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RELAXATION LENGTHS AND NONNEGATIVE SOLUTIONS IN NEUTRON TRANSPORT*)

by IVO MAREK

1. INTRODUCTION. FORMULATION OF THE PROBLEM

Some problems of mathematical physics lead to necessity of searching adequate theories for certain quantities involved in some physical models of investigated processes. Frequently these quantities, physical meaning of which is obvious from the intuition, can even be rigorously defined after an adequate methamatical theory has been developed. In [13] a very nice theory has been presented of the so called relaxation lengths of neutron distributions in moderators. In our note we make an attempt to extend this theory also to multiplying media.

As is well known, the theory of relaxation lengths is a theory connected with searching particular solutions of the Boltzmann equation in the form $e^{-\varkappa x} \psi(\mathbf{v})$ [12, 13]. Here $\varkappa$ is the reciprocal of one of the relaxation lengths; the largest among them is the diffusion length. In general, these relaxation lengths are complex [5]; they are real for nonmultiplying media, however, as shown in [10] for some special models and generally in [13] essentially using the detailed balance relation. The detailed balance relation will also be used in our study. Our further assumptions concerning the particular operators being involved in the Boltzmann equation, e.g. the elastic scattering operators are very general and include practically all physically possible situations. We would like to mention that certain restrictions are usually made in some papers devoted to similar investigations, e.g. certain assumptions concerning the symmetry and compactness of corresponding operators diminish the class of practical problems which are covered by the theory developed (see [2, 12, 13]).

We avoid the compactness and symmetry assumptions using slightly more general concept of a Radon–Nikolskii operator. The theory of this type of operators is very similar to the theory of compact operators and it was the main tool in our approach. The concept of Radon–Nikolskii operator was introduced in [15] in connection with some investigations concerning a heterogeneous analogue of the Paierl's integral equation [3, 4, 14, 18].

We shall consider the following time independent equation for the neutron distribution in a macroscopically uniform and isotropic medium without external sources

$$[\mathbf{v}\nabla + v\,\Sigma(v)]\,N(\mathbf{r}, \mathbf{v}) = \int_\Omega d^3\mathbf{v}' v' S(\mathbf{v}' \to \mathbf{v})\,N(\mathbf{r}, \mathbf{v}') + \nu(v)\,\chi(v) \int_\Omega d^3\mathbf{v}'\,\Sigma_f(v')\,N(\mathbf{r}, \mathbf{v}'), \tag{1.1}$$

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where the symbols used have the following meaning:

$\mathbf{r}$ and $\mathbf{v}$	space and velocity position vectors respectively
v	modulus of $\mathbf{v}$
$N(\mathbf{r}, \mathbf{v})$	the neutron density in the phase space
$S(\mathbf{v}' \to \mathbf{v})$	the macroscopic differential scattering cross section
$\Sigma(v)$	the total cross section
$\nu(v)$	the number of fission neutrons produced by a neutron having initial velocity modulus v
$\chi(v)$	the fission spectrum
$\Sigma_f(v)$	the cross section for fission
Ω	the velocity space, i.e. $\Omega = \omega \times [0, +\infty)$, where ω is the unit sphere in the threedimensional euclidean space.

The total cross-section $\Sigma(v)$ can be splitted into a sum of the absorbtion and scattering cross-sections

$$\Sigma(v) = \Sigma_a(v) + \Sigma_s(v)\,,$$

where

$$\Sigma_s(\mathbf{v}) = \Sigma_s(v) = \int_\Omega \mathrm{d}^3\mathbf{v}'\, S(\mathbf{v} \to \mathbf{v}')\,.$$

Because of isotropy of the medium under consideration the following relation (detailed balance relation) holds

$$vM(v)\, S(\mathbf{v} \to \mathbf{v}') = v'M(v')\, S(\mathbf{v}' \to \mathbf{v}), \tag{1.2}$$

where

$$M(v) = \left(\frac{m}{2k\pi T}\right)^{3/2} \exp\left\{-\frac{mv^2}{2kT}\right\},$$

(m, k, T being some positive constants).

The setting $\mathbf{r} = (x, 0, 0)$ and

$$N(\mathbf{r}, \mathbf{v}) = M(v)\, \psi(\mathbf{v})\, \mathrm{e}^{-\varkappa x}$$

together with relation (1.2) lead to the equation

$$[\Sigma(v) - \varkappa\mu]\, \psi(\mathbf{v}) = \int_\Omega \mathrm{d}^3\mathbf{v}'\, S(\mathbf{v} \to \mathbf{v}')\, \psi(\mathbf{v}') + \tag{1.3}$$

$$+ \frac{\nu(v)\,\chi(v)}{M(v)} \int_\Omega \mathrm{d}^3\mathbf{v}'\, \Sigma_f(v')\, M(v')\, \psi(\mathbf{v}'),$$

where $\mu = \dfrac{v_x}{v}$ and $\mathbf{v} = (v_x, v_y, v_z)$.

