NOTES

REFERENCES

It is not the intention to give a bibliography of articles on general topology, since compiling a complete list would be an enormous task, whereas a selective list, whether long or short, would be highly biased by personal judgements. For similar reasons we avoid remarks of a historical character. Thus, only some books on general topology and other topics are listed below. Many of these contain an extensive bibliography.

For various questions examined in this book, the well known treatise of N. Bourbaki may be consulted:

N. Bourbaki, Éléments de mathématique. Paris.

For questions concerning theory of classes and sets, in particular its axiomatic foundation, see e.g.:

H. Bachmann, Transfinite Zahlen. Berlin-Göttingen-Heidelberg, 1955.

A. Fraenkel, Abstract Set Theory. Amsterdam, 1953.

A. Fraenkel and Y. Bar-Hillel, Foundations of Set Theory. Amsterdam, 1958.

The following books on general topology may be mentioned (some of them, although basically concerned with different topics, contain material pertinent to questions considered in the present book):

C. Berge, Espaces topologiques, fonctions multivoques. Paris, 1959.

A. Csaszár, Grundlagen der allgemeinen Topologie. Budapest, 1963.

L. Gillman and M. Jerison, Rings of Continuous Functions. Princeton, 1960.

J. R. Isbell, Uniform Spaces. Providence, 1964.

J. L. Kelley, General Topology. New York, 1955.

K. Kuratowski, Topologie. Warszawa, I. 1958, II. 1961.

G. Nöbeling, Grundlagen der analytischen Topologie. Berlin, 1954.

R. Vaidanathaswamy, Set Topology. New York, 1960.

ON SOME NON-COMPRISABLE OBJECTS

As already indicated (cf. 3 F.13), it seems that there are no means, short of an introduction of an additional fundamental concept ("superclasses" or some such concept) to give a general definition, say, of a "regular multiplet" of classes. The reason lies in the fact that we cannot handle "infinitely many" non-comprisable objects simultaneously and, in particular, there are no such objects as "sequences of non-comprisable classes".

Therefore, it is necessary to define such objects only for a limited number of members. This can be done as follows.

Let a natural number k be actually given (for "practical" purposes, $k = 100$ will do). We define: x is a 2-tuple if and only if it is a pair; for $n \in \mathsf{N}$, $2 < n \leqq k$, x is an n-tuple if and only if there exist y and z such that $x = \langle y, z \rangle$ and z is an $(n - 1)$-tuple. This is, in fact, an abbreviation; written in full, the definition means, e.g. that x is a 5-tuple if and only if there exist s, t, u, v, w such that $x = \langle s, t, u, v, w \rangle$. Thus the expression defining the property of being an n-tuple becomes more involved with increasing n, and, as pointed out above, no mathematical induction on n is possible without the introduction of new concepts (or a mathematical induction within the logical system to which the expression "x is an n-tuple" belongs; however, this is a matter of mathematical logic and lies outside the scope of the present book).

Similarly we may define e.g. n-tuples of classes, $2 < n \leqq k$, as follows: x is a 2-tuple of classes if and only if $x = \langle y, z \rangle$, y, z being classes; x is a n-tuple of classes if and only if there exist y, z such that $x = \langle y, z \rangle$, z is an $(n - 1)$-tuple of classes, y is a class.

With k fixed once and for all, we may now give e.g. the following definition: x is a "regular $(\leqq k)$-multiplet of classes" if it is an n-tuple for some $n \leqq k$; for convenience we simply speak of a "regular multiplet of classes" instead of a $(\leqq k)$-multiplet.

Written in full, the definition would be as follows: x is a regular multiplet of classes, if either x is a pair of classes, or there exist classes v, y, z such that $x = \langle v, y, z \rangle$ or there exist classes u, v, y, z such that $x = \langle u, v, y, z \rangle$ or ... or there exist classes $u_1, \ldots, u_k$ such that $x = \langle u_1, \ldots, u_k \rangle$.

We shall now present the definitions of multiplets, deleting and enriching (see Section 7), etc.; explanations such as given above will now be formulated in a concise form.

Again, an actually given natural number k is conceived as fixed. We define n-multiplets and n-multiplets of classes (for $n \leqq k$), as well as $(\leqq k)$-multiplets of classes.

