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In: Miroslav Krbec and Jaromír Kuben (eds.): Proceedings of Equadiff 10, Czechoslovak International Conference on Differential Equations and Their Applications, Prague, August 27-31, 2001, [Part 1] Invited Lectures. Masaryk University, Brno, 2002. CD-ROM issued as a complement to the journal edition Mathematica Bohemica 2002/2. pp. 81--89.

Persistent URL: http://dml.cz/dmlcz/700311

Equadiff 10, August 27–31, 2001
Prague, Czech Republic

Equadiff 10 CD ROM
Invited Lectures, pp. 81–89

Probabilistic analysis of singularities for the 3-D Navier-Stokes Equations

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MSC 2000. 76D06, 35Q30, 60H15

Keywords. Singularities, Navier-Stokes equations, Brownian motion, stationary solutions

1 Introduction and preliminary remarks

The present note is a review of the paper [12] and some element from the related works [11], [17]. see also [18] for further results. For a few general references on stochastic Navier-Stokes equations see [2], [20], [21], [9], among many others, while for general references on Navier-Stokes equations on one side and infinite dimensional stochastic analysis on the other, see [19] and [5].

1.1 Could probability tell us something new about classical problems in fluid dynamics?

This difficult challenging problem has a few positive answers and works in progress. A first example is the ergodicity for the 2D stochastic Navier-Stokes equations, proved first by [10] under some assumptions on the noise, and later on by many authors under various sets of assumptions and with different techniques, see for instance [6], [1], [14], [22]. Some ergodic properties are often tacitly assumed in

This is author's version of the invited lecture.

statistical fluid mechanics, but a proof for the deterministic Navier-Stokes equations is still out of reach, in spite of the efforts spent on outstanding theories like the Ruelle-Bowen-Sinai one.

A second example is the probabilistic analysis of singularities for the 3D deterministic and stochastic Navier-Stokes equations developed in [11], [12]. This is the subject of the present note.

Finally we mention a number of other directions like the probabilistic representations of solutions to Navier-Stokes equations, the vortex method, probabilistic model of turbulence, statistical solutions of Foias-like equations, diffusion of passive scalars, stochastic vortex filaments. Without the aim to list contributions in all these fields, we mention only [4], [13], [15], [16], [7].

Two typical tools, beyond others, are employed: 1) irreducibility, 2) stochastic stationarity. Tool (1) is usually introduced by means of a noise forcing term in the Navier-Stokes equations. It is somewhat an idealization of the real behaviour of a fluid, but it captures in a sort of idealized limit the extreme variability observed in turbulent fluids. Tool (2) has some of the technical advantages of time-invariance, even if the single realizations (trajectories) may have a very complex time evolution.

1.2 3D Navier-Stokes equations and singularities

Consider the Navier-Stokes equation in a bounded regular domain $D \subset \mathbb{R}^3$

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla P = \nu \triangle u + f + \sigma \frac{\partial B}{\partial t} \\ \operatorname{div} \ u = 0, \quad u|_{\partial D} = 0, \quad u|_{t=0} = u_0 \end{cases} \tag{1}$$

Physically speaking, u is the velocity field, P the pressure, f a slowly varying forcing term, $\frac{\partial B}{\partial t}$ a fast fluctuating forcing term. The kinematic viscosity ν is assumed to be strictly positive, while the noise intensity $\sigma \geq 0$ may be equal to zero (deterministic case), depending on the theorem.

Before giving a rigorous definition of suitable weak solution, let us mention the concept of singular points. A point $(t, x) \in (0, \infty) \times D$ will be called regular if u is locally (essentially) bounded around it. Otherwise, the point (t, x) is called *singular*. The set of singular points of u will be denoted by $S(u)$. We have $S(u) \subset (0, \infty) \times D \subset \mathbb{R}^4$. The fundamental result of Caffarelli, Kohn and Nirenberg [3] tells us that the 1-dimensional Hausdorff measure of $S(u)$ is zero, when u is a suitable weak solution:

$$\mathcal{H}^1 (S(u)) = 0.$$

This result is a refinement of previous results of Scheffer. Whether $S(u)$ is empty or not is a main open problem. It is empty for time-invariant solutions. In a sense, we shall prove that it is empty also for stochastically stationary solutions.

