}_{\beta(a)}$, *such that the digram*



is commutative for each $a \leqq b$. Then there exists exactly one mapping h of $\varinjlim \mathcal{S}$ into $\varinjlim \mathcal{S}_1$ such that the diagram



is commutative for each $a \in A$, where ${}^a f$ and ${}^{\beta(a)} g$ denote the canonical mappings of $\mathcal{P}_a$ into $\varinjlim \mathcal{S}$ and $\mathcal{Q}_{\beta(a)}$ into $\varinjlim \mathcal{S}_1$, respectively.

Proof. Consider the underlying presheaves $|\mathcal{S}|$ and $|\mathcal{S}_1|$ and apply 40 A.29 to the families $\{|h_c| \mid c \in C\}$ and $\{|h_a| \mid a \in A\}$. The continuity is proved as in the preceding theorem; the details are left to the reader.

Corollaries. *Let $\mathcal{S}$ be a presheaf of spaces over $\langle A, \leqq \rangle$ and let $B \subset A$.* (a) *There exists exactly one mapping h of $\varprojlim \mathcal{S}$ into $\varprojlim \mathcal{S}_B$ such that, for each $b \in B$, the composite h with the canonical projection of $\varprojlim \mathcal{S}_B$ into $\mathcal{P}_b$ is the canonical projection of $\varprojlim \mathcal{S}$ into $\mathcal{P}_b$. This mapping is continuous.* (b) *There exists exactly one mapping k of $\varinjlim \mathcal{S}_B$ into $\varinjlim \mathcal{S}$ such that, for each $b \in B$, the composite of the canonical mappings of $\mathcal{P}_b$ into $\varinjlim \mathcal{S}_B$ followed by k is the canonical mapping of $\mathcal{P}_b$ into $\varinjlim \mathcal{S}$. This mapping is continuous.*

Proof. Put $\alpha = \beta = \{b \to b \mid b \in B\}$ and apply the theorem (in case (a) $\mathcal{S} = \mathcal{S}$ and $\mathcal{S}_1 = \mathcal{S}_B$, and in the case (b) $\mathcal{S} = \mathcal{S}_B$ and $\mathcal{S}_1 = \mathcal{S}$).

40 C.14. Remark to Corollaries of 40 C.13.

(a) The mapping h is an embedding provided that B is left cofinal in A and h is a homeomorphism whenever B is left cofinal and A is left-directed.

(b) If B is right cofinal in A, then the mapping k is a homeomorphism.

Proof. According to 40 A.30 applied to $|\mathcal{S}|$, if B is left cofinal then h is injective (because $|h|$ is injective) and if in addition A is left directed, then h is bijective (because $|h|$ is bijective); if B is right cofinal, then k is bijective (because $|k|$ is bijective). Hence it remains to show that h is a projective generating mapping if B is left cofinal, and k is an inductive generating mapping if B is right cofinal. Now $f'_b \circ h = f_b$, $k \circ {}^b f' = {}^b f$ $(b \in B)$, where f_b and f'_b denote the projections of $\varprojlim \mathcal{S}$ and $\varprojlim \mathcal{S}_B$ into $\mathcal{P}_b$ respectively, and ${}^b f$ and ${}^b f'$ denote the canonical mappings of $\mathcal{P}_b$ into $\varinjlim \mathcal{S}$ and $\varinjlim \mathcal{S}_B$ respectively; by virtue of 32 A.9 and 33 A.6 it is sufficient to show that $\{f_b \mid b \in B\}$ and $\{f'_b \mid b \in B\}$ are projective generating families and $\{{}^b f \mid b \in B\}$ and $\{{}^b f' \mid b \in B\}$ are inductive generating families. By definition of projective and inductive limits, the family $\{f'_b \mid b \in B\}$ is a projective generating family and $\{{}^b f' \mid b \in B\}$ is an inductive generating family. In general $\{f_b \mid b \in B\}$ need not be a projective generating family, but if B is left cofinal then $\{f_b\}$ is indeed a projective generating family. In fact, $\{f_a \mid a \in A\}$ is a projective generating family and both of the families $\{f_a \mid a \in A\}$ and $\{f_b \mid b \in B\}$ projectively generate the same closure, because $B \subset A$

and if $a \in A$ then $f_a = f_{ab} \circ f_b$ for some $b \leqq a$ with f_{ab} continuous. Similarly we can prove that $\{{}^b f' \mid b \in B\}$ in an inductive generating family.

