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On Factorization of Fefferman's Inequality

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Abstract. This paper is concerned with conditions for a weight function V in order that

$$\left(\int\limits_B u^2(x)V(x)\,dx\right)^{1/2} \le c\left(\int\limits_B (\nabla u(x))^2\,dx\right)^{1/2}, \qquad u \in W_0^{1,2}(B),$$

where B is a ball in $\mathbb{R}^N$. This inequality has found wide applications in many areas of analysis and this has been the reason for an effort to obtain various conditions, either sufficient or necessary and sufficient. Here we survey some of them and we also present a method, using decomposition of imbeddings between Sobolev and Lorentz-Orlicz spaces (and/or their weak counterpart). We state sufficient conditions in terms of a membership of the weight function V in Lorentz-Orlicz spaces and pay an attention to the so called 'size condition' in order to discuss applications to the strong unique continuation property for $|\Delta u| \le V|u|$ in dimensions 2 and 3.

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1 Introduction

Fefferman's inequality [F]

$$\left(\int\limits_{\mathbb{R}^N} u^2(x)V(x)\,dx\right)^{1/2} \le c\left(\int\limits_{\mathbb{R}^N} (\nabla u(x))^2\,dx\right)^{1/2}, \qquad u \in W^{1,2}(\mathbb{R}^N), \tag{1}$$

has turned out to be a very powerful tool to handle many topical problems in the PDEs including the strong unique continuation property (the SUCP in the sequel), distribution of eigenvalues and so on.

This is the preliminary version of the paper.

After making a short trip into the history, when we recall some of the most important results, our concern will be to establish efficient and manageable conditions for the function V, guaranteeing validity of a local version of (1), that is,

$$\left(\int_B u^2(x)V(x)\,dx\right)^{1/2} \le c\left(\int_B (\nabla u(x))^2\,dx\right)^{1/2}, \qquad u \in W_0^{1,2}, \tag{2}$$

where B is a bounded domain in $\mathbb{R}^N$, say, a ball, $|B| = 1$. We shall use a natural idea of a decomposition of the imbedding in (2) into an imbedding of $W_0^{1,2}$ into a suitable target space and an imbedding from this target into $L^2(V)$; we invoke imbedding theorems for the Sobolev space $W_0^{1,2}$ — the classical Sobolev theorem and a refinement in terms of Lorentz spaces in the role of target spaces in the dimension $N \ge 3$, and the limiting imbedding theorem due to Brézis-Wainger [BW] (see also [Zi], Lemma 2.10.5) in the dimension $N = 2$, which can be viewed as an analogous refinement of Trudinger's celebrated limiting imbedding [T]. The method suggested for proving (2) is a kind of a generator of n-dimensional Hardy inequalities or, alternatively, of weighted imbeddings $W_0^{1,2} \hookrightarrow L^2(V)$: general results of this nature will appear elsewhere. It is rather surprising that working with superpositions of imbeddings we do not lose much and that combining our conditions for validity of (2) with the conditions for the SUCP in Chanillo and Sawyer [CS] we recover or generalize some of known results about the strong unique continuation property for $|\Delta u| \le V|u|$ in dimensions 2 and 3. In fact all the above imbeddings of the Sobolev spaces are sharp in the scale of spaces considered and the same is true for the weighted imbeddings. In the latter case we shall use only Hölder's inequality, nevertheless, we actually use conditions which are necessary as well.

2 Recent history — a partial survey

Let us start with an observation that the theory of weighted imbeddings is by no means complete; only special problems have been fully solved. For instance the particular type of power weights has been considered in [OK] — powers of distance to the boundary of the domain in question (that is, power type weights after flattening the boundary using local coordinates). Passing to more general weights, a natural idea is to apply what is known for the behaviour of Riesz potentials in weighted spaces since (1) follows for a weight function V provided the boundedness of the Riesz potential of order 1 from the Lebesgue space L^2 into the weighted Lebesgue space $L_2(V)$ has been established. Let us observe that one of the peculiarities of the inequality (2) is that the powers at both sides are the same. Necessary and sufficient conditions have been found for the case of imbeddings of $W^{1,p}$ into $L_q(V)$, see Adams' inequality in [A] and Maz'ya [Ma], when $p < q$. If $p = q = 2$ and $N \ge 3$, then a necessary and sufficient condition is due to Kerman and Sawyer [KeSa]; it reads

