Historický vývoj geometrických transformací

English summary

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This monograph deals with the history of geometric transformations from ages up to the beginning of the 20th century. On the basis of original sources it refers to the first appearance and subsequent advancement of particular geometric transformations. It focuses especially on the following types of transformations: isometries, similarities, geometric motions, axial affinities, central dilatations, circular inversions, projections, stereographic projections, affine transformations, projective transformations and Cremona transformations. The monograph gives further details about those significant moments from the history of geometry when some new ideas in the context of geometric transformations arose for the first time. On the background of the world's development it follows reflections and the influence on investigated mathematical problems in the Czech lands as well. In this context selected parts of Czech textbooks on geometry following for the first time a new approach to some themes are commented in more detail.

The history of geometric transformations is very extensive, it has its roots in ancient times more than 2 500 years ago. Particular types of transformations were often found in the connection with solving some practical problems and were used not only in geometry, but also in other branches of human action. As an example we can mention application of the perspective in painting, using of a projection in civil engineering or construction of maps in cartography based on the stereographic projection.

Geometric transformations were at first regarded only as a relation between two geometric figures in a plane or space. Around the 18th century mathematicians began to reflect upon the process of transformation itself and wrote comprehensive works devoted to particular types of transformations. In this way the fundamentals and principles of a general theory were set down. In the 19th century mathematicians began to consider the possibility of classification of all known types of transformations. The first significant contribution was made by August Ferdinand Möbius (1790–1868) in his Barycentric Calculus and then this classification was completed by Felix Klein (1849–1925) in his Erlanger Programm.

Historical overview of geometric transformations

An isometry is the oldest example of geometric transformation. Ancient Greeks considered two geometric figures to be congruent if there existed a motion that carried a figure on the other one so that they coincided. Thalés of Miletos (7–6th century B.C.) used similar triangles to estimate for example the height of a pyramid. Euclid of Alexandria (about 325–265 B.C.) introduced in the first book of his *Elements* three elementary theorems on congruent triangles. Thalés referred to congruent figures as similar ones. We presume that the term *equal* for congruent figures was introduced by the Pythagoreans who thought that such figures consisted of equal numbers of points. Euclid and his followers called congruent figures *similar and equal*. Later the term *similar figures* acquired its more general meaning.

Geometric motions were used by the Pythagoreans quite often and systematically. For example they regarded a line as the trace of a moving point and a surface as the trace of a moving line. Euclid of Alexandria in his *Elements* set down those definitions of the sphere, the cone and the cylinder that involve a motion. On the other hand, Aristotle of Stageira (384–322 B.C.) condemned the use of motions in geometry. He regarded mathematical objects only as some abstractions of physical objects being therefore without any movement. According to his opinion, nothing that is continuous can be composed of indivisibles. It follows that one can not obtain a line by moving a point, a surface by moving a line, and a solid by moving a surface.

An axial affinity seems to be mentioned for the first time in Archimedes' treatise *On conoids and spheroids* dealing with the computation of the volumes of hyperboloids and paraboloids of rotation (*conoids*) and ellipsoids of rotation (*spheroids*). One proposition of this treatise states that the area of any ellipse is to that of the circle constructed upon its major axis as the minor axis to the major one. Archimedes in his proof used a right axial affinity.

The first appearance of the term *central dilatation* (homothety) can be found in Euclid's *Elements*. This transformation is implicitly involved in the solution of one geometric problem concerning a relation between two similar figures. A central dilatation was mentioned also by Apollonius of Perga (262–190 B.C.) in his treatise *On plane loci*. He was probably acquainted with basic properties of this transformation, but detailed information has not been preserved.

A circular inversion was considered for the first time by Apollonius in his treatise *On plane loci*. Later, general inversion in any regular conic section, i.e. not only the inversion in a circle but also the inversion in an ellipse, a hyperbola or a parabola, can be found in his subsequent treatise called *Treatise on conic sections* (Kōnika). In modern times many famous mathematicians, namely Jacob Steiner (1796–1863), Giusto Bellavitis (1803–1880) and August Ferdinand Möbius (1790–1868), made significant contributions to the theory of circular inversion.

