E. Binz

Recent results in the functional analytic investigations of convergence spaces

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 67--72.

Persistent URL: http://dml.cz/dmlcz/700743

RECENT RESULTS IN THE FUNCTIONAL ANALYTIC INVESTIGATIONS OF CONVERGENCE SPACES

E. BINZ

Mannheim

In the past fifteen years various generalizations of the notion of a topology have appeared in connection with several branches of mathematics, mainly with functional analysis, e.g. [16]. The generalization I have in mind is that of the so-called convergence structure introduced in [13] and [15].

During the Symposium Dr. Simon brought my attention to the following two papers of Katětov: On continuity structures and spaces of mappings, Comment. Math. Univ. Carolinae *6* (1965), 257–278; Convergence structures, Proceedings of the Second Prague Topological Symposium, 1966, 207–216. Those papers are indeed closely related to several papers listed in the References. I therefore would like to thank Dr. Simon very much.

A *convergence structure* Λ is a map from a non-empty set X into the power set $P(F(X))$ of the set of all filters (in the sense of Bourbaki [9]) $F(X)$ of X assigning to each point $p \in X$ a collection of filters $\Lambda(p)$ satisfying:

(i) The filter $\dot{p}$ generated by $\{p\}$ belongs to $\Lambda(p)$,
(ii) every filter Ψ finer than a member Φ of $\Lambda(p)$ belongs to $\Lambda(p)$ and finally
(iii) the infimum $\Phi \wedge \Psi$ of any two filters Φ and Ψ of $\Lambda(p)$ belongs to $\Lambda(p)$.

The filters in $\Lambda(p)$ are called the filters converging to p with respect to Λ or simply the filters converging to p. The set X together with Λ is called a convergence space or simply a space.

Every *topology* T on the set X will be interpreted as a convergence structure in the following way: For any point $p \in X$ let $T(p)$ be the set of all filters converging to p. Hence we say that the topology T is a convergence structure. We call a convergence structure *topological* if it is a topology.

Let me now construct an example of a space which is not a topological space. A map from a space X into a space Y is said to be continuous if for each point $p \in X$ the filter $f(\Phi)$ converges to $f(p)$ for any filter Φ converging to p. The collection of all continuous real-valued functions of the space X is denoted by $C(X)$. Now we endow $C(X)$ with the continuous convergence structure Λ_c defined as follows: For any function $f \in C(X)$ the set $\Lambda_c(f)$ consists of all filters Θ for which the filter $\Theta(\Phi)$ generated by

$$\{T(F) \mid T \in \Theta \text{ and } F \in \Phi\}$$

converges to $f(p)$ for any point $p \in X$ and any filter Φ converging to p. The set $C(X)$ together with Λ_c is denoted by $C_c(X)$. The latter space is, under the pointwise defined operations, an **R**-*convergence algebra*, meaning that the operations are continuous. For some basic facts on $C_c(X)$ one may consult [5]. For a completely regular topological space X, the continuous convergence structure Λ_c is a topology iff X is locally compact, in which case Λ_c coincides with the topology of compact convergence. Hence to give an example of a space which is not topological we choose in $C_c(X)$ the space X to be the rationals with the usual topology.

In this lecture I would like to present some of the recent results concerning the investigations of the relationship between a certain type of space X and $C_c(X)$. The class of spaces we restrict ourselves to is that of all c-embedded spaces. Let me briefly explain the notion of a *c-embedded space.* For any space X let $\mathrm{Hom}_c\, C_c(X)$ denote the collection of all real-valued continuous **R**-algebra-homomorphisms from $C(X)$ onto **R**, endowed with the continuous convergence structure. The map

$$i_X : X \to \mathrm{Hom}_c\, C_c(X)$$

defined by $i_X(p) = f(p)$ for all $p \in X$ and all $f \in C_c(X)$ is a continuous surjection [6]. A space X now is said to be c-embedded if i_X is a homeomorphism, i.e. a bicontinuous bijection. As examples of c-embedded spaces let me give $C_c(Y)$ for any space Y, any completely regular topological space (satisfying T_1) and any subspace of a c-embedded space.

The reason we restrict ourselves to the class of c-embedded spaces is the fact that any two c-embedded spaces X and Y are homeomorphic iff $C_c(X)$ and $C_c(Y)$ are bicontinuously isomorphic.

Now let me present the first result [17]:

Theorem 1. *For any c-embedded space X the following conditions are equivalent:*

(i) *X is locally compact;*

(ii) $C_c(X)$ *is topological.*

If $C_c(X)$ *is topological (X being c-embedded), then* Λ_c *is the topology of compact convergence.*

What means locally compact? A space is said to be *compact* if every ultrafilter converges to a unique point. We call a space X locally compact, if every convergent filter contains a compact subset of X.