$$\int_{\mathbb{R}^N} \left(\int_Q \frac{V(y)}{|x-y|^{N-1}}\, dy \right)^2 dx \le K \int_Q V(x)\, dx \tag{3}$$

for all dyadic cubes $Q \subset \mathbb{R}^N$, with a constant K independent of Q. This condition uses local potentials in an intrinsic way since it hangs on Sawyer's theorem on two weight inequalities for the maximal function from [Sa] and on the good-λ-inequality due to Muckenhoupt and Wheeden [MW]; the latter giving a link between an inequality for the corresponding Riesz potential and for the associated fractional maximal function. The condition (3) can sometimes be hard to verify since it involves the local potential of V, or, alternatively, the fractional integral of V. Hence various sufficient conditions, including those preceding [KeSa] are of importance.

The celebrated Fefferman's paper [F] gave a sufficient condition, which we describe in the following. Let us recall the definition of the *Fefferman-Phong class* F_p, $1 \le p \le N/2$. A function V belongs to F_p if

$$\|V\|_{F_p} = \sup_{\substack{x\in\mathbb{R}^N \\ r>0}} r^2 \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |V(y)|^p\, dy \right)^{1/p} < \infty.$$

Let us first formulate the basic result in the framework of the classes F_p.

Theorem 1 (Fefferman [F]). *Let $N \ge 3$, $1 < p \le N/2$, and $V \in F_p$. Then* (1) *holds.*

A particularly fine and elegant proof of (1) was given by Chiarenza and Frasca [CF].

It is worth observing that $F_{p_2} \subset F_{p_1}$ for $1 \le p_1 \le p_2 \le N/2$, and plainly $F_{N/2} = L^{N/2}$. Provided that we restrict ourselves to balls $B(x,r)$ with radius smaller than some $\varepsilon_0 > 0$ in the above definition the result can be identified with the *Morrey space* $\mathcal{L}^{p,N-2p}$. We recall that, for $0 < \lambda \le N$ and $1 \le p < \infty$, the Morrey space $\mathcal{L}^{p,\lambda}$ is the collection of all $V \in L^p_{\mathrm{loc}}$ such that

$$\|V\|_{\mathcal{L}^{p,\lambda}} = \sup_{\substack{x\in\mathbb{R}^N \\ 0<r\le r_0}} r^{-\lambda/p} \left(\int_{B(x,r)} |V(y)|^p\, dy \right)^{1/p} < \infty.$$

Inserting a 'hat function', that is, $u(x) = (r - |x|)\chi_{B(0,r)}$, $x \in \mathbb{R}^N$, into (1) is a standard way how to show that the weight V must belong to the Morrey space $\mathcal{L}^{1,1}$ in order (1) holds. Nevertheless, as is well known this is not sufficient. Thus Fefferman's theorem gives a sufficient condition in terms the Morrey spaces and further investigation shows that the situation near $\mathcal{L}^{1,N-2}$ is of rather delicate nature. Observe also that when passing to various refined conditions, then the constant C in (1) can depend on $\operatorname{supp} u$; this is quite sufficient for relevant applications.

For $f \in L^1_{\text{loc}}$, let us denote

$$\eta(f,\varepsilon) = \sup_{x\in\mathbb{R}^N} \int\limits_{|x-y|\le\varepsilon} \frac{|f(y)|}{|x-y|^{N-2}}\, dy.$$

The *Stummel-Kato* class is defined by

$$S = \{f;\ \eta(f,\varepsilon) < \infty \text{ for all } \varepsilon \text{ and } \eta(f,\varepsilon) \searrow 0 \text{ as } \varepsilon \searrow 0\}.$$

A *variant of the Stummel-Kato class*, sometimes denoted by $\widetilde{S}$ is defined as

$$\widetilde{S} = \{f;\ \eta(f,\varepsilon) < \infty \text{ for all } \varepsilon > 0\}.$$

Restriction of these spaces to a domain in $\mathbb{R}^N$, say, Ω can be done in an obvious way, namely, by considering $\chi_\Omega V$ instead of V in the above definitions.

It will be useful to give relations between the spaces considered up to now. They are discussed e.g. in Zamboni [Za], Di Fazio [DiF] (the first inclusion in (i)), Piccinini [Pi] (the statement in (iii) below) and Kurata [K]; the last quoted author considers also other variants of the Stummel-Kato class to get a background tailored for more general elliptic operators.