Every pair (of classes) is called a 2-multiplet (of classes). If $2 < n \leqq k$, then x is called an n-multiplet (respectively, as n-multiplet of classes) if there exist y, z and a natural number m, $1 \leqq m < n$, such that $x = \langle y, z \rangle$ and (i) if $m = 1$, then y is arbitrary (respectively, y is a class) and z is an $(n-1)$-multiplet (an $(n - 1)$-multiplet of classes), (ii) if $2 \leqq m \leqq n - 2$, then y is an m-multiplet (an m-multiplet of classes), z is an $(n - m)$-multiplet (an $(n - m)$-multiplet of classes), and (iii) if $m = n - 1$, then z is arbitrary (z is an arbitrary class), y is an $(n - 1)$-multiplet (an $(n - 1)$-multiplet of classes).

A $(\leqq k)$-multiplet (or simply a multiplet) of classes may be defined as follows: x is a $(\leqq k)$-multiplet of classes if it is an n-multiplet of classes for some n, $2 \leqq n \leqq k$.

As for deleting of objects, etc., the reader is requested to go through the pertinent part of Section 7; some observations made there are repeated below.

If we try to give exact definitions of deleting of objects, etc., for the non-comprisable case, we encounter serious difficulties. Namely, the definition of deleting of elements from a structure, or, conversely, of enriching of a structure, must necessarily be an "inductive" one. Such a definition can be given for comprisable structs and structures. However, for non-comprisable structs, e.g. for structs $\langle X, \alpha \rangle$ where neither X nor α are comprisable, the procedure described in Section 3 does not apply since it involves quite essentially the use of certain finite sequences which do not exist in the "non-comprisable case" since there are no "sequences of non-comprisable objects".

We must, however, bear in mind that the concept of an underlying struct and so on serve certain "practical" purposes. In many individual cases, structs are considered which are formed from a small number of objects which cannot or need not be represented as pairs any more; there occurs, "in practice", a rather small number of different kinds of structs. Therefore, it will be adequate to define the enriching of structures, the deleting of objects from a structure, etc., only for the case where it can be achieved by at most k elementary steps, k being a fixed natural number. Of course, k being a fixed actually given number, it is then possible to define the enrichment, etc., of structures achieved in k steps at most. This will now be done.

If $\zeta = \langle x, y \rangle$, ζ' is an "immediate extension (under formation of pairs)" of ζ if there exists a z such that $\zeta' = \langle x', y' \rangle$ and either $x' = \langle z, x \rangle$, $y' = \langle z, y \rangle$ or $x' = \langle x, z \rangle$, $y' = \langle y, z \rangle$.

We shall say that $\zeta^* = \langle x^*, y^* \rangle$ is a "($\leqq k$)- extension (relative to the formation of pairs)", or simply, since k is fixed, an "extension (relative to the formation of pairs)" of $\zeta = \langle x, y \rangle$ if there exist pairs $\zeta_1, \zeta_2, \ldots, \zeta_{k-1}$ such that the following holds: either ζ_1 is an immediate extension of ζ or $\zeta_1 = \zeta$; either ζ_2 is an immediate extension of ζ_1 or $\zeta_2 = \zeta_1$; ...; either ζ^* is an immediate extension of ζ_{k-1} or $\zeta^* = \zeta_{k-1}$.

We shall say that ξ is obtained from η by deleting α if either ξ, η, α are elements and $\langle \xi, \eta \rangle \in \vartheta_\alpha$ where ϑ_α is the relation described in Section 3, or $\langle \xi, \eta \rangle$ is an extension (relative to the formation of pairs) of some $\langle x, \langle x, \alpha \rangle \rangle$ or $\langle x, \langle \alpha, x \rangle \rangle$. If $\alpha, \beta, \ldots, \vartheta$ are actually given, we shall say that ξ is obtained from η by deleting $\alpha, \beta, \ldots, \vartheta$ if ξ is obtained from η by succesively deleting (in the above sense) $\alpha, \beta, \ldots, \vartheta$. Then we shall also say that η is obtained by enriching ξ with $\vartheta, \ldots \ldots, \beta, \alpha$.