A projection was extensively used in ancient Greece and Rome. Marcus Vitruvius Pollio, the famous Roman architect of the first century B.C., in his *Ten books on architecture* (De architectura libri decem) described three types of projections frequently used by architects; they were called *ichnography*, *orthography* and *scenography*.

A stereographic projection is one of the most important projections which had its application already in ancient times. Claudius Ptolemy (about 85–165) in his treatise *Representation of the sphere in the plane* (Aplōsis epiphaneias sphairas) described the projection of the celestial sphere on the equilateral plane. His another treatise *On projection* (Peri analēmmatos) deals with the orthogonal projection of the celestial sphere on the horizontal plane which served to solve various problems of spherical astronomy. This projection have been still used to project the surface of the Earth onto a plane, that means for making maps in cartography. Angles between curves are invariant under a stereographic projection and therefore such maps are especially useful for seamen.

Both a central dilatation and an axial affinity are only special examples of affine transformations, the most general one-to-one transformations of the plane which carry straight lines again onto straight lines. They preserve parallelism of straight lines. Systematic treatment of affine transformations was made by Leonhard Euler (1707–1783). In 1748, he used for the first time the term *affine* which referred to the fact that although geometric figure and its affine image are not strictly similar, they are nevertheless related.

Projective transformations are still more general than affine ones. To define a projective transformation of a plane we must add to the plane points at infinity which serve as points of intersection of parallel lines. This step arises from the requirement to preserve one-to-one character of such correspondence between points. The concept of a point at infinity was introduced by Johannes Kepler (1571–1630) in his treatise *Astronomiae pars optica* in 1604. In the chapter on conics, he defined the foci of conics being those points such that straight lines drawn from these points to the point of tangency of the tangent to the conic form with it equal angles. In the case of a parabola, he placed its second focus into a point at infinity. Projective transformations were subsequently systematically investigated by Girard Desargues (1591–1661). He used the term *involution* to denote such transformation that coincides with its inverse. He was the first mathematician to consider polar transformations relative to a conic. Girard Desargues also proved the theorem on homologous triangles which is today known as Desargues' theorem.

Barycentric Calculus

August Ferdinand Möbius was one of the leading German mathematicians of the 19th century. Together with Julius Plücker (1801–1868) he was the prominent representative of algebraic projective geometry. He developed the barycentric calculus containing the first important example of homogeneous coordinates and made significant contribution to the classification of geometries. He is also remembered as one of the founders of topology.

August Ferdinand Möbius (1790–1868) entered the University of Leipzig in 1809 to study mathematics, physics and astronomy. For one year he studied theoretical astronomy at the University of Göttingen. In 1816, he became an associate professor of astronomy and higher mechanics in Leipzig, but only in 1844, he was appointed as a full professor. In Leipzig, Möbius served more than fifty years as an astronomer and later as a director of the local astronomical observatory.

Möbius was the first mathematician who described affine and projective transformations by means of homogeneous coordinates in his work *Der barycentrische Calcul* [Barycentric Calculus] from 1827. Basic idea of Möbius' barycentric coordinates is the following. Consider any fixed triangle. Coordinates of an arbitrary point in the plane are represented by masses which must be placed in the vertices of the triangle so as this point is the centre of gravity of these masses. The analytical approach made it possible to characterize points at infinity and operate with imaginary points as well.

Möbius in his *Der barycentrische Calcul* introduced a general principle of geometric transformation including algebraic characterization of geometric transformations, he paid special attention to these types of transformations: *an isometry*, *a similarity*, *an affinity* and *a collineation*. Subsequently, he discussed relationships between particular types of transformations. Homogeneous coordinates enabled him to describe a great number of affine and projective properties of planar or spatial geometric figures. Möbius proved for instance that proportions of oriented lengths, areas and volumes remain invariant under affine transformations.

Cremona transformations

Luigi Cremona was a leading Italian mathematician. He had a great influence on Italian geometry being one of the founders of the new Italian mathematical school. He was mainly interested in projective and algebraic geometry and discovered graphical methods for solving problems in statics as well. Many propositions on synthetic geometry were revised and improved by him. Birational transformations of a projective plane or space, later known as Cremona transformations, have been still used for studying rational curves and surfaces.