Proposition 2. *The following statements are true:*

(i) $\mathcal{L}^{1,\lambda} \subset S \subset \widetilde{S} \subset \mathcal{L}^{1,N-2}$, $\lambda > N-2$.
(ii) $L^{N/2,\infty} \subset F_p$ *for every* $1 \le p < N/2$, *where the former space denotes the weak* $L^{N/2}$ *space (the Marcinkiewicz space).*
(iii) *For each* $p \ge 2$ *and each* $0 < \lambda < n$, *there exists a function* $f \in \mathcal{L}^{p,\lambda} \setminus L^q$ *for every* $q > p$.
(iv) *For every sufficiently small* $p > 1$ *there exists a function* $f \in F_p \setminus L^{N/2,\infty}$.
(v) $S(\Omega) \subset F_1(\Omega)$, *and* $L^{N/2}(\Omega)$ *is incomparable with* $S(\Omega)$.

Let us observe that (ii) gives a sufficient condition for the validity of (1) in terms of another scale of function spaces, namely, of the weak Lebesgue spaces. We shall come to use of more general Lorentz spaces later in this paper.

Employing the class $\widetilde{S}$, it is possible to prove (see [Za]):

Theorem 3. *Let* $V \in \widetilde{S}$. *Then for every* $r > 0$ *there is* C_r *depending only on* $\eta(V,r)$ *and* N *such that*

$$\int\limits_{\mathbb{R}^N} u^2(x)V(x)\, dx \le C_r \int\limits_{\mathbb{R}^N} |\nabla u(x)|^2\, dx$$

holds for every $u \in C_0^\infty$ *supported in* $B(0,r)$.

A reader can find further results in Chang, Wilson and Wolff [CWW], who consider a certain Orlicz variant of Morrey spaces. An interesting Orlicz spaces type refinement of the well-known Adams' inequality [A], has recently appeared in Ragusa and Zamboni [RZ].

The inequalities (1), (2) and further weighted imbeddings certainly deserve further study aimed at obtaining necessary and sufficient conditions or to get as close as possible to them; at the same time it is desirable that these conditions are described in a manageable way.

3 The size condition and some applications

For the sake of applications we shall pay a special attention the so called '*smallness condition*' or the '*size condition*' (see (5) below), playing a important role in the study of the strong unique continuation property. We shall restrict ourselves to a differential inequality arising from the Schrödinger operator, namely, $|\Delta u| \le V|u|$.

Let us recall that a locally integrable function u is said to have a *zero of infinite order* at x_0 if

$$\lim_{r\to 0_+} r^{-k} \int\limits_{|x-x_0|<r} |u(x)|^2\,dx = 0$$

for all $k = 1, 2, \ldots$. If every solution of a given differential equation, with a zero of infinite order, vanishes identically, then the corresponding operator is said to satisfy the *strong unique continuation property* (the SUCP). As to non-analytic setting of the problem let us recall that in 1939 Carleman [C] proved that the operator $-\Delta + V$ has the strong unique continuation property provided $V \in L^\infty_{\mathrm{loc}}$, that is, he showed that under this assumption a solution of the equation $-\Delta u + V(x)u = 0$ with a zero of infinite order vanishes identically. There is a lot of results concerning the SUCP, with various assumptions on the potential V and also on coefficients in the case of a more general elliptic operator in question. Here we shall go along the lines of sufficient conditions in terms of integrability of the potential with no apriori assumptions on its pointwise behaviour.

Let us first recall Jerison and Kenig [JK], Stein [St], where the SUCP is proved for $V \in L^{N/2}_{\mathrm{loc}}$ or for V locally small in the Marcinkiewicz space $L^{N/2,\infty}$, $N \ge 3$, and Pan [Pa] with the pointwise growth condition $V(x) \le M/|x|^2$, $N \ge 2$, and without the size conditions for V.

Wolff [W] has constructed counterexamples for $N = 3$ and $N = 2$, showing that the assumption about the local smallness of the imbedding norm in (1) cannot be removed in general. For $N = 2$ there is the result due to Gossez and Loulit [GL] with the sufficient condition $V \in L^1 \log L$ for the SUCP.

Theorem 4 (Wolff [W]). *The following statements are true:*

(1) *There exists a function $u : \mathbb{R}^3 \to \mathbb{R}^1$, smooth and not identically zero, vanishing at infinite order at the origin and such that $|\Delta u| \le V|u|$ with $V \in L^{3/2,\infty}$.*

(2) *There exists a function* $u : \mathbb{R}^2 \to \mathbb{R}^1$, *smooth and not identically zero, vanishing at infinite order at the origin and such that* $|\Delta u| \le V|u|$ *with* $V \in L^1$.