UNDECIDABILITY OF SOME SIMPLE ASSERTIONS

In 15 B.12, the assertion that $(\mathsf{N} \times \mathsf{N}) \cap \mathsf{N} = \emptyset$ occurred as an added assumption (and consistently in the following text). This might seem surprising, and some explanation is probably appropriate.

It has been remarked at several points that nothing else can be asserted about objects such as the natural numbers, than follows from the defining axioms (e.g. cf. 3 D.1, remark). However, many simple assertions about the natural numbers are independent of the axioms; one of these is the assertion that $\mathsf{N} \times \mathsf{N}$ and N are disjoint.

Indeed, first let ϱ be the relation $\{x \to \langle \emptyset, x \rangle\}$; denote $\emptyset$ by 0, let N be the least ϱ-saturated set containing 0, and let $s = \varrho_N$. Then $N, 0, s$ satisfy the Peano axioms (cf. 3 C.4), and both $\langle 0, 0 \rangle = s\,0 \in N$ and $\langle 0, 0 \rangle \in N \times N$.

Secondly, let N^* consist of all (x) with $x \in N$; let $0^* = (0) = (\emptyset)$, and let s^* consist of all $\langle (x), (y) \rangle$ with $x \in N$, $y = sx$. Then, evidently, $N^*, 0^*, s^*$ satisfy the Peano axioms and $(N^* \times N^*) \cap N^* = \emptyset$.

For this reason $(\mathsf{N} \times \mathsf{N}) \cap \mathsf{N} = \emptyset$ is indeed an assumption and not a generally valid property. A logically less offensive procedure would be to speak not about N but rather some other set (e.g. the set of all (n) with $n \in \mathsf{N}$); however, the more current formulation has been preferred here.

Similar situations arise at several other points. Having the above remarks in mind, the reader will find no difficulty in finding the necessary more precise formulations.

ON GENERAL CONTINUITY STRUCTURES

In the introductory remarks to Chapter III, we have mentioned continuity structures. Three types of these, namely closure structures, proximities and uniformities, have been studied extensively in Chapters III—VII. In fact, there are other continuity structures that been investigated by various authors. However, with the exception of "syntopogene spaces" (A. Csaszár), there does not exist a systematic theory of any type of such structures, although their importance in mathematics may be no less than that of topological spaces. Therefore, we give a brief account of a possible approach to the examination of general continuity structures (for a somewhat more detailed exposition and also some references see M. Katětov, Allgemeine Stetigkeitsstrukturen, Proc. Int. Congr. Math. 1962, pp. 473—479). Let us try to express the intuitive idea of continuity or of a continuous structure in a more or less exact manner. It seems natural that such a structure is given if it is known, for certain "variable objects" $x(t)$, $y(t)$ whether $y(t)$ approximates

$x(t)$ arbitrarily well or not. Thus, a closure structure on a set P is determined as soon as it is known, for every $x \in P$ and every $M \subset P$, whether or not x may be indefinitely approximated by points of P. A semi-uniformity on P is given as soon as it is known, for any set T of pairs $\langle x, y \rangle$, $x \in P$, $y \in P$, whether T contains "arbitrarily small" pairs $\langle x, y \rangle$, i.e. pairs $\langle x, y \rangle$ with y arbitrarily close to x. Let us, however, point out that those objects the approximation of which is considered need not be elements of the set P in question; we may consider e.g. the approximation of subsets (of P) by subsets.

Clearly, the idea of the possibility of an "arbitrarily close approximation" is connected with that of "arbitrarily small objects"; if the objects $x(t)$, t variable, admit of an arbitrarily close approximation by certain $y(t)$, then we may say that $\langle x(t), y(t) \rangle$ may become arbitrarily small. Thus it seems intuitively appropriate to conceive the idea of a general continuity structure as based on that of "arbitrarily small objects". In this sense, it seems fairly reasonable to consider general topology, roughly speaking, as a general theory of infinitesimal quantities.