Our investigations will be carried out in the Hilbert space $\mathscr{H}$ consisting of classes of equivalent functions on Ω with respect to the Lebesgue measure with the inner product

$$(\varphi, \psi) = \int_{\Omega} \mathrm{d}^3\mathbf{v} v M(v)\, \varphi(\mathbf{v})\, \psi(\mathbf{v}).$$

Symbolically equation (1.3) can be written as

$$[\Sigma - \varkappa\mu]\, \psi = (S + F)\, \psi, \tag{1.4}$$

the meaning of the symbols used being obvious.

It is reasonable to assume that

$$0 < \varkappa^* = \inf\{\Sigma(v) : v \in [0, +\infty)\}$$

and that for any $\mathbf{v} \in \Omega$ there exists an $\varepsilon(v)$ such that $S(\mathbf{v} \to \mathbf{v}') > 0$ for $|\mathbf{v} - \mathbf{v}'| < \varepsilon(v)$.

Let us summarize some of the properties of the operators S and F.

(a) S is compact for the monoatomic gas model [16].

(b) For liquid and solid materials the operator S can be split into two parts $S = S_e + S_{in}$, where S_e is a bounded and S_{in} a compact operator respectively. Furthermore, the operator $K = \Sigma^{-1/2}\, S\, \Sigma^{-1/2}$ is compact for liquids [12].

(c) For solid media neither the operator S_e nor K are compact. The operator S_e can be expressed as (see [19])

$$S_e\psi = \int_{\Omega} \mathrm{d}^3\mathbf{v}' \psi(\mathbf{v}') \frac{A(\mathbf{v}, \mathbf{v}')}{v'^2}\, \delta(v - v'), \tag{1.5}$$

for an amorphous solid and as

$$S_e\psi = \int_{\Omega} \mathrm{d}^3\mathbf{v}' \psi(\mathbf{v}') \left\{\frac{A(\mathbf{v}, \mathbf{v}')}{v'^2} + \frac{1}{v'^4} \sum_{\tau \neq 0}^{\tau < 2v'} B(\tau, \mathbf{v}, \mathbf{v}')\, \delta\left(1 - \frac{\tau^2}{2v'^2} - \mu_0\right)\right\} \tag{1.6}$$

for a crystaline solid. The expressions A and B introduced in (1.5) and (1.6) are some bounded functions, the summation index τ ranges over the magnitudes of vectors of the reciprocal lattice and μ_0 is the angle between $\mathbf{v}$ and $\mathbf{v}'$.

(d) It follows from the experiments that the function χ in (1.3) can be expressed as

$$\chi(v) = k \exp\left\{-\frac{v^2}{1{,}93}\right\} \sinh v\sqrt{1{,}15}$$

for the case of ${}_{92}U^{235}$. A similar form χ has also for other fission materials (see [1]). Therefore, in general,

$$\chi(v) = a(v) \exp\left\{-\frac{v^2}{2b}\right\} \sinh cv,$$

where the positive constants b and c depend weakly on the sort of the fission material and $\chi/M \in \mathscr{H}$.

(e) The function $\nu(v)$ is bounded [19].

(f) It is reasonable to assume [9] that $\Theta \in \mathscr{H}$, where $\Theta(v) = \sqrt{M(v)}\, \Sigma_f(v)$.

It is obvious that F has one dimensional range; consequently, since according to (d) – (f) it is bounded, F is compact and so is

$$L = \Sigma^{-1/2} F \, \Sigma^{-1/2}.$$

The noncompactness of K in the case of solids is a serious complication in investigating the spectrum of the equation (1.4). Our basic tool of avoiding this difficulty is the following property of S.

(g) The operator S is a Radon–Nikolskii operator, i.e. $S = U + V$, where U is a compact and V a bounded operator such that for the spectral radii $r(S)$ and $r(V)$ we have that $r(S) > r(V)$; herein we may set $U = S_{in}$ and $V = S_e$.