Chanillo and Sawyer [CS] considered the classes F_p for $p > (N-1)/2$ and proved the SUCP for potentials V which have locally small F_p-norm in the sense that

$$\limsup_{r\to 0} \|V\chi_{B(y,r)}\|_{F_p} \le \varepsilon(p,N) \qquad \text{for all } y \in \mathbb{R}^N, \tag{4}$$

where $\varepsilon(p,N)$ is a sufficiently small constant. Since $L^{N/2,\infty} \subset F_p$ for all $p < N/2$ (see Proposition 2) this gives a result for V in a larger class than in [JK], [St], however, with the size constraint, this time in the F_p class; again the value of the constant appearing in the size condition is not specified.

If $N \le 3$, then a condition for the SUCP in terms of the local smallness of the constant C in (1) appears; more specifically:

Theorem 5 (Chanillo, Sawyer [CS]). *Let us assume that* $N = 2$ *or* $N = 3$ *and that* Ω *is a bounded open and connected subset of* $\mathbb{R}^N$. *Let* $T(V)$ *denote the imbedding in* (1). *If*

$$\limsup_{r\to 0_+} \|T(V\chi_{B(x,r)})\| \le \varepsilon \tag{5}$$

with a sufficiently small $\varepsilon > 0$ *for all* $x \in \Omega$, *then any solution* $u \in W^{2,2}_{loc}$ *of the inequality* $|\Delta u| \le V|u|$ *in* Ω *has the* SUCP.

It turns out that the size condition can be effectively verified in some cases. We shall consider the scale of Lorentz spaces in the dimension 3, and for $N = 2$ we present a general theorem, including [GL] as a special case. Proofs will appear elsewhere (see [KrSc]).

We shall need some basic facts from the Orlicz, Lorentz-Zygmund and Orlicz-Lorentz spaces theory. Let us agree that all the spaces in the sequel will be considered on a ball $B \subset \mathbb{R}^N$ with the unit measure, $N \ge 2$, or on the interval $(0,1)$; we shall usually omit the appropriate symbol for the domain since it will be clear from the context.

We shall also need a finer scale of spaces, which includes Orlicz spaces in a rather same manner as Lorentz spaces include Lebesgue spaces. We refer to Montgomery-Smith [M-S].

Let us recall that an even and convex function $\Phi : \mathbb{R} \to [0,\infty)$ such that $\lim_{t\to 0} \Phi(t) = \lim_{t\to\infty} 1/\Phi(t) = 0$ is called a *Young function*. A general reference for the (non-weighted) theory of Orlicz spaces is [KR], more general modular spaces are subject of [Mu].

Let Φ and Ψ be Young functions. For a function g even on $\mathbb{R}^1$ and positive on $(0,\infty)$ let us put

$$\widetilde{g}(t) = \begin{cases} 1/g(1/t), & t > 0, \\ \widetilde{g}(-t), & t < 0, \\ g(0), & t = 0. \end{cases}$$

Let V be a weight in B and let f_V^* denote the non-increasing rearrangement of f with respect to the measure $V(x)\,dx$. An *Orlicz-Lorentz space* $L^{\Phi,\Psi}(V)$ is the set of all measurable f on B for which the *Orlicz-Lorentz functional*

$$\begin{aligned}\|f\|_{\Phi,\Psi;V} &= \|f_V^* \circ \widetilde{\Phi} \circ \widetilde{\Psi}^{-1}\|_{\Psi} \\ &= \inf\{\lambda > 0;\ \int_0^\infty \Psi\left(\frac{f_V^*(\widetilde{\Phi}(\widetilde{\Psi}^{-1}(t)))}{\lambda}\right) dt \le 1\}\end{aligned} \tag{6}$$

is finite. A measurable function f defined on B belongs to a *weak Orlicz* (or *Orlicz-Marcinkiewicz*) space $L^{\Phi,\infty}(V)$ if its *Orlicz-Marcinkiewicz functional*

$$\|f\|_{\Phi,\infty;V} = \sup_{\xi>0} \widetilde{\Phi}^{-1}(\xi) f_V^*(\xi) \tag{7}$$

is finite. If $V \equiv 1$, we shall simply write $L^{\Phi,\Psi}$ and $L^{\Phi,\infty}$ instead of $L^{\Phi,\Psi}(1)$ and $L^{\Phi,\infty}(1)$, resp.