A singularity corresponds to a local concentration of energy. The global kinetic energy cannot blow up: for $\sigma = 0$, $\frac{1}{2} \int_D |u(t, x)|^2 \, dx$ (plus dissipation energy) is bounded by $\frac{1}{2} \int_D |u_0(x)|^2 \, dx$ plus the work done by the body forces, and the same result is true (with a more involved inequality) also for $\sigma > 0$ under reasonable assumptions on B. However, energy may concentrate, it may be transferred to smaller scales, and the (energy)/(unite volume) may blow up at some point: $\frac{1}{r^3}\int_{B_r(x_0)} |u(t,x)|^2 dx \to \infty$ as $r \to 0$ (this is an open problem). This problem is similar to the concentration of energy in finite regions that can be seen for Hamiltonian systems of ∞-many particles. Here and below we denote the ball of center x_0 and radius r by $B_r(x_0)$ or simply by B_r.

Roughly speaking, the idea of the blow-up control is the following one. On one side, we have a local energy balance of the form

$$\frac{d}{dt}\int_{B_r} \frac{|u|^2}{2} + \nu \int_{B_r} |\nabla u|^2 \le \int_{\partial B_r} \frac{|u|^2}{2} u \cdot n + \text{work done by forces}$$

which says that the local variation (possible concentration) of kinetic energy $\frac{d}{dt}\int_{B_r} \frac{|u|^2}{2}$, plus the local dissipation $\nu \int_{B_r} |\nabla u|^2$, are controlled by the energy flux $\int_{\partial B_r} \frac{|u|^2}{2} u \cdot n$ plus work terms. On the other side, we have the Sobolev inequality

$$\int_{B_r} |u|^3 \le C \left(\int_{B_r} |\nabla u|^2\right)^{\frac{3}{4}} \left(\int_{B_r} |u|^2\right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(\int_{B_r} |u|^2\right)^{\frac{3}{2}}$$

which allows us to control terms of the order of the energy flux by local kinetic and dissipation energy. These two tools together give rise to iterative *nonlinear* relations for the previous quantities, on a sequence of nested balls B_{r_n}. The resulting inequalities may be *closed* if some quantity is small. The criterium discovered by Caffarelli, Kohn and Nirenberg is that

$$\limsup_{r\to 0} \frac{1}{r} \int_{t-r^2}^{t+r^2} \int_{B_r(x)} |\nabla u|^2 = 0 \tag{2}$$

(or just smaller than a certain universal constant) implies (t,x) regular. Having established this fact, it is not difficult to prove that $\mathcal{H}^1(S(u)) = 0$.

How probability may enter this problem?

1) As for ∞-many particle Hamiltonian systems, one could try to prove a good result in a stationary regime and for many initial conditions with respect to a probability measure. This is the content of this note.

2) Perhaps the emergence of singularities requires a great degree of organization (only special fluid configurations may produce singularities). Perhaps this coherence is broken by the noise. We cannot solve this problem with a true understanding of the geometry of emerging singularities. We can only prove that in the presence of noise that activates all modes, our results hold true for most initial conditions.

1.3 Suitable weak solutions

The *martingale* suitable weak solutions for the Navier-Stokes system are solutions of a stochastic differential equation driven by an additive noise which satisfy almost surely a local balance of energy. Let H be the Hilbert space

$$H = \{u : D \to \mathbb{R}^3 \,|\, u \in \left(L^2(D)\right)^3, \,\mathrm{div} u = 0, \,(u \cdot n)\,|_{\partial D} = 0\}$$

where n is the outer normal to ∂D (see for example Temam [19]), and V be the space of all $u \in \left(H^1(D)\right)^3 \cap H$ such that $u|_{\partial D} = 0$. Define the (Stokes) operator $A : D(A) \subset H \to H$ as $Au = \mathcal{P}\triangle u$, where $\mathcal{P}$ is the orthogonal projection from $\left(L^2(D)\right)^3$ onto H and $D(A) = \left(H^2(D)\right)^3 \cap V$.

