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In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 123--126.

Persistent URL: http://dml.cz/dmlcz/700727

SEQUENTIAL ENVELOPE AND SUBSPACES OF THE ČECH-STONE COMPACTIFICATION

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A convergence $\mathfrak{L}$ for a non-empty set L is a set of pairs $(\{x_n\}, x)$ where $\{x_n\}$ is a sequence of points of L and $x \in L$, satisfying the axioms of Fréchet:

$(\mathscr{L}_0)$ If $(\{x_n\}, x) \in \mathfrak{L}$ and $(\{x_n\}, y) \in \mathfrak{L}$, then $x = y$.

$(\mathscr{L}_1)$ If $x_n = x$ for each positive integer n, then $(\{x_n\}, x) \in \mathfrak{L}$.

$(\mathscr{L}_2)$ If $(\{x_n\}, x) \in \mathfrak{L}$ and $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$, then $(\{x_{n_i}\}, x) \in \mathfrak{L}$.

A convergence $\mathfrak{L}$ can be enlarged to a convergence $\mathfrak{L}^*$ for L satisfying the axiom of Urysohn:

$(\mathscr{L}_3)$ If each subsequence $\{x_{n_i}\}$ of a sequence $\{x_n\}$ contains a subsequence $\{x_{n_{i_j}}\}$ such that $(\{x_{n_{i_j}}\}, x) \in \mathfrak{L}$, then $(\{x_n\}, x) \in \mathfrak{L}$.

For each $A \subset L$ denote

$$\lambda A = \{x : (\{x_n\}, x) \in \mathfrak{L},\ \bigcup(x_n) \subset A\}\,.$$

Then $(L, \mathfrak{L}, \lambda)$ or briefly (L, λ) is a closure space and it is called a convergence space. Notice that $\mathfrak{L}$ and $\mathfrak{L}^*$ induce the same convergence closure λ; operator λ^{ω_1} defined inductively for $\xi \leqq \omega_1$ by $\lambda^\xi A = \bigcup\limits_{\eta<\xi} \lambda(\lambda^\eta A)$, $A \subset L$, is a topology for L and $\lambda^{\omega_1} A$ is the smallest sequentially closed set in L containing A. If $\lambda^{\omega_1} A = L$, then we say that A is sequentially dense in L.

If $\langle(L_\alpha, \mathfrak{L}_\alpha, \lambda_\alpha), \alpha \in I\rangle$ is a non-empty family of convergence spaces, then $(L, \mathfrak{L}, \lambda)$, where $L = \prod\limits_{\alpha\in I} L_\alpha$, $(\{\langle x(\alpha), \alpha \in I\rangle_n\}, \langle x(\alpha), \alpha \in I\rangle) \in \mathfrak{L}$ whenever $(\{x_n(\alpha)\}, x(\alpha)) \in \mathfrak{L}^*_\alpha$, $\alpha \in I$ (coordinatewise star convergence), and λ is the induced convergence closure for L, is a convergence space (cf. [5]). We shall call it a *star convergence Cartesian product space.*

In what follows the set of all continuous functions on (L, λ) is denoted by $\mathscr{C} = \mathscr{C}(L)$, the subset of all bounded continuous functions by $\mathscr{C}^*$, and a subset of $\mathscr{C}$ by $\mathscr{C}_0$.

Definition. We say that a convergence space (L, λ) has the *property p* with respect to $\mathscr{C}_0$ if

(p) For each two sequences $\{x_n\}$, $\{y_n\}$ of points of L such that

$$(\lambda \bigcup(x_n)) \cap (\lambda \bigcup(y_n)) = \emptyset$$

there is a function $f \in \mathscr{C}_0$ such that

$$\lim f(x_n) = \lim f(y_n)$$

does not hold.

Note 1. If in (p) we have $y_n = y$, $n = 1, 2, \ldots$, then we speak about the $\mathscr{C}_0$ *sequential regularity* (Cf. [6]).

Note 2. The property p is topological.

Theorem 1. *The properties p with respect to $\mathscr{C}$ and $\mathscr{C}^*$ are equivalent.*[1])

Theorem 2. *The property p is star convergence Cartesian productive and hereditary with respect to sequentially closed subspaces.*

Theorem 3. *A convergence space associated*[2]) *with a normal or a realcompact topological space has the property p.*

Theorem 4. *A convergence space has the property p if and only if it is homeomorphic to a sequentially closed subspace of a star convergence Cartesian product of the real lines.*

Theorem 5. *A $\mathscr{C}_0$ sequentially regular convergence space (L, λ) has the property p with respect to $\mathscr{C}_0$ if and only if it cannot be $\mathscr{C}_0$ embedded*[3]) *as a sequentially dense proper subspace in any sequentially regular convergence space.*

It is well known that the Čech-Stone compactification (Hewitt realcompactification) of a completely regular topological space is the maximal completely regular space in which the original one is $\mathscr{C}^*$ embedded ($\mathscr{C}$ embedded) dense subspace. The sequential envelope of a sequentially regular convergence space introduced by J. Novák ([4]) is the maximal sequentially regular convergence space in which the original space is $\mathscr{C}^*$ embedded sequentially dense subspace. More generally, following [6]:

[1]) We speak simply of the property p in this case.