For brevity and in accordance with a general usage we shall often use only the major part of a Young function (that is, functions equivalent to the Young function in question in a neighbourhood of infinity) in symbols for spaces.

Let us observe that $L^{\Phi,\Phi} = L^{\Phi}$, the *Orlicz space*. If $\Phi(t) = |t|^p$ and $\Psi(t) = |t|^q$, then $L^{\Phi,\Psi} = L^{p,q}$, the *Lorentz space*, $L^{\Phi,\infty} = L^{p,\infty}$, the Marcinkiewicz space; analogously for the weighted variants.

Special cases of the Orlicz-Lorentz spaces are the Lorentz-Zygmund spaces, that is, logarithmic Lorentz spaces, investigated by Bennett and Rudnick [BR]. For $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}^1$, the Lorentz-Zygmund space $L^{p,q}(\log L)^\alpha$ consists of functions f with the finite functional

$$\|f\|_{L^{p,q}(\log L)^\alpha} = \left(\int_0^1 [t^{1/p}(\log(e/t))^\alpha f^*(t)]^q \,\frac{dt}{t}\right)^{1/q}, \qquad \text{for } q < \infty,$$

$$\|f\|_{L^{p,\infty}(\log L)^\alpha} = \sup_{0<t<1} t^{1/p}\,(\log(e/t))^\alpha\, f^*(t), \qquad \text{for } q = \infty$$

(we put $t^{1/\infty} = 1$). It is easy to check that these spaces increase with decreasing p, increasing q and decreasing α.

Note that later we shall also need the spaces of the form $L^{\exp t^{r'}, t^r}$, where $1/r + 1/r' = 1$. It turns out that they coincide (see [EdKr]) with spaces characterized by the integral condition (used e.g. in [BW] and in [Zi], Lemma 2.10.5)

$$\int_0^1 \left(\frac{f^*(t)}{\log(e/t)}\right)^r dt < \infty,$$

which equal to $L^{\infty,r}(\log L)^{-1}$ in the [BR] notation. Also, the Zygmund space $L \log L$ equals to $L^{1,1} \log L$ and it is nothing but $L^{t \log t, t \log t}$.

Remark 6. We recall that $L^{p_1,q_1}(\log L)^{\alpha_1} \subset L^{p_2,q_2}(\log L)^{\alpha_2}$ if any of the following conditions holds:

(i) $p_1 > p_2$;
(ii) $p_1 = p_2$, $q_1 > q_2$, and $\alpha_1 + 1/q_1 > \alpha_2 + 1/q_2$;
(iii) $p_1 = p_2 < \infty$, $q_1 \leq q_2$, and $\alpha_1 \geq \alpha_2$;
(iv) $p_1 = p_2 = \infty$, $q_1 \leq q_2$, and $\alpha_1 + 1/q_1 \geq \alpha_2 + 1/q_2$

(see [BR], Theorems 9.1 and 9.3 and 9.5).

Remark 7. According to the limiting imbedding theorem due to Brézis and Wainger [BW] we have, for $N = 2$,

$$W_0^{1,2} \hookrightarrow L^{\infty,2}(\log L)^{-1}. \tag{8}$$

The latter space, as was observed above, is the Orlicz-Zygmund space $L^{\exp t^2, t^2}$, a space smaller than $L^{\exp t^2} = L^{\exp t^2, \exp t^2}$, and this interpretation of the target space in (8) gives a natural analogue to the (sublimiting) imbeddings of Sobolev spaces into Lebesgue spaces and their Lorentz refinements.

4 Decomposition of imbeddings

Let us recall our agreement that for the sake of simplicity we shall suppose that the domain B is a ball, $|B| = 1$. We shall usually omit the symbol of the domain. We are seeking for sufficient conditions for (2) and (5); we shall even find a condition stronger than (5), namely,

$$\lim_{\delta \to 0} \sup_{\substack{A \subset B \\ |A| < \delta}} \|T(V\chi_A)\| = 0. \tag{9}$$

First we shall separately consider the scale of Lorentz spaces.

Theorem 8 ([KrSc]). *Let $N \geq 3$. and $V \in L^{N/2,r}$, $N/2 \leq r < \infty$. Then (2) and (9) hold.*

We shall pass to Lorentz-Zygmund spaces and present a theorem, establishing a general sufficient condition for (2) and various sufficient conditions for (9); let us observe that the situation is not straightforward since three parameters can change. The first parameter will be kept fixed, equal to 1; its changes lead to changes too big for the fine tuning we need.