We give now some exact definitions. Let Z be a set and let $\Gamma \subset \exp Z$ be such that (1) if $S \in \Gamma$, $S \subset S_1 \subset Z$, then $S_1 \in \Gamma$; (2) $S_1 \cup S_2 \in \Gamma$ implies that $S_1 \in \Gamma$ or $S_2 \in \Gamma$, (3) $\emptyset \notin \Gamma$. Then Γ will be called a J-structure on Z (observe that conditions (1)—(3) seem to be intuitively indispensable properties of the collection of all sets containing "arbitrarily small elements"). The J-morphisms are defined as mappings $f: \langle Z, \Gamma \rangle \to \langle Z', \Gamma' \rangle$, Γ' being a J-structure on Z', such that $S \in \Gamma$ implies $f[S] \in \Gamma'$.

Now let $\mathscr{M}$ denote the category of all sets (see **13 B.1**) and let Φ be a covariant functor on $\mathscr{M}$ into $\mathscr{M}$. If X is a set, then any J-structure Γ on the set ΦX assigned to X under Φ will be called a Φ-structure on X, and $\langle X, \Gamma \rangle$ will be called a Φ-space. A mapping $F: \langle X, \Gamma \rangle \to \langle X', \Gamma' \rangle$ will be called a Φ-morphism or a Φ-continuous mapping if $\Phi f: \langle \Phi X, \Gamma \rangle \to \langle \Phi X', \Gamma' \rangle$ is a J-morphism.

Let Q denote the functor on $\mathscr{M}$ into $\mathscr{M}$ under which every set X is assigned its "square" $X \times X$. Let us consider Q-structures Γ on a set P such that, for every $x \in P$, $\langle x, x \rangle \in \Gamma$. It is not difficult to show that closure spaces, proximity and uniform spaces may be considered as special cases of Q-spaces. Other important, though scarcely investigated (for some results and references see e.g. M. Katětov, On continuity structures and spaces of mappings, Comm. Math. Univ. Carol. *6* (1965), 257—278) types of spaces are obtained if we consider the functor P which assigns to a mapping $f: X \to Y$ the mapping of $\exp X$ into $\exp Y$ which transforms $M \subset X$ into $f[M]$ (under this functor, a set X is assigned its exponential $\exp X$). There does not exist any systematic theory of Φ-spaces as yet. However, some concepts can be given an adequate definition without difficulty. Thus, if X is a set and f_a are mappings of certain Φ-spaces Y_a into X, then there is a (unique) finest Φ-structure on X under which all f_a are Φ-continuous; this structure is said to be inductively generated by the mappings f_a. In a similar manner, a projectively generated Φ-structure may be defined. In this way we also obtain, for Φ-spaces, a general definition of a subspace and a quotient, of the sum and product of spaces and so on (of course, it is necessary to examine, for each type of space, to what extent these definitions are in agreement with the intuitive ideas).

We conclude with a mention of two further types of Φ-structures: first, A. Csaszár's syntopogene spaces are equivalent, in a sense which can be easily specified, with a certain kind of $(Q \circ P)$-spaces; secondly, if Λ is the functor under which every set X is assigned the module ΛX of (finite) formal linear combinations $\Sigma \lambda_i x_i$ where λ_i are real, $x_i \in X$, then the Λ-spaces provide a type of continuity structure which is rather different from those indicated above and possesses various interesting connections with certain problems of functional analysis.

CATEGORIAL CHARACTER OF SOME NOTIONS AND RESULTS

Many concepts, such as e.g. product, projective limit, projective generating family, topological modification, which have been defined in each situation independently of each other, may be defined in terms of category theory. Here several basic definitions are introduced and fundamental results are sketched. As the formulations of earlier definitions and theorems, and also their proofs, have been arranged to point out their categorial character, the illustrations are suppressed to some of those which show further appearance of the definitions introduced. It should be noted that all counter-examples needed can be constructed on the basis of the material introduced earlier. The reader is, however, invited to confront the definitions and results of this Note with the corresponding definitions and results made earlier for the subcategories of the category **C** of all closure spaces, the category **P** of all proximity spaces, and the category **U** of all semi-uniform spaces. Of course, e.g. the category **C** of all closure spaces has the continuous mappings as morphisms, the closure spaces as objects, and the categorial composition is the composition of mappings.