Luigi Cremona (1830–1903) studied for a degree in civil engineering at the University of Pavia. During his study he took an active part in political events of the revolution of 1848 which led to the unification of Italy in 1859. He joined the Free Italy battalion and was fighting against the Austrian occupation. In 1857, Cremona was appointed as a professor at the grammar school in Pavia. He stayed there for three years and wrote a number of mathematical papers on curves using methods of projective geometry. In 1859, he began to teach at the Lyceum of Saint Alexander in Milan. Next year, he was appointed as a full professor at the University of Bologna and served there until 1867. During that time he published about 45 mathematical papers including his most important work on transformations of plane curves. He also developed the theory of birational transformations, later known as *Cremona transformations*.

In 1867, Cremona removed to the Polytechnical Institute of Milan where he received the Professor title in 1872. This period is thought to be the time of Cremona's gratest productivity. He wrote many articles on such diverse geometric topics as conic sections, plane curves, developable surfaces, third- and fourth-degree surfaces, projective geometry and also on graphical statics. In 1873, he became a director and a professor of graphical statics of the newly established Polytechnical School of Engineering in Rome. In 1877, Cremona was appointed to the Chair of higher mathematics at the University of Rome.

In the second half of the 19th century, mathematicians all over the world were interested in the study of geometric transformations. They especially drew their attention to higher degree transformations, the main stress was laid upon birational transformations. *A Cremona transformation* is defined as any birational transformation of a projective space over a field. The particular examples of Cremona transformations are *the circular inversion* or *quadratic birational transformations*.

Both the circular inversion and quadratic birational transformations are transformations of the second degree. In his treatise *Sulle trasformazioni geometriche delle figure piane* [Geometric Transformations of Plane Figures] published in 1863 and 1865, Cremona was engaged in the question on how to construct general geometric transformation of an arbitrary degree, i.e. how to construct a transformation which carries straight lines into curves of an arbitrary degree. He derived two fundamental equations which must be held for the number of points common to all those curves. Amongst the most important properties we should mention that under Cremona transformations neither the degree nor the class of algebraic curves remain invariant in general, but the genus of algebraic curves does. Cremona transformations of plane algebraic curves have been still used for the reduction of their singularities.

Luigi Cremona had a great influence on the development of projective geometry. His ideas had a world wide response and were reflected in many contributions. Some mathematicians tried to follow and deepen his results. Some of them, namely Arthur Cayley (1821–1895), Alfred Clebsch (1833–1872) and Rudolf Sturm (1841–1919), made a significant contribution to the theory of birational transformations.

In the second half of the 19th century several mathematicians in the Czech lands were also engaged in the problems of birational transformations. The most important one was certainly Emil Weyr (1848–1894) who gained the doctor's degree in philosophy at the University of Leipzig in 1869. In 1870, he was appointed as a private docent of modern geometry at the Prague University and next year he received the post of an associate professor of mathematics at the Czech Polytechnic in Prague. In 1875, he was invited as a full professor to the Vienna University in order to give there a lecture on modern geometry which was successfully developed in Prague.

At first, Emil Weyr was interested in one-to-one geometric transformations. In his extensive works from 1869 and 1870 he showed that besides bijective relationships expressing one-to-one projective transformation between two figures there exist also transformations under which more points of one figure correspond to only one point of the second figure. He pointed out that such transformations are in a close connection with the theory of algebraic curves and surfaces.

In the school year 1870/71 Emil Weyr spent a study stay in Milan during which he established a lifelong friendship and a cooperation with Italian geometers, especially with Luigi Cremona. He took part in his lectures and they discussed various geometric problems. Emil Weyr was the first Czech geometer to have understood the essential significance of Cremona's geometric work on birational transformations for further development of projective geometry. After his arrival back to Prague he translated two most important Cremona's books into Czech language. Together with his younger brother Eduard Weyr (1852–1903) he completed the first Czech mathematical textbook on projective geometry *Základové vyšší geometrie* [Foundations of Higher Geometry] which was published in three volumes since 1871. This textbook was written on the basis of French and German ones, it contains both elementary and special subject matter enriched with their own scientific results.