For compact c-embedded spaces we have [8]:

Theorem 2. *For any c-embedded space X the following conditions are equivalent:*

(i) *X is compact*;
(ii) *X is compact and topological*;
(iii) $C_c(X)$ *carries a norm topology.*

In a c-embedded convergence space any compact subspace is topological. Hence any c-embedded locally compact space is the inductive limit (in the category of convergence spaces) of compact topological spaces. We shall soon meet an example of a c-embedded locally compact space which is not topological.

The next two results concern completely regular topological spaces. We would like to convert the topological term "normal" of a completely regular topological space X into a functional analytic term of $C_c(X)$. The space X is normal iff the restriction map

$$r : C(X) \to C(A)$$

is surjective for any (non-empty) closed subset A of X. Let $I(A)$ denote the ideal in $C(X)$ of all functions vanishing on A. Hence we have the following commutative diagram:

$$\begin{array}{ccc} C_c(X) & \xrightarrow{\ r\ } & C(A) \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \bar{r}} & \\ C(X)/I(A) & & \end{array}$$

where π denotes the canonical projection and $\bar{r}$ the map induced by r. Hence r is surjective iff $\bar{r}$ is surjective. Now let us endow $C(X)/I(A)$ with the finest [3] of all convergence structures for which π is continuous. This space will be denoted by $C_c(X)/I(A)$. Then $\bar{r}$ is a homeomorphism onto a subspace of $C_c(A)$.

For any space Y the convergence algebra $C_c(Y)$ is complete [7], i.e. every Cauchy-filter (in the obvious sense) converges. Now there is a Stone-Weierstrass theorem [7] saying that for any completely regular topological space Y any complete subalgebra of $C_c(Y)$ inducing the topology and containing the constants is all of $C(Y)$.

Since $\bar{r}(C_c(X)/I(A))$ induces the topology of A and contains the constants, $\bar{r}$ is surjective iff $C_c(X)/I(A)$ is complete. Since any closed proper ideal in $C_c(X)$ is of the form $I(A)$ for some closed (non-empty) subset $A \subset X$, we have:

Theorem 3. *Let X be a completely regular topological space. Then X is normal iff $C_c(X)/I$ is complete for any closed (proper) ideal in $C_c(X)$.*

The next two theorems are due to W. A. Feldman [12].

Theorem 4. *Let X be a completely regular topological space. The following conditions are equivalent*:

(i) *X is metrizable and separable*;
(ii) $C_c(X)$ *is second countable.*

Second countable means the following:

There is a system S of at most countably many subsets of $C_c(X)$ such that to each filter in $C_c(X)$ there exists a coarser one (still convergent) with a basis of members of S.

To find a functional analytic equivalent of the term *metric* one may consult [2].

Theorem 5. *Let X be a c-embedded space. The following conditions are equivalent*:

(i) *X is Lindelöf;*
(ii) *$C_c(X)$ is first countable.*

The notion of Lindelöf is based on the notion of a *covering system.* A system S of subsets of X is a covering system if in every convergent filter in X there is a member of S. The space X is Lindelöf if to every covering system S there is a countable covering system S' refining (defined in the obvious way) S.

First countable means simply that to any convergent filter there exists a coarser one (still converging) with a countable basis.

We now turn our attention to some functional analytic properties of $C_c(X)$, namely to those of *duality*. Let $L_cC_c(X)$ be the c-dual space, i.e. the space of all continuous linear real-valued functions carrying the continuous convergence structure. The next three theorems are due to H. P. Butzmann [10], [11].

Theorem 6. *For any convergence space X the canonical map*

$$j : C_c(X) \to L_cL_cC_c(X)$$

(defined by $j(f) = l(f)$ for all $l \in L_cC_c(X)$ and all $f \in C(X)$) is a bicontinuous isomorphism, i.e. $C_c(X)$ is c-reflexive.

This theorem is based on the following two theorems:

Theorem 7. *Let X be a c-embedded space. The locally convex topology on $C(X)$ generated by all continuous seminorms of $C_c(X)$ is the topology of compact convergence.*

Theorem 8. *For any space X the* **R**-*vector space generated by $i_X(X)$ in $L_cC_c(X)$ is dense in $L_cC_c(X)$.*

The theory of c-duality for general convergence spaces is, except for certain special classes of such spaces [14], not developed at all. With the intention to develop such a theory my assistents Dr. H. P. Butzmann, Dr. K. Kutzler and myself began to study the c-dual spaces of topological **R**-vector spaces. Here are some of the results the proof of which can be found in [4] and [11].

Theorem 9. *A convergence* **R**-*vector space F is a c-dual space of some topological vector space iff the following conditions hold*:

(i) *F is locally compact;*
(ii) *all compact subsets in F are topological and*
(iii) *F has point-separating continuous linear functionals.*

Now we can easily give an example of a locally compact c-embedded convergence space which is not topological. Theorem 9 applied to any infinite dimensional locally convex separated vector space E asserts that the c-dual L_cE is locally compact. Clearly L_cE is c-embedded and not topological.