For convenience, given a category $\mathscr{K}$, obj $\mathscr{K}$ denotes the class of all objects of $\mathscr{K}$; $f \in \mathscr{K}$ means that $f \in |\mathscr{K}|$, i.e. that f is a morphism of $\mathscr{K}$, and the composition of $\mathscr{K}$ is denoted by $\circ$. The units will also be called the identity morphisms. The symbol **Ens** denotes the category of all sets and their mappings.

By a forgetful functor of a category $\mathscr{K}_1$ of structs into another one $\mathscr{K}_2$ we mean a functor $\mathscr{F}$ which assigns to each morphism f the transposed mapping $\mathscr{F}f$ such that the domain and range carriers of $\mathscr{F}f$ are the underlying structs of the domain and range carriers of f; e.g.: the forgetful functor of **C** into **Ens** assigns to each continuous mapping the underlying mapping of the underlying sets.

Products, sums and limits

Let $\mathscr{K}$ be a category. A presheaf in $\mathscr{K}$ over a quasi-ordered set $\langle A, \leqq \rangle$ is a pair $\mathscr{S} = \langle \{K_a \mid a \in A\}, \{k_{ab} \mid a \leqq b\} \rangle$ such that $k_{ab} : K_a \to K_b \in \mathscr{K}$, all k_{aa} are units, and $k_{bc} \circ k_{ab} = k_{ac}$ for each $a \leqq b \leqq c$. A family $\{k_a : K \to K_a\}$ is compatible with $\mathscr{S}$ if $k_b = k_{ab} \circ k_a$ for each $a \leqq b$. We shall say that $\kappa = \langle K, \{k_a\} \rangle$ is a projective limit of $\mathscr{S}$, and write $\kappa = \varprojlim \mathscr{S}$ or $\kappa = \lim \operatorname{proj} \mathscr{S}$, if $\{k_a\}$ is compatible with $\mathscr{S}$, and for any $\{l_a : L \to K_a\}$ compatible with $\mathscr{S}$ there exists a unique $k : L \to K$ with $l_a = k_a \circ k$ for each a. If $\{K_a \mid a \in A\}$ is a family in obj $\mathscr{K}$, then $\mathscr{S} = \langle \{K_a\}, \{e_a \mid a \in A\} \rangle$ is a presheaf in K, and any projective limit $\kappa = \langle K, \{k_a\} \rangle$ of $\mathscr{S}$ is called a product of $\{K_a\}$; we write $\kappa = \Pi\{K_a\}$, and K is called the product of $\{K_a\}$ under $\{k_a\}$. The reader is invited to define the concept of the inductive limit of a presheaf and of the sum of a family of objects. If $\langle K, \{k_a\} \rangle = \lim \operatorname{proj} \mathscr{S}$, then K is also called the projective limit of $\mathscr{S}$ and similarly for products, inductive limits and sums.

A projective limit of a presheaf, in particular a product of a family of objects, is determined up to an isomorphism in K; more precisely, if

$$\langle L, \{l_a\} \rangle = \lim \operatorname{proj} \mathscr{S} = \langle K, \{k_a\} \rangle ,$$

then there exists a unique isomorphism $i : L \to K$ with $l_a = k_a \circ i$ for each a. Therefore it is sometimes convenient to select one limit in the class of all projective limits of any presheaf $\mathscr{S}$, which is then defined to be the projective limit of $\mathscr{S}$, and similarly for products. E.g. $\Pi\{\mathscr{P}_a\}$, as introduced earlier, is one of the products of a family $\{\mathscr{P}_a\}$ of sets, closure spaces, etc. Of course, $\Pi\{\mathscr{P}_a\}$ is the product under the family of all projections.

A category is said to be projective-limit-admitting if any presheaf in $\mathscr{K}$ has a projective limit. The terms product-admitting (or also completely productive), inductive-limit-admitting, sum-admitting (also completely summative) and limit-admitting are evident.