Erlanger Programm

The Erlanger Programm is the title of Klein's famous lecture presented at the University of Erlangen in October 1872. It brought his unified definition of a geometry based on the transformation groups. He showed that all geometries can be conveniently classified by means of group of geometric transformations and invariant theory. Klein's basic idea is that every geometry can be characterized by a group of transformations which preserve elementary properties of the given geometry.

Felix Klein (1849–1925) was one of the leading German mathematicians of the second half of the 19th century. In 1872, he was appointed as a full professor at the University of Erlangen. Three years later he moved to the Technische Hochschule in Munich. During the period from 1875 to 1880, he published about seventy papers which covered group theory, theory of algebraic equations and function theory, all from a characteristically geometric viewpoint. In 1880, Felix Klein was offered the new Chair of Geometry at the University of Leipzig. However, during the autumn 1882, he mentally collapsed and fell into a depression. His career as a top mathematician was over. He stayed in Leipzig until 1886 when he moved back to Göttingen where he established a world-known mathematical centre. Felix Klein was interested not only in geometry but also in group theory, theory of algebraic equations and function theory. At the University of Göttingen he lectured on various parts of mathematics and physics. In 1876, Felix Klein became the chief editor of the mathematical journal *Mathematische Annalen* founded in 1868. This journal specialized mainly in complex analysis, algebraic geometry and invariant theory.

The essence of the classification of various geometries consists in the following. As is well known, Euclidean geometry considers the properties of figures that do not change under any motions; equal figures are defined as those that can be transferred onto one another by a motion. But instead of motions one may choose any other collection of geometric transformations and declare as equal those figures that are obtained from one another by transformations from this collection. This approach leads to another geometry which studies the properties of figures that are invariant under such transformations.

The relation between two figures must really be an equivalence; this means that it is a reflexive, symmetrical and transitive relation. It follows that the set of geometric transformations must be closed with respect to the composition of transformations, it must include the identity and the inverse of every transformation must be involved as well. In other words, the set of transformations must form a group.

The theory that studies the properties of figures preserved under all transformations of any given group is called the geometry of this group. The choice of distinct transformation groups leads to distinct geometries. Thus, the analysis of the group of motions leads to the common Euclidean geometry. When the motions are replaced by affine or projective transformations, the result is the affine or the projective geometry. Felix Klein proved in his work that starting from projective transformations that carry a certain circle or any other regular conic into itself, one comes to the non-Euclidean geometry.

In Klein's revealing conception of geometry the theory of groups played an essential role. The term *group* arose in mathematics from the theory of algebraic equations of higher degree. Within the framework of investigation whether those equations are solvable or not, mathematicians began to operate with groups of permutations.

The geometry known in ancient times was called Euclidean geometry and it dominated mathematics for over 2 000 years. It had its roots in Euclid's *Elements* written about 300 B.C. In this remarkable book Euclid systematically arranged ancient mathematical results into a logical sequence and set up the basis of deductive approach to mathematics. The most important Euclid's contribution was the insistence that all mathematical results must be established deductively on the basis of explicit axioms and postulates.

Until the 19th century Euclidean geometry was the only type of geometry which was widely known and studied by many important mathematicians. By this time new mathematical methods for the study of synthetical geometry have been found and developed, e.g. analytic geometry, algebraic geometry, descriptive geometry, differential geometry, projective geometry, and so on.

The attempts over many centuries to prove the fifth Euclidean postulate led to the discovery of non-Euclidean geometries in the early 19th century. The famous question on the independence of this postulate was solved by Nikolai Ivanovich Lobatchevsky (1792–1856) and János Bolyai (1802–1860). Both Lobatchevsky and Bolyai considered the Euclidean parallel axiom to be independent of the other axioms of Euclidean geometry and developed the entirely new geometry based upon an alternative to Euclid's parallel axiom in which there exist two different straight lines going through a given point lying outside a given straight line having no common point with this straight line.