[2]) A sequence converges to a point whenever it is eventually in each topological neighborhood of the point.

[3]) A subspace (L, λ) of a convergence space (L', λ') is $\mathscr{C}_0(L)$ embedded in (L', λ') if each $f \in \mathscr{C}_0$ has a continuous extension onto the whole space.

A convergence space (S, σ) is called a $\mathscr{C}_0$ sequential envelope of a $\mathscr{C}_0$ sequentially regular convergence space (L, λ) if

1° (L, λ) is a sequentially dense subspace of (S, σ).

2° (L, λ) is $\mathscr{C}_0$ embedded in (S, σ) and (S, σ) is $\overline{\mathscr{C}}_0(S)$ sequentially regular, where

$$\overline{\mathscr{C}}_0(S) = \{f : f \in \mathscr{C}(S),\ f/L \in \mathscr{C}_0\}\,.$$

3° There is no convergence space (S', σ') containing (S, σ) as a proper subspace and fulfilling 1° and 2° with regard to (L, λ) and (S', σ').

Theorem 6. *A convergence space (S, σ) is a $\mathscr{C}_0$ sequential envelope of a $\mathscr{C}_0$ sequentially regular convergence space (L, λ) if and only if*

(i) *(L, λ) is a sequentially dense $\mathscr{C}_0$ embedded subspace of (S, σ).*

(ii) *(S, σ) has the property p with respect to $\overline{\mathscr{C}}_0(S)$.*

The Čech-Stone compactification and the Hewitt realcompactification can substantially differ, e.g. $\beta N \neq N = \upsilon N$. In comparison with this fact Theorem 1 yields the following

Theorem 7. *$\mathscr{C}^*$ sequential and $\mathscr{C}$ sequential envelopes of a sequentially regular convergence space are homeomorphic and the homeomorphism leaves the original space pointwise fixed.*

In the sequel let (L, λ) be a sequentially regular convergence space, $\tilde{\lambda}$ the completely regular modification[4]) of λ, and (P, u) the Čech-Stone compactification of $(L, \tilde{\lambda})$, where P is regarded as the set of all z-ultrafilters on $(L, \tilde{\lambda})$. For every infinite cardinal $\aleph_\alpha$ let $P_{\aleph_\alpha} \subset P = P_{\aleph_0}$ be the set of all z-ultrafilters having the $\aleph_\alpha$-intersection property (i.e. the intersections of less than $\aleph_\alpha$ members of the z-ultrafilter are non-empty). For example $(P_{\aleph_1}, u/P_{\aleph_1})$ is the Hewitt realcompactification of $(L, \tilde{\lambda})$ (cf. [2]). Observe that if (P, π) is the convergence space associated with (P, u), then $(Q, \pi/Q)$ is associated with $(Q, u/Q)$ for each non-empty subset Q of P.

V. Koutník pointed out ([3]) that the $\mathscr{C}^*$ sequential and hence, by Theorem 5, the $\mathscr{C}$ sequential envelope of (L, λ) is the smallest sequentially closed subset in the Čech-Stone compactification of $(L, \tilde{\lambda})$ which contains L endowed with the associated convergence and convergence closure.

Theorem 8. *Denote $S = \pi^{\omega_1}L$. Then $P_{\aleph_0} \supset P_{\aleph_1} \supset S$ and $(S, \pi/S)$ is a sequential envelope of (L, λ).*

There are examples such that $P_{\aleph_1} \neq S$ and $S - P_{\aleph_2} \neq \emptyset$. Consequently, Theorem 8 is in this direction the best possible result. On the other hand, it is easy

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4) $\tilde{\lambda}$ is the finest of all completely regular topologies for L coarser than λ (cf. [3]).

to verify that if $2^{\operatorname{card} L} = \aleph_\alpha$ then $P_{\aleph_{\alpha+1}} = L$. Hence the following problems might be of interest.

Problem 1. Find the least cardinal $\aleph_\alpha$ depending on card L such that $S \supset P_{\aleph_\alpha}$.

Problem 2. Is there the least cardinal $\aleph_\alpha$ such that $S \supset P_{\aleph_\alpha}$ whenever (L, λ) is a sequentially regular convergence space?

References

[1] *E. Čech:* Topological spaces. Prague, 1966.

[2] *L. Gillman and M. Jerison:* Rings of continuous functions. Princeton, 1960.

[3] *V. Koutník:* On convergence topologies. General Topology and its Relations to Modern Analysis and Algebra, II (Proc. Second Prague Topological Sympos., 1966). Academia, Prague, 1967, 226—228.

[4] *J. Novák:* On the sequential envelope. General Topology and its Relations to Modern Analysis and Algebra (I) (Proc. (First) Prague Topological Sympos., 1961). Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962, 292—294.

[5] *J. Novák:* On convergence spaces and their sequential envelopes. Czechoslovak Math. J. *15* (1965), 74—100.

[6] *J. Novák:* On some problems concerning the convergence spaces and groups. General Topology and its Relations to Modern Analysis and Algebra (Proc. Kanpur Topological Conf., 1968). Academia, Prague, 1971, 219—229.

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