Now we proceed to an investigation of properties invariant under projective and inductive limits.

40 C.15. Theorem. *If a class K of spaces is hereditary and closed under products, then K is closed under the formation of projective limits, that is, the projective limit of a presheaf of spaces from K belongs to K. If a class K of spaces is closed under sums and under the formation of quotient spaces, then K is closed under the formation of inductive limits, that is, the inductive limit of a presheaf of spaces from K belongs to K.*

Proof. By 40 C.2 the projective limit of a presheaf of spaces $\{\mathscr{P}_a\}$ is a subspace of the product space $\Pi\{\mathscr{P}_a\}$ and the inductive limit of $\{\mathscr{P}_a\}$ is a quotient of the sum space $\Sigma\{\mathscr{P}_a\}$.

Corollary. *The classes of all topological spaces, semi-separated spaces, separated spaces, regular spaces and uniformizable spaces are closed under the formation of projective limits. The classes of all discrete spaces, quasi-discrete spaces, locally connected spaces and spaces admitting a determining sequential relation are closed under the formation of inductive limits.*

Up to now we have considered no class of spaces invariant under products and formation of closed subspaces but not of arbitrary subspaces. Nevertheless there are important classes with these properties, e.g. the class of all compact spaces. Therefore, for the sake of completeness, we shall prove the following proposition which gives a sufficient condition for a projective limit of a presheaf of spaces to be closed in the corresponding product space.

40 C.16. *If $\mathscr{S} = \langle\{\mathscr{P}_a\}, \{f_{ab}\}\rangle$ is a presheaf of separated spaces over $\langle A, \leqq\rangle$, then the projective limit of $\mathscr{S}$ is a closed subspace of the product space $\Pi\{\mathscr{P}_a \mid a \in A\}$.*

Proof. The projective limit of $\mathscr{S}$ is the subspace of $\Pi\{\mathscr{P}_a \mid a \in A\}$ consisting of all points x such that $(f_{ab} \circ \pi_a)\, x = \pi_b x$ for each $a \leqq b$ ($\{\pi_a\}$ is the family of all projections of the product). Since all mappings under question are continuous and all spaces $\mathscr{P}_a$ are separated, $\varprojlim \mathscr{S}$ is closed by 27 A.7.

Corollary. *Assume that a class K of separated spaces is closed under formation of products and each closed subspace of each space from K belongs to K. Then the projective limit of every presheaf of spaces from K belongs to K.*

The inductive limit of a presheaf of topological spaces need not be a topological space. For example, let A be a monotone ordered subset of some $\mathbf{C}(P)$ such that each closure of A is topological but the least upper bound of A is not topological. Consider the presheaf $\mathscr{S} = \langle\{\langle P, a\rangle\}, \{f_{ab}\}\rangle$ over A where each f_{ab} is the identity mapping of $\langle P, a\rangle$ onto $\langle P, b\rangle$. By 40 C.5 $\varinjlim \mathscr{S}$ is homeomorphic with $\langle P, \sup A\rangle$, and consequently $\varinjlim \mathscr{S}$ is not a topological inductive limit of a presheaf of topological spaces. It is to be noted that, similarly, the inductive limit of a presheaf of uniform spaces need not be a uniform space, and also the inductive limit of a presheaf of uniformizable proximity spaces need not be a uniformizable proximity space. Therefore one can introduce, in a natural way, the notion of an inductive limit in the uniform sense and also the corresponding notion for presheaves of uniformizable proximities.

40 C.17. Topological inductive limit. The *topological inductive limit of a presheaf $\mathscr{S}$ of a topological closure spaces* is defined to be the topological modification of $\varinjlim \mathscr{S}$. It follows that the underlying set of the topological inductive limit is $|\varinjlim \mathscr{S}| = \varinjlim |\mathscr{S}|$ and the closure of the topological inductive limit is topologically inductively generated by the family of all canonical mappings defined in the obvious manner. The reader is asked to verify that all preceding results for inductive limit remain valid for topological inductive limits of presheaves of topological spaces. In what follows we restrict ourselves to inductive limits.

The concluding part is devoted to an examination of the consequences of an important additional assumption upon the base of a presheaf, namely of left directedness or right directedness. Twice these assumptions have already been employed as sufficient conditions: in 40 A.16 it has been shown that $\{\langle a, x\rangle \to \langle b, y\rangle \mid f_{ac}x = f_{bc}y$ for some $c\}$ is an equivalence whenever the base is right-directed, and in 40 C.14 it has been shown that the canonical mapping of the projective limit of a presheaf of spaces over a left-directed set into the projective limit of a restricted presheaf over a left cofinal subset is a homeomorphism. For convenience we shall introduce some terminology.