Theorem 10. *For any topological* **R**-*vector space E the canonical map*

$$j : E \to L_cL_cE$$

maps E homeomorphically onto a dense subspace of the complete locally convex **R**-*vector space* L_cL_cE *iff E is locally convex.*

Another branch of our studies is devoted to an extension of *Pontryagin's duality theory* for locally compact Abelian groups. The main result, due to H. P. Butzmann, links the c-dual space of an **R**-convergence vector space E of a certain type with the group Γ_cE (carrying the continuous convergence structure) of all continuous group homomorphisms of E into the unit circle T. It allows us to describe an extension of Pontryagin's duality theory for certain groups:

Theorem 11. *Let E be an* **R**-*convergence vector space in which for any filter* Φ *in E converging to zero the filter* $[-1, 1]\cdot\Phi$ *generated by* $\{[-1, 1]\cdot F \mid F \in \Phi\}$ *converges to zero, too. Then the canonical projection* $\pi : \mathbf{R} \to T$, *sending each* $\lambda \in \mathbf{R}$ *into* $e^{2\pi i\lambda}$ *induces a bicontinuous group isomorphism*

$$\pi^* : L_cE \to \Gamma_cE\,,$$

defined by $\pi^*(l) = \pi \circ l$ *for all* $l \in L_cE$. *Moreover, the canonical group homomorphism*

$$j'_E : E \to \Gamma_c\Gamma_cE$$

(defined by $j'_E(p)(g) = g(p)$ *for all* $p \in E$ *and all* $g \in \Gamma_cE$*) is a bicontinuous group isomorphism iff E is c-reflexive. Hence* $j'_{C_c(X)}$ *is a bicontinuous group isomorphism for any space X.*

For a given topological **R**-*vector space E the group homomorphism* j'_E *is a bicontinuous bijection iff E is locally convex and complete.*

References

[1] *E. Binz and H. H. Keller:* Funktionenräume in der Kategorie der Limesräume. Ann. Acad. Sci. Fenn. Ser. A I. *383* (1966), 1—21.

[2] *E. Binz and K. Kutzler:* Über metrische Räume und $C_c(X)$. Ann. Scuola Norm. Sup. Pisa *26* (1) (1972), 197—223.

[3] *E. Binz and W. A. Feldman:* A functional analytic description of normal spaces. Canad. J. Math. *24* (1) (1972), 45—49.

[4] *E. Binz, H. P. Butzmann and K. Kutzler:* Über den *c*-Dual eines topologischen Vektorraumes. Math. Z. *127* (1972), 70—74.

[5] *E. Binz:* Convergence spaces and convergence function algebras. Proc. Internat. Sympos. on Topology and its Applications (Herceg-Ňovi, 1968). Savez Društava Mat. Fiz. i Astronom., Belgrade, 1969, 87—92.

[6] *E. Binz:* Zu den Beziehungen zwischen *c*-einbettbaren Limesräumen und ihren limitierten Funktionenalgebren. Math. Ann. *181* (1969), 45—52.

[7] *E. Binz:* Notes on a characterization of function algebras. Math. Ann. *186* (1970), 314—326.

[8] *E. Binz:* Kompakte Limesräume und limitierte Funktionenalgebren. Comment. Math. Helv. *43* (1968), 195—203.

[9] *N. Bourbaki:* Topologie générale. Chapitre I, 3ème ed. Act. Sci. Ind. 1142, Paris, 1961.

[10] *H. P. Butzmann:* Dualitäten in $C_c(X)$. Ph. D. Thesis, University of Mannheim, W. Germany.

[11] *H. P. Butzmann:* Über die *c*-Reflexivität von $C_c(X)$. Comment. Math. Helv. *47* (1972), 92—101.

[12] *W. A. Feldman:* Topological spaces and their associated convergence function algebras. Ph. D. Thesis, Queen's Univ., Kingston, Canada.

[13] *H. R. Fischer:* Limesräume. Math. Ann. *137* (1959), 269—303.

[14] *H. Jarchow:* Duale Charakterisierung der Schwartz-Räume. Math. Ann. *196* (1972), 85—90.

[15] *H. J. Kowalsky:* Limesräume und Komplettierung. Math. Nachr *12* (1954), 301—340.

[16] *G. Marinescu:* Espaces vectoriels pseudotopologiques et théorie des distributions. Deutsch. Verlag Wissensch., Berlin, 1963.

[17] *M. Schroder:* Continuous convergence in a Gelfand theory for topological algebras. Ph. D. Thesis, Queen's Univ., Kingston, Canada.