Generation

In what follows let $\mathscr{F}$ be a fixed covariant functor of a category $\mathscr{K}$ into a category $\mathscr{M}$. A morphism $k \in \mathscr{K}$ will be termed an $\mathscr{F}$-unit or also an $\mathscr{F}$-identity if $\mathscr{F}k$ is a unit in $\mathscr{M}$. The class of all $\mathscr{F}$-units forms a subcategory $\mathscr{B}$ of $\mathscr{K}$. The relation $\mathbf{E}\{\langle \mathrm{D}k, \mathrm{E}k\rangle \mid k \in \mathscr{B}\}$, denoted by $\leqq_{\mathscr{F}}$ or simply $\leqq$, is a quasi-order on obj $\mathscr{K}$. To avoid unnecessary difficulties, the functor $\mathscr{F}$ is assumed to be amnestic, i.e., by definition,

(*) $k_i : K \to K_1$, $i = 1, 2$, $\mathscr{F}k_1 = \mathscr{F}k_2 \Rightarrow k_1 = k_2$, and

(**) $\leqq$ is an order.

It should be noted that a functor $\mathscr{F}$ satisfying (*) is called faithful. It follows from (*) that a morphism $k \in \mathscr{K}$ is uniquely determined by the following three data: $\mathscr{F}k$, $\mathrm{D}k$ and $\mathrm{E}k$. Therefore the following convention may be introduced: If $m = \mathscr{F}k$, then the symbol $m : \mathrm{D}k \to \mathrm{E}k$ stands for k. For convenience a triple $\langle m, K, K_1 \rangle$ with $m : \mathscr{F}K \to \mathscr{F}K_1 \in \mathscr{M}$, denoted by $m : K \to K_1$, will be called an $\mathscr{F}$-pseudomorphism.

It follows from (*) that any $\mathscr{F}$-identity is a bimorphism.

It should be noted that the forgetful functors which have been considered, e.g. of **C**, **P** or **U** into the category of sets, are amnestic.

Definition. A family $\{k_a : K \to K_a\}$ in $\mathscr{K}$ is termed an $\mathscr{F}$-projective generating family if $K' \leqq K$ for each $K' \in$ obj $\mathscr{K}$ such that $\mathscr{F}k_a : K' \to K_a \in \mathscr{K}$ for each a. $\mathscr{F}$ is termed projectively generative if, for each family $\{m_a : M \to M_a\}$ in $\mathscr{M}$ and any $K_a \in$ obj $\mathscr{K}$ with $\mathscr{F}K = M_a$, there exists a $K \in$ obj $\mathscr{K}$ such that $\{m_a : K \to K_a\}$ is a projective generating family. The reader is invited to formulate the definitions of the following concepts: $\mathscr{F}$-projective progeny of a subclass of obj $\mathscr{K}$, $\mathscr{F}$-inductive generating family, $\mathscr{F}$-inductive progeny of a subclass of obj $\mathscr{K}$, inductively generative $\mathscr{F}$. The functor $\mathscr{F}$ is termed generative if $\mathscr{F}$ is both projectively and inductively generative.

It should be remarked that $\mathscr{B}$ only enters into the definition of projective and inductive generating families; therefore these concepts may be introduced for $\mathscr{K}$ and a given subcategory $\mathscr{B}$ of $\mathscr{K}$ (subject to certain condition).

The indicated manner of the introducing of these concepts may be the starting point of an investigation of categories of continuous structures. Intuitively, $\mathscr{B}$ is a sufficiently large class of bijective "continuous" mappings.

The interpretation of generation adopted here leads naturally to a more restrictive definition as follows: $\{k_a : K \to K_a\}$ is an $\mathscr{F}$-projectively generating family in the strong sense if the following condition is fulfilled:

If $K' \in$ obj $\mathscr{K}$, $m : \mathscr{F}K' \to \mathscr{F}K \in \mathscr{M}$ and if $\mathscr{F}k_a \circ m : K' \to K_a$ is a morphism for each a, then $m : K' \to K$ is a morphism.

A description of some interrelations between generation and generation in the strong sense will be preceded by two theorems illustrating the importance of the latter concept.

Theorem on strong associativity. Suppose that $\{k_a : K \to K_a\}$ is a family in $\mathscr{K}$ such that each K_a is projectively generated in the strong sense by a family $\{k_{ab} : K_a \to K_{ab}\}$. Then $\{k_a\}$ is a projective generating family (in the strong sense) if and only if the family $\{k_{ab} \circ k_a\}$ is projectively generating (in the strong sense, respectively).

