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DELAY MAKES PROBLEMS IN POPULATION MODELLING

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The basic population model $x' = ax$, although it is very simple, plays an important role in the history of modelling. Contrary to mathematical models in physics, it is not designed to determine quantities. Here the mathemtics only exhibits a tendency. The purpose of such a model is not prediction, but insight [2].

The same is true for most population models. Important are the qualitative properties, such as stability of equilibrium states, existence of periodic solutions, etc.

Intuitively it is clear that the history of development of a modelled biological system is very important for his state in the future. Hence the use of delayed differential equations seems to be appropriate. Moreover, using the delay one can describe age and also the spatial structure of the population [3]. Generally speaking, a model involving the delay allows more complex types of behavior than models without delay.

By modelling usually the following assumptions are made:

1) Different histories determine different solutions.

2) Suitable choice of the history allows the solution to attain any prescribed value at a given time.

The first problem is a problem of injectivity of the shift operator (or solution map) [1]. Take e.g. the equation

$$x'(t) = -ax(t-1)[1-x(t)]$$

which arises in a natural way from a model

$$y'(t) = a(y - y^2)$$

of self-limited population. Then for every initial function φ satisfying the condition $\varphi(0) = 1$ we have $x(t) \equiv 1$ for $t \geq 0$.

Let $C(-1,0)$ be the space of continuous mappings from $[-1,0]$ to the set R of real numbers. For the equation

$$x'(t) = f(x(t-1)) \tag{1}$$

define an operator $T : C(-1,0) \to C(-1,0)$ by

$$T(\varphi)(t) = x_\varphi(t+1), \qquad t \in [-1,0]$$

where x_φ denotes the solution of (1) determined by the initial function $\varphi \in C(-1,0)$. Such an operator T is called shift operator or solution map for the equation (1). The above quoted example exhibits an equation for which the shift operator is not injective. The following result states that this is actually the typical case for equation (1).

Theorem 1. Let C be the metric space of continuous mappings from R to R, equipped with the metric $\rho(f,g) = \min\{1, \sup_x |f(x) - g(x)|\}$. Let $H \subset C$ be the set of those mappings $f \in C$, for which there are initial functions $\varphi \neq \psi$ with $\varphi(0) = \psi(0)$, generating the same solution $x_\varphi(t) = x_\psi(t)$ of (1) for $t \geq 0$. Then $C \setminus H$ is nowhere dense in C.

In other words, the shift operator for the solution (1) is generically not injective.

Proof. The set M of those $f \in C$, which attain strong local maxima at certain points, is clearly open and dense in C. Hence it suffices to show that $M \subset H$.

Choose, for any given $f \in M$, points $a < c < b$ with $f(a) = f(b) < f(c) = \max\{f(t);\ t \in [a,b]\}$. For any $d > 0$ choose a set $M(d)$ of points

$$a = a_0 < a_1 < \ldots < a_k = c = b_k < \ldots < b_1 < b_0 = b$$

(k is suitable integer) such that $f(a_i) = f(b_i)$ for any i, and

$$|f(t) - f(s)| < d \text{ whenever } t, s \in [a_j, a_{j+1}] \text{ or } t, s \in [b_{j+1}, b_j]$$

for some j. Let $\varphi(d), \psi(d) \in C(-1,0)$ be such that $\varphi(d)(i/k - 1) = a_i$, $\psi(d)(i/k - 1) = b_i$, and $\varphi(d)$, $\psi(d)$ be linear on every of the intervals $[i/k - 1, (i+1)/k - 1]$, $i = 0,\ldots,k$.

We follow the above construction for every $d = 1/n$, $n = 1,2,\ldots$. Without loss of generality we may assume that $M(1/n) \subset M(1/(n+1))$

$$\lim_{n\to\infty} \varphi(1/n) = \varphi \quad \text{and} \quad \lim_{n\to\infty} \psi(1/n) = \psi$$

uniformly. Moreover, $\varphi(0) = \psi(0)$ and $f(\varphi(t)) = f(\psi(t))$ for every $t \in [-1,0]$, i.e. φ and ψ are the desired initial functions, q.e.d.

Modifying the above argument for equation with continuously differentiable right side, we obtain the next

Theorem 2. Let C^1 be the set of continuously differentiable functions from R to R, with the usual C^1-netric. Let H^1 be the set of those f from C^1, for which there are initial functions $\varphi \neq \psi$ with $\varphi(0) = \psi(0)$, generating the same solution of (1). Then both

$$\text{Int } H^1 \neq \phi \quad \text{and} \quad \text{Int } (C^1 \setminus H^1) \neq \phi$$

Remark. Theorem 2 can be generalized to the equation

$$x'(t) = f(x(t), x(t-1)) \tag{2}$$

where f is continuously differenatiable. We conjecture that also Theorem 1 can be generalized to this case, although we are not able to give any proof.

Next consider the second problem. Given some $(\bar{t},\bar{x}) \in R^2$, where $\bar{t} > 0$, does there exist an initial function $\varphi \in C(-1,0)$ such that $x_\varphi(\bar{t}) = \bar{x}$, where x_φ is the solution of the equation (1)? THis is the problem of pointwise completeness of (1). In [4] we recently gave a sufficient condition for pointwise completeness, which for (1) can be formulated as follows:

Theorem 3. Let $\bar{t},\bar{x} \in R$ be given. If $f \in C$ satisfies the Lipschitz condition

$$|f(x) - f(y)| \le L|x - y| \quad \text{for any} \quad x,y \in R$$

and

$$L \cdot \bar{t} < 1 \tag{3}$$

then there is a function $\varphi \in C(-1,0)$ such that $x_\varphi(\bar{t}) = \bar{x}$.

Note that theorem is true also for $\bar{x} \in R^n$.

The following example shows that the condition (3) in general cannot be omitted.

Example. Let $f(x) \equiv 0$ for $x \leq 1$, and $f(x) = 1 - x$ for $x > 1$. Let $\varphi \in C(-1,0)$ be an initial function. Let x_φ be the corresponding solution of (1). Then $x_\varphi(t)$ is non-increasing in $[0,\infty)$, hence we have $x_\varphi(t) \geq x_\varphi(1)$, for $t \in [0,1]$. Since $x_\psi(t) \leq 1$ for $t > 1$, where $\psi(t) = x_\varphi(1) =$ const for $t \in [-1,0]$, we have $x_\varphi(2) \leq 1$. This is true for arbitrary φ.

The above quoted properties of delayed differential equations does not seem to be good for modelling. Theorem 1 e.g. indicates a structural non-stability for models involving the equation (1). But also the well-known Volterra's - Lotka's predator-prey differenatial model is structurally unstable though its role among the known models is of great importance.

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