Definition 1. A martingale suitable weak solution is a process (u, P) defined on a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}, (B_t)_{t\geq 0}\right)$, where B is a Brownian motion adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ with values in $D(A^\beta)$, for a $\beta > 0$, such that

$$\omega \in \Omega \mapsto (u(\omega), P(\omega)) \in L^2(0, T; H) \times L^{\frac{5}{3}}_{loc}((0, T) \times D)$$

is a measurable mapping and there exists a set $\Omega_0 \subset \Omega$ of full probability such that for each $\omega \in \Omega_0$

$$u(\omega) \in L^\infty\left(0, T; L^2(D)\right) \cap L^2\left(0, T; H^1(D)\right), \quad P(\omega) \in L^{\frac{5}{3}}_{loc}((0, T) \times D),$$

the new variables $v(\omega) = u(\omega) - z(\omega)$ and $\pi(\omega) = P(\omega) - Q(\omega)$ satisfy the modified Navier-Stokes equations (4) below in the sense of distributions over $(0, T) \times D$, where (z, Q) is the solution to the Stokes problem (3) below. Moreover the following local energy inequality has to hold for all $\omega \in \Omega_0$

$$\int_D |v(t,\omega)|^2\varphi + 2\int_0^t \int_D \varphi|\nabla v(\omega)|^2 \leq \int_0^t \int_D |v(\omega)|^2 \left(\frac{\partial \varphi}{\partial t} + \triangle\varphi\right)$$

$$+ \int_0^t \int_D \left(|v(\omega)|^2 + 2v(\omega)\cdot z(\omega)\right)((v(\omega) + z(\omega)) \cdot \nabla\varphi)$$
$$+ 2\int_0^t \int_D \varphi z(\omega) \cdot ((v(\omega) + z(\omega)) \cdot \nabla)\, v(\omega) + \int_0^t \int_D 2\pi v(\omega) \cdot \nabla\varphi$$

for every smooth function $\varphi : \mathbb{R}^3 \times D \to \mathbb{R}$, $\varphi \geq 0$, with compact support in $(0, T] \times D$.

It is worth noticing that such solutions exist. A proof of this claim is given in [17]. Also, the concept of martingale solution is equivalent to the one of statistical solution, as given by Foias, Temam, and others. In the previous definition we did not insist on the regularity properties of the auxiliary variables (z, Q); see [12] for the details.

2 Main results

2.1 Extention of C-K-N theorem to stochastic Navier-Stokes equations

Theorem 2. *Assume* $f \in L^p((0,T \times D))$, $B \in C^{\frac{1}{2}-\varepsilon}\left([0,T]; D(A^{\frac{1}{4}+\beta})\right)$ *for some* $p > \frac{5}{2}$ *and* $\beta > \varepsilon > 0$. *Let* u *be a suitable weak solution of the deterministic Navier-Stokes equation forced by* $f + \frac{\partial B}{\partial t}$. *Then* $\mathcal{H}^1(S(u)) = 0$.

The interpretation of this statement may be: fast (distributional in time) fluctuations of the forces do not deteriorate the (upper) estimate on singularities.

The result is expressed for an individual solution of the deterministic Navier-Stokes equation forced by $f + \frac{\partial B}{\partial t}$, but if we replace the deterministic distribution $\frac{\partial \omega}{\partial t}$ by the white noise with path having the regularity specified by the theorem and interpret the equation as a stochastic equation, and if u denotes a stochastic process solving that equation in the sense of martingale suitable weak solutions, then we have $\mathcal{H}^1(S(u)) = 0$ with probability one.