2. THE SPECTRUM.

We say that $\varkappa$ belongs to the spectrum of the equation (1.4) if $\lambda = 0$ is in the spectrum of the operator

$$\Sigma - S - F - \varkappa\mu = C(\varkappa).$$

Now we use the terminology and some results of GOCHBERG–KREJN paper [7]. Let A be a densely defined linear operator. A is called normally solvable if $Ax = y$ has solution if and only if $y'(y) = 0$ for all $y' \in \mathscr{N}^+$, where $\mathscr{N}^+$ is the defect subspace of A. Let us set $\beta_A = \dim \mathscr{N}^+$ and $\alpha_A = \dim \mathscr{N}$, where $\mathscr{N}$ is the null-space of A. A point λ is called $\boldsymbol{F}$-point of A if $A - \lambda I$ is normally solvable and $\beta_{A-\lambda I}$ and $\alpha_{A-\lambda I}$ are finite. The set of all $\boldsymbol{F}$-points of A is called the $\boldsymbol{F}$-set of A; it is denoted by $\boldsymbol{F}_A$.

According to Theorem 3.4 in [7] telling us that the $\boldsymbol{F}$-sets of $A + B$ and A, where B is an A-compact perturbation, are identical $\boldsymbol{F}_{A+B} = \boldsymbol{F}_A$, and because of compactness of F we may claim that $\lambda = 0$ is an $\boldsymbol{F}$-point of $C(\varkappa)$ if and only if $\lambda = 0$ is an $\boldsymbol{F}$-point of $B(\varkappa) = \Sigma - S - \varkappa\mu$. Furthermore, Theorem 3.6 in [7] claims that if $A(\varkappa)$ is a closed operator-function analytically depending on $\varkappa$ in a connected region G for which O is $\boldsymbol{F}$-point for all $A(\varkappa)$ then everywhere in G $\dim \mathscr{N}(A(\varkappa)) = d$ with a possible at most countable exceptional set $\{\varkappa_j\}$, where $\dim \mathscr{N}(A(\varkappa_j)) > d$; $\mathscr{N}(A(\varkappa))$ being the null-space of $A(\varkappa)$.

Using the properties of $B(\varkappa) = \Sigma - S - \varkappa\mu$ derived in [12] and the above remark we obtain the following results.

Theorem 2.1. *Let*

$$\alpha = \inf\{((\Sigma - S)\,\varphi, \varphi) : (\varphi, \varphi) = 1\}$$

be positive. Let $\varkappa \in \Psi$, where

$$\Psi = \{\lambda : \operatorname{Im} \lambda > 0\} \cup \{\lambda : \operatorname{Im} \lambda < 0\} \cup (-\alpha, \alpha) .$$

Then either $(\Sigma - S - F - \varkappa\mu)^{-1}$ *exists as a bounded everywhere defined operator in* $\mathscr{H}$ *or* $\varkappa$ *is an isolated eigenvalue of* (1.4) *with a finitedimensional eigenspace and finite-dimensional defect subspace.*

Theorem 2.2. *Let* $K = \Sigma^{-1/2} S \Sigma^{-1/2}$ *be compact and let* $\Psi' = \{\lambda : \operatorname{Im} \lambda \neq 0\} \cup \cup (-\varkappa^*, \varkappa^*)$. *Then the inverse* $(\Sigma - S - F - \varkappa\mu)^{-1}$ *exists and is an analytic function of* $\varkappa$ *for* $\varkappa \in \Psi'$ *with a possible exception of a set of isolated eigenvalues* $\varkappa_j$, *where it has poles. Hence equation* (1.4) *has nontrivial solution for each such* $\varkappa_j$.

On the other hand, the inverse of $C(\varkappa)$ *does not exist as a bounded densely defined transformation for* $\varkappa$ *belonging to* $(-\infty, -\varkappa^*] \cup [\varkappa^*, +\infty)$.

Theorem 2.3. *Let*

$$\sup \left\{ \frac{\Sigma_e(v)}{\Sigma(v)} ; v \in [0, +\infty) \right\} = 1 - \gamma ,$$

where $0 < \gamma < 1$ *does not depend on* v. *Then the inverse* $(\Sigma - S - F - \varkappa\mu)^{-1}$ *exists and is an analytic function of* $\varkappa$ *for* $\varkappa \in \psi'$ *with a possible exception of a set of isolated eigenvalues* $\varkappa_j$, *where it has poles. Hence equation* (1.4) *has a nontrivial solution for each such* $\varkappa_j$.

On the other hand, the inverse of $\Sigma - S - F - \varkappa\mu$ *does not exist as a bounded densely defined transformation for* $\varkappa \in (-\infty, -\varkappa^*] \cup [\varkappa^* + \infty)$.

Similarly as in [13] a question remains open whether a continuously distributed spectrum may extend beyond the bounds $\pm\varkappa^*$ into a possible gap $(-\varkappa^*, -\gamma\varkappa^*] \cup \cup [\gamma\varkappa^*, \varkappa^*)$.