40 C.18. Convention. A left-directed (right-directed) presheaf of sets or spaces is a presheaf of sets or spaces over a left-directed (right-directed) set.

For presheaves of sets we shall prove the following result:

40 C.19. *Let $\mathscr{S} = \langle\{P_a\}, \{f_{ab}\}\rangle$ be a presheaf of sets over $\langle A, \leqq\rangle$. If $\mathscr{S}$ is left-directed and F is a finite subset of $\varprojlim \mathscr{S}$, then there exists an α in A such that the restriction of the projection f_α of $\varprojlim \mathscr{S}$ into P_α to F is injective. If $\mathscr{S}$ is right-directed and F is a finite subset of $\varinjlim \mathscr{S}$, then there exists an α in A and a finite subset F_1 of P_α such that the restriction of the canonical mapping ${}^\alpha f$ of P_α into $\varinjlim \mathscr{S}$ to F_1 is injective and maps F_1 onto F.*

Proof. I. Suppose that F is a finite subset of $\varprojlim \mathscr{S}$. For any two distinct points x and y of F we can choose an $a = a(x, y)$ in A so that $f_a x \neq f_a y$. If $\langle A, \leqq\rangle$ is left-directed (in particular, non-void), we can choose an α in A preceding all $a(x, y)$. Clearly α possesses the required property. — II. Now let F be a finite subset of $\varinjlim \mathscr{S}$. For each x in F we can choose an $a = a_x$ in A so that ${}^a f y_x = x$ for some y_x in P_a. If $\langle A, \leqq\rangle$ is right-directed then there exists an α in A following each a_x, $x \in F$. Let F_1 be the set of all $f_{a_x\alpha} y_x$, $x \in F$. Obviously F_1 and α possess the required properties.

If x is a point of the projective limit of a presheaf of closure spaces $\{\mathscr{P}_a\}$ over $\langle A, \leqq\rangle$ and if $\{f_a\}$ is the family of projections, then, by virtue of 32 A.6, the collection $\mathscr{U}$ of all sets of the form $f_a^{-1}[U]$, U a neighborhood of $f_a x$ in $\mathscr{P}_a$ and $a \in A$, is a local sub-base at x in $\varprojlim \mathscr{S}$. Indeed, $\varprojlim \mathscr{S}$ is, by definition, projectively generated by the family $\{f_a\}$. On the other hand, $\mathscr{U}$ need not be a local base at x; e.g. consider a product of non-accrete spaces. Sometimes it may be useful to know that $\mathscr{U}$ is a local base provided that $\langle A, \leqq\rangle$ is left-directed, as stated in the following proposition.

40 C.20. *Let $\mathscr{S} = \langle\{\mathscr{P}_a\}, \{f_{ab}\}\rangle$ be a left-directed presheaf of closure spaces over $\langle A, \leqq\rangle$ and let $\{f_a\}$ be the family of all canonical projections. Then, for each x in $\varprojlim \mathscr{S}$, the collection $\mathscr{U}_x$ of all sets of the form $f_a^{-1}[U]$, where U varies over all neighborhoods of $f_a x$ and a varies over all $a \in A$, is a local base at x. If all spaces $\mathscr{P}_a$ are topological then ($\varprojlim \mathscr{S}$ is topological and) the collection of all sets of the form $f_a^{-1}[U]$, U open in $\mathscr{P}_a$, $a \in A$, is an open base for $\varprojlim \mathscr{S}$.*

Proof. As we noted above, $\mathscr{U}_x$ is a local sub-base at x. To prove that $\mathscr{U}_x$ is a local base at x it remains to show that $\mathscr{U}_x$ is a filter base, that is, the intersection of any two sets V_1 and V_2 from $\mathscr{U}_x$ contains a $V \in \mathscr{U}_x$. Let $V_i = f_{a_i}^{-1}[U_i]$, $i = 1, 2$, where U_i is a neighborhood of $f_{a_i}x$ in $\mathscr{P}_{a_i}$. Since $\langle A, \leqq\rangle$ is left-directed, we can choose an a in A preceding both a_1 and a_2. The mappings f_{aa_1} being continuous, the set $U = (f_{aa_1}^{-1}[U_1] \cap f_{aa_2}^{-1}[U_2])$ is a neighborhood of $f_a x$ in $\mathscr{P}_a$. Put $V = f_a^{-1}[U]$. From $f_{a_i} = f_{aa_i} \circ f_a$ we obtain $V \subset V_1 \cap V_2$. The second statement is in an immediate consequence of the first.