About the proof, that is quite long, we only notice that one has to introduce the auxiliary Stokes system

$$\begin{cases} \dfrac{\partial z}{\partial t} + \nabla Q = \nu \triangle z + f + \sigma \frac{\partial B}{\partial t} \\ \text{div } z = 0, \quad z|_{\partial D} = 0, \quad z|_{t=0} = 0. \end{cases} \tag{3}$$

Then $v = u - z$, $\pi = P - Q$ satisfy the equation

$$\begin{cases} \dfrac{\partial v}{\partial t} + ((v+z) \cdot \nabla)(v+z) + \nabla \pi = \nu \triangle v \\ \text{div } v = 0, \quad v|_{\partial D} = 0, \quad v|_{t=0} = u_0. \end{cases} \tag{4}$$

After the necessary preliminary results on z are established, one has to adapt the proof of [3] to this new equation. See [12] for details.

2.2 Improvement for stationary solutions

We want to study stationary solutions for the Navier-Stokes equations, stationary in the sense of probability or ergodic theory. In the spirit of ergodic theory, we will speak of probability measures on the space $\mathcal{S}$ of all trajectories, invariant for the time shift.

The space $\mathcal{S} \subset L^2_{loc}([0,\infty); H) \times C\left([0,\infty); D(A^\beta)\right)$ (for a $\beta > 0$) contains all the trajectories of the stochastic processes which are solutions to the stochastic Navier-Stokes equations, namely the set of all pairs (u, B), where B is a trajectory of the fast fluctuating forcing term and u is a suitable weak solution in each time interval $[0,T]$, of the Navier-Stokes equation forced by $\partial_t B$. In this setting the pressure P is treated as an auxiliary scalar field.

We define a metric on $\mathcal{S}$ in the following way. If $\left(u^1,B^1\right)$ and $\left(u^2,B^2\right)$ are in $\mathcal{S}$, the distance between them $d\left((u^1,B^1),(u^2,B^2)\right)$ is defined as

$$\sum_{n=1}^{\infty}2^{-n}\left(1\wedge\int_0^n\left|v^1-v^2\right|^2dt\right)^{1/2}+\sum_{n=1}^{\infty}2^{-n}\left(1\wedge\sup_{(0,n)}\left|B^1-B^2\right|\right). \quad (5)$$

Let $C_b\left(\mathcal{S}\right)$ be the space of all bounded real continuous functions on $\mathcal{S}$ with the uniform topology, let $\mathcal{B}$ be the Borel σ-algebra of $(\mathcal{S},d)$ and $M_1\left(\mathcal{S}\right)$ be the set of all probability measures on $(\mathcal{S},\mathcal{B})$. Let $\tau_t:\mathcal{S}\to\mathcal{S}$, $(t\geq 0)$ be the time shift on $\mathcal{S}$, defined as

$$\tau_t\left(u,B\right)(s)=\left(u(s+t),B(t+s)-B(t)\right)$$

A statistical suitable weak solution of Navier-Stokes equations is simply a probability measure $\mu\in M_1\left(\mathcal{S}\right)$. The classical case where $(B_t)_{t\geq 0}$ is a Brownian motion is recovered simply by assuming that the marginal law of μ on the second component of $\mathcal{S}$ is the law of the Brownian motion itself. It is not difficult to re-interpret this concept by means of stochastic processes u, B satisfying the Navier-Stokes equations. To this purpose it is sufficient to consider the canonical process defined on $\mathcal{S}$, under the law μ.

Definition 3. A probability measure $\mu\in M_1\left(\mathcal{S}\right)$ is time-stationary if $\tau_t\mu=\mu$ for all $t\geq 0$. We say that μ has finite mean dissipation rate if

$$\int_{\mathcal{S}}\left[\int_0^T\int_D\left|\nabla u\right|^2dx\,dt\right]\mu(d(u,B))<\infty$$

for all $T>0$.

A proof of the existence of a stationary solution is given in [17], when the marginal measure of μ in the second component is the law of a Brownian motion. But we believe that this result may be proved also in the case when the marginal measure is the law of some other process with stationary increments, such as fractional Brownian motions, etc.