In October 1872, Felix Klein in his inaugural lecture *Vergleichende Betrachtungen über neuere geometrische Forschungen* [A Comparative Review of Recent Researches in Geometry] addressed at the Philosophical Faculty of the University of Erlangen presented his unified way of the classification of various geometries. It has later become known and famous as the Erlanger Programm.

The whole Erlanger Programm consists of ten chapters. Fundamental ideas of Klein's classification of various geometries are presented in the first chapter where the following definition of such a geometry is stated:

> *Have a geometric space and some transformation group. A geometry is the study of those properties of the given geometric space that remain invariant under the transformations from this group. In other words, every geometry is the invariant theory of the given transformation group.*

Felix Klein emphasizes that the transformation group can be an arbitrary group.

In the second chapter of the Erlanger Programm, Felix Klein defines an ordering of geometries in such a way that he transfers the inclusion relation among various transformation groups to the corresponding geometries. Replacing some transformation group by another transformation group in which the original group is involved, only a part of the former geometric properties remains invariant. The passage to a larger group or a subgroup of a transformation group makes it possible to pass from one type of geometry to another. In this way the Erlanger Programm codified a simple, but very important principle of ordering of particular geometries.

In the Erlanger Programm, Felix Klein set forth a unified conception of geometry that was far broader and more abstract than any one contemplated previously. It served to validate non-Euclidean geometries and define new geometries as well. Although it does not comprise some important branches of geometry, e.g. Riemannian geometry, it had a substantial stimulating effect on the subsequent development of geometry.

From 1908 till 1911, Felix Klein published his *Elementarmathematik vom höheren Standpunkte aus* [Elementary Mathematics from an Advanced Standpoint], the three-volumed textbook containing among others fundamental ideas of his Erlanger Programm.

He considered at first *special linear substitutions*, namely a translation, a rotation, a reflection in a point, a reflection in a line and similarity transformations. Geometry was subsequently defined as the invariant theory of these linear substitutions. He mentioned those geometric properties that remain invariant under such transformations; these properties are the fundamentals of metric geometry. In the following text, he derived another types of geometry, e.g. affine geometry, projective geometry, geometry of reciprocal radii and topology which he called *die Analysis situs*.

In the first half of the 20th century some mathematicians tried to follow Klein's results connecting with the classification of geometries. Partially successful efforts were made by Oswald Veblen (1880–1960) and Élie Cartan (1869–1951) to extend and generalize Klein's definition so as to include geometries that lie outside Klein's original Erlanger Programm.

Klein's definition of a geometry, which assigns the key role to the notion *geometric transformation*, has an interesting reflection in physics. It is namely connected with the problems how to formulate physical laws independently of coordinate system.

Classical mechanics is based upon the so-called Galilean principle of relativity which states that no mechanical experiment can differentiate between particular innertial systems. From this point of view we can say that all physical properties are preserved under those transformations of the physical system which impart a constant velocity to this system. These transformations are called Galilean transformations. In other words, the physical properties can be described as those properties which remain invariant under Galilean transformations. Since Galilean transformations can be easily shown to form a group, this description identifies the mechanics with a certain geometry of a three-dimensional space determined by the convenient choice of the group of transformations.

At the beginning of the 20th century, modern physics has replaced the Galilean principle of relativity by the so-called Einstein principle of relativity which is the basis of the special theory of relativity. In this case it is therefore necessary to replace Galilean transformations by more complicated Lorentz transformations which also form a group. Thus, when we pass from the classical mechanics of Galileo Galilei and Isaac Newton to Einstein's theory of relativity, we are actually changing our geometrical view of the surrounding world and this geometrical system, in the full agreement with Klein's point of view, is determined by the choice of the group of transformations which preserve physical laws holding in the framework of the considered branch of physics.

Meraner Programm

The Meraner Programm is the programme for restructuring of mathematical and natural historical subject matter at secondary schools which was formulated in Merano in 1905.

In the later years of his career, Felix Klein was interested in the teaching of mathematics at German schools as well. He was fighting for its modernization and he made efforts for incorporation of the latest knowledge of mathematical science to classes at secondary schools and universities. With Klein's full support, the first Department of Mathematics Education was established at the University of Göttingen in 1886. The idea about additional education of mathematics teachers by means of lectures and holiday courses arose at that time. First courses under Klein's leadership took place in 1892.