The assumption of right directedness does not have topological consequences for neither the projective nor for the inductive limit.

Remark. It has already been shown that the product of a family $\{\mathscr{P}_a \mid a \in A\}$ of sets or spaces is the projective limit of the presheaf $\langle\{\mathscr{P}_a\}, \{f_{ab}\}\rangle$ over the set A ordered by the identity relation J_A with identities as connecting mappings. Similarly, the sum is evidently "almost" the inductive limit of this presheaf. On the other hand, if $\mathscr{P}_a = \mathscr{P}$ for each a, then, in general, there exists no left directed presheaf $\{\mathscr{Q}_c\}$ such that $\mathscr{Q}_c = \mathscr{P}$ for each c and $\Pi\{\mathscr{P}_a\} = \varprojlim \{\mathscr{Q}_c\}$. For example, if the cardinal of $|\mathscr{P}|$ is finite, say n, then the cardinal of $\varprojlim \{\mathscr{Q}_c\}$ is at most n by 40 C.19. On the other hand, the cardinal of $\Pi\{\mathscr{P}_a\}$ is infinite if the index set is infinite and $\mathscr{P}$ has at least two elements. The same is true for the inductive limit.

40 C.21. It is easily seen that the product is the projective limit of finite partial products and the sum is "almost" the inductive limit of finite partial sums. Indeed, let $\{\mathscr{P}_a \mid a \in A\}$ be a family of spaces, and consider the presheaf $\mathscr{S} = \langle\{\mathscr{Q}_c\}, \{f_{cd}\}\rangle$ over $\langle C, \supset\rangle$ where C is the collection of all non-void finite subsets of A, $\mathscr{Q}_c = \Pi\{\mathscr{P}_a \mid a \in C\}$ and f_{cd} is the canonical mapping of $\mathscr{Q}_c$ onto $\mathscr{Q}_d$, i.e. $f_{cd}x$ is the restriction of x to d for each x. It is easily seen that $\Pi\{\mathscr{P}_a\} = \varprojlim \mathscr{S}$. For the sum the construction is analogous. Clearly $\mathscr{S} = \langle\{\mathscr{Q}_c\}, \{f_{cd}\}\rangle$ is a presheaf over $\langle C, \subset\rangle$, where C is again the collection of all finite subsets of A, $\mathscr{Q}_c = \Sigma\{\mathscr{P}_a \mid a \in C\}$ and f_{cd} is the identity mapping of $\mathscr{Q}_c$ into $\mathscr{Q}_d$. Evidently there exists a one-to-one mapping h of $\varinjlim \mathscr{S}$ onto $\Sigma\{\mathscr{P}_a \mid a \in A\}$ such that the element of $\varinjlim \mathscr{S}$ containing $\langle a, \langle a, x\rangle\rangle \in |Q_{(a)}|$ is carried into $\langle a, x\rangle$ for each $a \in A$ and $x \in \mathscr{P}_a$.

40 C.22. Theorem. *Let K be a class of closure spaces. The following three conditions on a closure space $\mathscr{P}$ are equivalent:*

(a) *$\mathscr{P}$ is homeomorphic with the projective limit of a left-directed presheaf of subspaces of finite products of spaces from K;*

(b) *$\mathscr{P}$ admits an embedding into the product of a family of spaces from K;*

(c) *The class $\mathscr{F}$ of all continuous mappings of $\mathscr{P}$ into finite products of spaces from K distinguishes points and for each subset X of $\mathscr{P}$ and each $x \in P - \overline{X}$ there exists a φ in $\mathscr{F}$ such that $\varphi x \notin \overline{\varphi(X)}$.*

Proof. We shall prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). I. The implications (a) $\Rightarrow$ (b) follows from the facts that the projective limit is a subspace of the corresponding product, and the product of products of spaces from a given class of spaces is homeomorphic with the product of a family of spaces from the same class.