The possible complex eigenvalues of (1.4) are contained in the strip $\{\lambda : |\operatorname{Im} \lambda| \leqq \leqq \eta \, \| L \|\}$, where η does not depend on L.

3. NONNEGATIVE EIGENSOLUTIONS

In this section we shall examine the existence and uniqueness of nonnegative solutions of equation (1.4) which are of particular interest of the physicists.

Let us mention that the existence of nonnegative solutions of equation (1.4) is guaranteed for certain $\varkappa$'s by the fact that the operators S and F leave invariant the cone [11] of nonnegative functions in $\mathscr{H}$. The uniqueness is a consequence of irreducibility of S. We use the following concept of irreducibility introduced by I. Sawashima [17].

A bounded operator T leaving invariant a generating and normal cone $\mathscr{K}$ in a Banach space $\mathscr{X}$ is called irreducible (more precisely, $\mathscr{K}$-irreducible, and, originally semi-nonsupporting), if for every pair $x \in \mathscr{K}$, $x \neq 0$, $x' \in \mathscr{K}'$, $x' \neq 0$, where $\mathscr{K}'$

is the adjoint cone [11], there exists a positive integer $p = p(x, x')$ such that $x'(T^p x) > 0$.

Our main result is based on the following lemma.

Lemma. *Let T be a Radon–Nikolskii operator leaving invariant a generating and normal cone $\mathcal{K}$ in a Banach space $\mathcal{X}$. Then there exists an eigenvector of T corresponding to the spectral radius $r(T) : Tx_0 = r(T)\, x_0$, $x_0 \in \mathcal{K}$, $x_0 \neq 0$.*

If, moreover, T is irreducible then the eigenspace belonging to $r(T)$ is onedimensional and there are no other eigenvectors of T lying in $\mathcal{K}$ being linearly independent of x_0.

The existence proof of this lemma is based on the fact that each spectral point ν of a Radon–Nikolskii operator $T = U + V$ for which $|\nu| > r(V)$ is a pole of the resolvent operator $(\lambda I - T)^{-1}$. The final part of the Lemma is proved in [17].

Theorem 3.1. *Let us assume that $r(K + L) \leqq 1$, Then there exists the smallest $\varkappa_0 \in [0, \varkappa^*]$ for which $(\Sigma - S - F - \varkappa_0 \mu)$ is not boundedly invertible. If $\varkappa_0$ is an eigenvalue of equation (1.4), then to this eigenvalue there corresponds a uniquely determined eigensolution ψ_0 and this eigensolution is nonnegative.*

Furthermore, there exists a positive constant ν such that if for $\nu(v)$ in (1.3) we have $\nu(v) \leqq \nu$ for all $v \in [0, +\infty)$, then $\psi^{\pm}$ are the only nonnegative eigensolutions of equation (1.3) when $\varkappa$ ranges over the complex plane; herein $\psi^+(\mathbf{v}) = \psi_0(\mathbf{v})$ and $\psi^-(\mathbf{v}) = \psi_0(-\mathbf{v})$.

Finally, with no restriction on $r(K + L)$, there exists a nonnegative constant τ such that if $\nu(v_0) \geqq \tau$ for a suitable $v_0 \in [0, +\infty)$ then there are no nonnegative solutions of equation (1.3) for any complex $\varkappa$.

The existence of the constants ν and τ mentioned in Theorem 3.1 is a consequence of some continuity arguments; hence we have no quantitative bounds for these constants. A rough bound for τ can be obtained exploiting the fact that nonnegative eigensolutions of (1.3) cannot exist whenever $r(L + K) \geqq 1 + \dfrac{\varkappa^0}{\varkappa^*}$, where $\varkappa^0 \in [0, \varkappa^*]$ the smallest value for which $(\Sigma - S - \varkappa^0 \mu)$ is not boundedly invertible. For the existence of $\varkappa^0$ see [13].

Let us note that if in particular $r(K) = 1$ and L is the zero-operator, then the conclusion of Theorem 3.1 completes the uniqueness result for solid moderators conjectured in [13].

Conclusions. The preceding results imply the following:

If the fission in the medium is sufficiently weak then:

(i) *The properties of the relaxation lengths are very similar to those in the corresponding nonmultiplying medium characterized by the same cross-sections.*

(ii) *The possible complex relaxation lengths are located in a strip around the real axis the thickness of the strip being linearly dependent on the norm of the fission operator.*

(iii) *The diffusion length is (up to the symmetry) the only of the relaxation lengths to which there corresponds a nonnegative eigensolution.*

If the fission in the medium under consideration is sufficiently strong then:

(iv) *There are no nonnegative eigensolutions of the relaxation lengths equation* (1.3); *this physically means that the concept of diffusion length is empty in this case.*

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