The main result in the framework of the statistical solutions is the following one. Let μ be a stationary solution. We assume that the marginal of μ in the second component is concentrated on the space $C^{\frac{1}{2}-\varepsilon}([0,\infty);D(A^{\frac{1}{4}+\beta}))$ for some $\beta>\varepsilon>0$.

Theorem 4. *Let μ be a stationary solution as above, then for every time $t\geq 0$ the set of singular points at time t is empty for μ-almost every trajectory.*

In other words, if $S_t\left(u\right)$ denotes the set of all $x\in D$ such that (t,x) is a singular point for the function u, then for all given $t\geq 0$, the set $S_t\left(u\right)$ is empty for μ-almost all trajectories (u,B).

About the proof (see [12]), by stationarity and finite mean dissipation rate we have that

$$\int_{\mathcal{S}}\left[\frac{1}{r}\int_{t-r^2}^{t+r^2}\int_D\left|\nabla u\right|^2\right]\mu(d(u,B))=Cr$$

for some constant $C > 0$, so $\frac{1}{r}\int_{t-r^2}^{t+r^2}\int_D |\nabla u|^2$ converges to zero as $r \to 0$ with probability one, by an argument based on Borel-Cantelli lemma and the monotonicity in r of the previous integral. Notice that the result is true for all $\sigma \geq 0$, hence it is uniquely due to the stationarity and not to the presence of noise.

2.3 Final results for a.e. initial conditions

¿From this theorem, a regularity result for almost any initial condition can be deduced. First we define a measure on the space H of initial conditions given by a stationary solution μ. Since weak solutions are continuous from $[0,\infty)$ to H with the weak topology, the map

$$p_0 : (u, B) \in \mathcal{S} \to u(0) \in H$$

is well defined and measurable. Hence it is possible to consider the image measure of μ with respect to p_0. Denote by μ_0 such measure on H. In a heuristic sense, μ_0 is an invariant measure in H for the Navier-Stokes equations, but we cannot state this in the usual sense since the Navier-Stokes equations does not define a dynamical system or a Markov semigroup (one may use the concept of infinitesimal invariance).

One can prove that μ disintegrates with respect to μ_0 (see the details in [12]):

$$\mu(.) = \int_H \mu(.|u(0) = u_0)\ \mu_0(du_0).$$

For μ_0-a.e. $u_0 \in H$, the measure $\mu(.|u(0) = u_0)$ is a statistical solution of the Navier-Stokes equations with initial condition u_0.

As a consequence of the previous theorem one can prove that (see [12]):

Corollary 5. *For every $t \geq 0$, for μ_0-a.e. $u_0 \in H$,*

$$S_t(u) = \emptyset \qquad \mu(.|u(0) = u_0)\text{-a.s.}$$

The interpretation is that we do not see singularities at any given time t, not only in the stationary regime (the theorem of the previous section) but also for μ_0-a.e. initial condition. Hence only special (with respect to μ_0) initial conditions may produce a certain kind of singular behaviour.

The weak point of the previous theorem could be that μ_0 is concentrated on a very poor set, like a point or a periodic orbit. In the case of a single point it means that μ was the delta Dirac mass over a time-invariant solution, and therefore the absence of singularities is a well know fact (easy consequence of the result of [3]). It is therefore interesting to know that under suitable assumptions of non-degeneracy of the noise the support of μ_0 is H. This is our first theorem in which B cannot be just a deterministic function. We assume that it is a Brownian motion in H (see [5] for the definition). We also assume that this noise force directly acts on all Fourier components, namely that the covariance is injective. Presumably this condition can be weakened. It implies a form of *irreducibility* of the dynamic, proved in [8], which implies that trajectories visit all open sets of H with positive probability.

Theorem 6. *Assume that $\sigma > 0$ and B is a Brownian motion in H with injective covariance, and with trajectories satisfying the regularity conditions of the previous theorems. Then the support of μ_0 is the full space H:*

$$\mathrm{supp}\,(\mu_0) = H.$$

Therefore the set of initial conditions having the property of the corollary is rich. This result is due to the noise, while all the previous ones hold true also for $\sigma = 0$.

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