Around the turn of the 19th and 20th century, *the International Congresses of Mathematicians* were held in Zürich (1897), Paris (1900) and Heidelberg (1904). The main invited speakers at these Congresses have been those whose contributions to mathematics were considered in particularly high esteem. At the Congress in Zürich, Felix Klein performed the lecture *Zur Frage des höheren mathematischen Unterrichts* [To the Question on the Teaching of Higher Mathematics]. Thanks to Klein's impulse, *the International Section for the Teaching of Mathematics* was established at the Congress in Paris.

In 1904, *the Meeting of German Naturalists and Physicians* took place in Breslau (Wrocław). On this occasion, *German Committee on the Instruction of Mathematics and Natural Sciences* was established; German mathematician August Gutzmer (1860–1924) was appointed its chairman. Felix Klein set forth his own proposal for the reform of mathematical and physical education to this Committee. Consequently, the Committee elaborated a programme of the reform of secondary education in mathematics which was performed, discussed and afterwards accepted during the next Meeting of German Naturalists and Physicians in Merano in 1905. This programme has later become known as *the Meraner Programm.*

The Meraner Programm set down an essential significance to mathematics in the secondary education, the main aim of mathematics was found out in the development of intellectual and logical abilities. Functional thinking should be more encouraged; the notion of function was supposed to become the central point of all mathematics education. Selected parts of the infinitesimal calculus were recommended into the subject matter at higher classes. Factually, the Meraner Programm laid down these requirements for the teaching of mathematics at secondary schools:

- to support the development of spatial abilities,
- to incorporate the notion of function, the infinitesimal calculus and groups of geometric transformations into the subject matter,
- to reduce formalism and abstract subject matter,
- to solve some practical exercises from the common life,
- to develop the relations among particular subjects.

Fundamental ideas of the Meraner Programm have later become the basis of many other reforms which brought out some changes of the mathematical subject matter at secondary schools.

In 1906, the Meeting of German Naturalists and Physicians took place in Stuttgart. Its central topic consisted in an endeavour to incorporate elements of the infinitesimal calculus into the subject matter at secondary schools. Next Meeting took place in Dresden in 1907. Participants accepted a recommendation to lay a stress upon mathematical applications at secondary schools. In 1908, Felix Klein was appointed the President of the International Commission on Mathematical Instruction. This Commission submitted a recommendation to enrich instruction at secondary schools with selected parts of the projective geometry, the set theory and the group theory. A great number of publications about the teaching of mathematics at all school grades was edited under Klein's leadership.

Czech mathematics education was much influenced by the all-European reform movement. In the Czech lands, *the Union of Czech Mathematicians and Physicists* was the main organizer of the reform movement. This organization arose in 1869 from *the Association for Free Lectures on Mathematics and Physics* founded in Prague in 1862. It mediated not only foreign experiences, but initiated some reform activities as well. It gained a great recognition for the development of the modern Czech mathematical and physical literature containing secondary schools' textbooks. Due to its professional and organizational activity, the level of the teaching of mathematics at Czech secondary schools reached out the level of the prominent European countries.

As a reaction to the Meraner Programm, *the Marchet's reform* was declared in the Austro-Hungarian Empire, i.e. also in the Czech lands, in 1909. It resulted in the acceptance of the new curriculum for secondary schools. The conception of function theory was made the center of the teaching of mathematics. For the first time, elementary functions and some parts of the infinitesimal calculus were involved into the subject matter at secondary schools. Concerning the teaching method at secondary schools determined by the decree of the Ministry of Education, the heuristic method was stressed.

It was subsequently inevitable, modern mathematics textbooks for secondary schools according to the new curriculum to create. The Union of Czech Mathematicians and Physicists was tasked with this intention and therefore it named afterwards Czech mathematicians and mathematics teachers Ladislav Červenka (1874–1947), Miloslav Valouch (1878–1952), Bohumil Bydžovský (1880–1969) and Jan Vojtěch (1879–1953) as the authors of new textbooks. During the period 1910 to 1912, they have written textbooks on arithmetic, algebra and geometry for all school grades of particular types of secondary schools. These new textbooks were of very high quality for that time, they were reprinted many times and were used up to 1950s. Among their merits it is also worth to point out that they introduced the unified terminology and symbolism into secondary schools.