Theorem 9 ([KrSc]). *Let $N = 2$.*

(1) *The inequality (2) holds provided $V \in L^{1,\infty}(\log L)^2$.*

(2) *Let* $V \in L^{1,s}(\log L)^{\beta}$, *where either*

$$0 < s \leq 1, \qquad \beta \geq 1, \tag{10}$$

or

$$1 < s < \infty, \qquad \beta \geq 2 - 1/s, \tag{11}$$

or

$$s = \infty, \qquad \beta > 2. \tag{12}$$

Then (2) *and* (9) *hold.*

Remark 10. The *proofs* of Theorems 8 and 9 can be carried out making use of the refined Sobolev imbedding $W^{1,2} \hookrightarrow L^{2N/(N-2),s}$ for $N \geq 3$ and of the refined limiting imbedding in (8) for $N = 2$ together with conditions (necessary and sufficient) for the imbeddings of weighted Orlicz-Lorentz spaces, taking, moreover, care about the quantitative behaviour of norms of the imbeddings. The details can be found in [KrSc].

Remark 11. The space $L^{1,\infty}(\log L)^2$ can be identified with the Orlicz-Marcinkiewicz space $L^{t\log^2 t,\infty}$ and $L^{1,s}(\log L)^{\beta}$, $0 < s < \infty$, with $L^{t\log^{\beta} t, t^s}$. This can be checked easily. Indeed, considering for instance $V \in L^{1,\infty}(\log L)^2$, that is, if we have $\sup\limits_{0<t<1} t(\log(e/t))^2 V^*(t) < \infty$, then $\widetilde{F}^{-1}(t) = t(\log(e/t))^2$ near the origin, hence $F(\xi) \sim \xi(\log(e/\xi))^2$ for large values of ξ.

By way of applications we give a sufficient condition for the SUCP, relying on the SUCP theorem in [CS] invoked earlier.

Corollary 12 ([KrSc]). *The following statements are true:*

(1) *Let* $N = 3$. *Let* $V \in L^{3/2,r}$, $3/2 \leq r < \infty$. *Then the inequality* $|\Delta u| \leq V|u|$ *has the* SUCP *in* $W^{2,2}_{loc} \cap W^{1,2}_0$.
(2) *Let* $N = 2$. *Let* $V \in L^{1,s}(\log L)^{\beta}$, *where* s *and* β *satisfy any of the conditions* (10)–(12). *Then the inequality* $|\Delta u| \leq V|u|$ *has the* SUCP *in* $W^{2,2}_{loc} \cap W^{1,2}_0$.

Remark 13. The statement in (1) actually says that the size condition from Stein [St] is fulfilled under the given conditions.

If $V \in L^{1,s}(\log L)^{\beta}$, where s and β satisfy either (10) or (12), then $V \in L^{1,1}(\log L)^1$ and we recover the SUCP theorem due to Gossez and Loulit [GL]. Concerning (11) one can construct functions, which show that $L^{1,1}(\log L)^1$ and $L^{1,s}(\log L)^{2-(1/s)}$ are incomparable for $1 < s < \infty$.

Indeed, if $V(\alpha, .)$, $0 < \alpha \leq 1$, is such that

$$V^*(\alpha, t) = \frac{1}{t}\left(\log(e/t)\right)^{-2}\left(\log\left(\log(e/t)\right)\right)^{-\alpha}, \qquad \text{for } t \text{ small,}$$

then $V(\alpha,.) \notin L^{1,1}(\log L)^1$ and if $s > 1/\alpha$, then $V(\alpha,.) \in L^{1,s}(\log L)^{2-(1/s)}$. On the other hand, if $V(\tau,.)$, $0 < \tau < 1$, is such that $V^*(\tau,t) = \chi_{(0,\tau)}(t)$, then

$$\|V(\tau,.)\|_{L^{1,1}(\log L)^1} = \tau(2 - \log\tau), \qquad 0 < \tau < 1.$$

Going through some calculation one can check that

$$\lim_{\tau\to 0} \frac{\|V(\tau,.)\|^s_{L^{1,s}(\log L)^{2-(1/s)}}}{\|V(\tau,.)\|^s_{L^{1,1}(\log L)^1}} = \infty.$$

Therefore $L^{1,1}(\log L)^1$ is not continuously imbedded into $L^{1,s}(\log L)^{2-(1/s)}$ and by the closed graph theorem we get $L^{1,1}(\log L)^1 \not\subset L^{1,s}(\log L)^{2-(1/s)}$.

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