Theorem on construction of products. Assume that $\{K_a\}$ is a family of objects of $\mathscr{K}$ and M is the product of $\{\mathscr{F}K_a\}$ (in $\mathscr{K}$) under $\{m_a\}$. Then K is the product of $\{K_a\}$ (in $\mathscr{K}$) under $\{m_a : K \to K_a\}$ if and only if $\{m_a : K \to K_a\}$ is a projective generating family in the strong sense.

Corollary. If $\mathscr{M}$ is product-admitting and $\mathscr{F}$ is projectively generative in the strong sense, then $\mathscr{K}$ is product-admitting and $\mathscr{F}$ is product-preserving (the definition is obvious).

It should be remarked that, without any assumption on the amnestic $\mathscr{F}$, one can prove that if K is the product of $\{K_a\}$ (in $\mathscr{K}$) under $\{k_a\}$ then there exists no $K' \geqq K$, $K' \neq K$, such that all $\mathscr{F}k_a : K' \to K_a$ are morphisms; thus K is a maximal object with the property that all the above pseudomorphisms are morphisms. On the other hand, $\{k_a\}$ need not be a projective generating family.

For most of the forgetful functors considered earlier, the generation and the generation in the strong sense coincide. All these results can be obtained from the following result. We shall say that $\mathscr{F}$ has the factorization property if for any $k \in \mathscr{K}$ and any factorization $\mathscr{F}k = m_2 \circ m_1$ there exist k_i with $k = k_2 \circ k_1$ and $\mathscr{F}k_i = m_i$.

Theorem. Projective generation and projective generation in the strong sense coincide provided one of the following two conditions is fulfilled:

(a) $\mathscr{F}$ is projectively generative in the strong sense.

(b) $\mathscr{F}$ is inductively generative and has the factorization property.

The first statement is evident; to prove the second, assume that $\{k_a : K \to K_a\}$ is a projectively generating family and $m : \mathscr{F}K' \to \mathscr{F}K$ is a morphism such that all $\mathscr{F}k_a \circ m : K \to K_a$ are morphisms. Consider the inductively generating mapping $m : K' \to L$, and factorize each $\mathscr{F}k_a \circ m :$ $: K' \to K_a$.

It is an unsolved problem to define the concept of a subobject (corresponding to the notion of a subspace of a closure space, etc.) and a quotient of an object. It is clear that a subobject of K is necessarily a monomorphism of K; on the other hand, this requirement does not guarantee all that is needed.

It turns out, however, that having defined subobjects in the category $\mathscr{M}$, the subobjects in $\mathscr{K}$ (with respect to $\mathscr{F}$) may usefully be defined as the projective generating mappings k (preferably in the strong sense) such that $\mathscr{F}k$ are subobjects, and similarly for quotients.

Modification and reflections

The concepts such as e.g. topological modification, sequential modification or uniformizable modification were introduced by means of the order $\leqq_{\mathscr{F}}$ which is determined by the corresponding forgetful functor $\mathscr{F}$. In general, given $\mathscr{F} : \mathscr{K} \to \mathscr{M}$ as in the preceding Note, and given a subcategory $\mathscr{L}$ of $\mathscr{K}$, the upper (lower) $\mathscr{F}$-modification of a $K \in \operatorname{obj} \mathscr{K}$ in $\mathscr{L}$ is defined as the upper (lower) modification of K in the subclass $\operatorname{obj} \mathscr{L}$ of the ordered class $\langle \operatorname{obj} \mathscr{K}, \leqq_{\mathscr{F}} \rangle$; then these concepts depend essentially on $\mathscr{F}$. A closely related notion will be introduced.

Definition. Let $\mathscr{L}$ be a subcategory of $\mathscr{K}$. A reflection of a $K \in \operatorname{obj} \mathscr{K}$ into $\mathscr{L}$ is a pair $\langle L, t \rangle$ such that $t : K \to L$, and that any $f : K \to L_1$, $L_1 \in \operatorname{obj} \mathscr{L}$, admits a unique factorization $f = g \circ t$ with g in $\mathscr{L}$; L is called a reflection of K (under t). Similarly, a coreflection of K in $\mathscr{L}$ is a pair $\langle L, t \rangle$ such that $t : L \to K$, $L \in \operatorname{obj} \mathscr{L}$ and that any $f : L_1 \to K$, $L_1 \in \operatorname{obj} \mathscr{L}$, admits a unique factorization $f = t \circ g$ with g in $\mathscr{L}$.