Comparing these new textbooks with those used until that time they differ particularly in the endeavour after explaining and motivating of the subject matter. All new knowledge is deduced on the basis of students' previous experience which leads to the logical ordering of the subject matter. They put a special emphasis on the development of mathematical theories, adequate to the students' age, on the logical deducing and critical attitude to the obtained results. New textbooks are based on the cyclic ordering of the subject matter, everywhere, where it is pertinent, the instruction is illustrated by geometric point of view. Relationships among algebra, analysis and geometry are pointed out as often as possible.

Transformations at the turn of the 19th and 20th century

The new approach to geometric transformations which turned up at the beginning of the 20th century had its roots in the modern axiomatic deductive system. In order to prove any statement in a deductive system, one must show that this statement is a necessary logical consequence of axioms and the statements proved previously. From this point of view Euclid's *Elements* had become an important model for rigorous mathematical proofs. Later, some gaps and inconsistencies were found in Euclid's *Elements*. In the connection with the discovery of the non-Euclidean geometry, mathematicians felt a need to revise and modificate this geometrical system. First attempts at the axiomatic approach to the study of geometry were made by Moritz Pasch (1843–1930) and afterwards, axiomatic system in geometry was systematically developed by David Hilbert (1862–1943).

Moritz Pasch published his *Vorlesungen über neuere Geometrie* [Lectures on Modern Geometry] in 1882. In this book he set up the basis of the projective geometry independent of the fifth Euclid's parallel axiom. He regarded a point, an abscissa and a part of a plane as primitive notions. He was the first mathematician who pointed out that Euclid in his proofs had been operating with those properties of the betweenness relation that had not been mentioned in his system. That's why Moritz Pasch at first implicitly defined the betweenness relation for collinear points and then formulated basic properties of this relation which form the fundamental principle of any metric geometry up to present days.

In his book he defined two geometric figures being congruent if there exists another geometric figure which coincides with both these figures. In the following text, he formulated ten axioms on congruent figures describing basic properties of such figures. The axioms implicitly involve the statement that the congruence is an equivalence, it means it is a reflexive, symmetrical and transitive relation. One axiom concerns the betweenness relation among collinear points, another one is today known as the so-called *Archimedes' axiom.* According to another axiom there exist just two congruences in a plane (direct and non-direct) that carry any given triangle onto a congruent one having prescribed the image of one side of this triangle.

In 1899, David Hilbert in his treatise *Grundlagen der Geometrie* [Foundations of Geometry] introduced the first absolute system of twenty axioms on Euclidean geometry. He considered points, lines and planes as primitive, non-definable geometric objects being in a certain mutual relation. All possible relations among points, lines and planes are described according to axioms. He divided the axioms into five groups: axioms of connection, axioms of betweenness, parallel axioms, axioms of congruence and axioms of continuity.

David Hilbert published his another treatise called *Ueber die Grundlagen der Geometrie* [On the Foundations of Geometry] in 1902. He introduced there a new axiomatic system of planar geometry based upon the notion of group and presented general definition of a geometric motion for the first time.

Conclusion

The theory of geometric transformations is very significant and inspirational up to present days although its theoretical background has already been completed. Geometric transformations play an important role in mathematics because they bring dynamics into geometry. They make it possible to change both a position and a shape of any geometric figure. Geometric figures can namely translate, rotate, turn over, contract, enlarge or however deform according to the prescribed rules under geometric transformations. This process can be used for solving more difficult mathematical problems so as they can be conveniently transformed into simple ones.

In the recent years, more complicated geometric transformations are studied by means of computer technology. Modern mathematical computer programmes enable to study transformations of an arbitrary degree including their analytic representations and graphical outcome. Moreover, it is interesting that geometric transformations are today the most frequently used operations in the computer graphics.