II. Now suppose (b). Let Φ be an embedding of $\mathscr{P}$ into a product $\mathscr{Q} = \Pi\{\mathscr{Q}_a \mid a \in A\}$ where $\mathscr{Q}_a \in K$ for each a. If π_B denotes the projection of $\mathscr{Q}$ onto $\mathscr{Q}_B = \Pi\{\mathscr{Q}_a \mid a \in B\}$, then clearly the mappings $\pi_{(a)} \circ \Phi$, $a \in A$, distinguish the points of $\mathscr{P}$ and belong to $\mathscr{F}$, and, if $x \notin \overline{X}$, then we can take $\pi_B \circ \Phi$ as φ where B is a suitable finite subset of A.

III. It remains to show that (c) implies (a). Suppose (c) and choose a subset $\mathscr{F}_1$ of $\mathscr{F}$ so that condition (c) remains true if $\mathscr{F}$ is replaced by $\mathscr{F}_1$, and consider the set A of all finite subsets of $\mathscr{F}_1$. The set A is left-directed by the inverse inclusion $\supset$ (because $\emptyset \in A$ and hence $A \neq \emptyset$). By our assumption, the range carrier of each $\varphi \in \mathscr{F}_1$ is a product

$$\mathbf{E}^*\varphi = \Pi\{\mathscr{Q}_c \mid c \in C_\varphi\}$$

where $\mathscr{Q}_c \in K$ and C_φ is a finite set. For each $a \in A$ put

$$\mathscr{P}_a = \Pi\{\mathscr{Q}_c \mid c \in C_\varphi, \varphi \in a\}$$

and, if $a \supset b$, let f_{ab} be the canonical projection of $\mathscr{P}_a$ onto $\mathscr{P}_b$. Clearly $\mathscr{S} = \langle\{\mathscr{P}_a\}, \{f_{ab}\}\rangle$ is a presheaf over $\langle A, \supset\rangle$. For each a in A let φ_a be the mapping of $\mathscr{P}$ into $\mathscr{P}_a$ which assigns to each point x of $\mathscr{P}$ the point $\{\mathrm{pr}_c\, \varphi x \mid c \in C_\varphi, \varphi \in a\}$ of $\mathscr{P}_a$. Clearly $\{\varphi_a\}$ is a projective family of mappings compatible for $\mathscr{S}$. Now we are prepared to define the required presheaf. For each a let $\mathscr{P}'_a$ be the subspace $\varphi_a[|\mathscr{P}|]$ of $\mathscr{P}_a$, and for each $a \supset b$, let f'_{ab} be the restriction of f_{ab} to a mapping of $\mathscr{P}'_a$ into $\mathscr{P}'_b$ (this restriction exists, since $\{\varphi_a\}$ is compatible for $\mathscr{S}$). It is obvious that $\mathscr{S}' = \langle\{\mathscr{P}'_a\}, \{f'_{ab}\}\rangle$ is a presheaf over $\langle A, \supset\rangle$. We shall prove that $\varprojlim \mathscr{S}'$ is homeomorphic to $\mathscr{P}$. Denote by φ'_a, $a \in A$ the restriction of φ_a to a mapping of $\mathscr{P}$ into (in fact, onto) $\mathscr{P}'_a$. Since $\{\varphi_a\}$ is compatible for $\mathscr{S}$, $\{\varphi'_a\}$ is compatible for $\mathscr{S}'$. By 40 C.5 there exists a continuous mapping f of $\mathscr{P}$ into $\varprojlim \mathscr{S}'$ such that $f_a \circ f = \varphi'_a$ for each a in A, where f_a, as usual, is the canonical projection of $\varprojlim \mathscr{S}'$ into $\mathscr{P}'_a$. Since all mappings φ'_a are surjective, f is also such. If x and y are two distinct points of $\mathscr{P}$

then, by our assumption, there exists a φ in $\mathscr{F}_1$ such that $\varphi x \neq \varphi y$ and hence $\varphi'_a x \neq \varphi'_a y$ for $a = (\varphi)$. As a consequence $fx \neq fy$, which shows that f is injective. It remains to prove that f^{-1} is continuous. It is enough to show that $x \in (|\mathscr{P}| - \overline{X})$ implies $fx \notin \overline{f[X]}$. Suppose $x \in |\mathscr{P}| - \overline{X}$. By our assumption we can choose a φ in $\mathscr{F}_1$ so that $x \notin \overline{\varphi[X]}$ (in $\mathbf{E}^*\varphi$), and hence $x \notin \overline{\varphi'_a[X]}$ for $a = (\varphi)$. Since the projection f is continuous and $f \circ f_a = \varphi'_a$, necessarily $fx \notin \overline{f[X]}$; this concludes the proof.