Theorem. Let $\mathscr{F} : \mathscr{K} \to \mathscr{M}$ and $\mathscr{L}$ be as above. If $\langle L, t \rangle\rangle$ is a reflection of K in $\mathscr{L}$ and t is an $\mathscr{F}$-identity, then L is the upper $\mathscr{F}$-modification of K in $\mathscr{L}$. Conversely, if L is the upper modification of K in $\mathscr{L}$, t is the $\mathscr{F}$-identity morphism of K into L, then $\langle L, t \rangle$ is a reflection of K in $\mathscr{L}$ provided that either of the following two conditions is fulfilled:

(a) If $k : K \to K_1 \in \mathscr{K}$, $L \in \operatorname{obj} \mathscr{L}$, then there exists an $L_2 \in \operatorname{obj} \mathscr{L}$ such that $L_2 \geqq K$ and $\mathscr{F}k : L_2 \to L_1$ is a morphism.

(b) If $m : M \to M_1$, $\mathscr{F}L_1 = M_1$ and $L_1 \in \operatorname{obj} \mathscr{L}$, then there exists a projectively generating mapping $m : L_2 \to L_1$ with L_2 in $\operatorname{obj} \mathscr{L}$.

Examples. Let U be the category of all separated uniform spaces and let U_0 be the subcategory of all complete spaces. If $\mathscr{P}_0 \in \operatorname{obj} U_0$ is a completion of an $\mathscr{P} \in \operatorname{obj} U$, then $\langle \mathscr{P}_0, \mathrm{J} : \mathscr{P} \to \mathscr{P}_0 \rangle$ is a reflection of $\mathscr{P}$ in U_0. Similarly $\langle \beta\mathscr{P}, \mathrm{J} : \mathscr{P} \to \beta\mathscr{P} \rangle$ is a reflection of the object $\mathscr{P}$ from the category of all separated uniformizable spaces in the subcategory of all compact spaces.

The reader is invited to introduce the concepts of projectively and inductively generating families in a subcategory $\mathscr{L}$ of $\mathscr{K}$, and to carry over the results of 33B and 35D.

In conclusion, the basic properties of the reflection are given. If both $\langle L_1, t_1 \rangle$ and $\langle L_2, t_2 \rangle$ are reflections of a K in $\mathscr{L}$, then there exists an isomorphism i with $t_2 = i \circ t_1$; thus a reflection is determined up to an isomorphism. If $L \in \operatorname{obj} \mathscr{L}$, then $\langle L, e_L \rangle$ is a reflection of L in $\mathscr{L}$. Assume that $\mathscr{L}$ is reflective, i.e., each $K \in \operatorname{obj} \mathscr{K}$ has a reflection $\langle L_K, t_K \rangle$ in $\mathscr{L}$; we may assume that t_K is a unit of $K \in \operatorname{obj} \mathscr{L}$. For each $f : K \to K_1 \in \mathscr{K}$ there exists a unique $\mathscr{G}f$ in $\mathscr{L}$ with $t_{K_1} \circ f = \mathscr{G}f \circ t_K$, and $\mathscr{G}$ is clearly a covariant functor of $\mathscr{K}$ into $\mathscr{L}$ with $\mathscr{G} \circ \mathscr{G} = \mathscr{G}$.

T h e o r e m. Let $\mathscr{L}$ be reflective in $\mathscr{K}$ and $\mathscr{G}$ be as above. If L is a product of $\{L_a\}$ in $\mathscr{L}$, then L is a product of $\{L_a\}$ in $\mathscr{K}$. If K is a product of $\{L_a\}$ in K with $L_a \in \operatorname{obj} \mathscr{L}$, and if $\langle L, t \rangle$ is a reflection of K in $\mathscr{L}$, then t is an isomorphism and L is a product of $\{L_a\}$ in $\mathscr{L}$. Consequently, if $\mathscr{K}$ is completely productive, or equivalently, product-admitting, then so is $\mathscr{L}$, $\mathscr{L}$ is product-preserving and, up to an isomorphism, product-stable.