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ALGEBRAS OF GERMS OF FOURIER TRANSFORMS[1])

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1. Introduction. We denote by $A(G)$, as usual, the algebra of all functions on a locally compact abelian group G that are Fourier transforms of functions in L^1 of the dual group of G. Thus, $A(G)$ is a subalgebra of the algebra $C(G)$ of all continuous complex valued functions on G with pointwise operations. Let $I_0(G)$ be the ideal of functions in $A(G)$ that vanish on a neighborhood (depending upon the function) of the identity 0 in G. The title of this paper refers to the algebra $A(G)/I_0(G)$.

It is known [8] that if G and G' are locally isomorphic groups then the respective algebra of germs $A(G)/I_0(G)$ and $A(G')/I_0(G')$ are isomorphic. J. P. Kahane asked [1, p. 352] whether, conversely, isomorphism of the algebras of germs implies local isomorphism of the groups. Spector [8] observed that this is so in case the algebra of germs is isomorphic with the complex field, i.e., that the group must then be discrete. In the present paper we prove that for a locally connected metrizable group G, if $A(G)/I_0(G)$ is isomorphic with $A(T^n)/I_0(T^n)$ then G is locally isomorphic with T^n. Here, T^n denotes the torus group of n dimensions and includes the possibility $n = \omega$, the infinite-dimensional torus.

The technique of the proof is to show that if $A(G)/I_0(G)$ is isomorphic with $A(T^n)/I_0(T^n)$, then G is n-dimensional. The known structure theorem for the component of the identity of such a group [6, p. 170] yields the desired local isomorphism. Thus, the technique consists of extracting information about the topology of G from the algebraic structure of $A(G)/I_0(G)$, and it provides a partial solution of Kahane's problem when the topology of G in a neighborhood of 0 determines its local group structure. Although we have not yet succeeded in doing so, it appears plausible that the requirements of metrizability and of local connectivity can both be dispensed with, that is, that the hypothesis that $A(G)/I_0(G)$ is isomorphic with $A(T^n)/I_0(T^n)$ already implies that G is metrizable and locally connected.

The analogous question of extracting the local topology of a space X from the ring $C(X)$ of all continuous functions in place of $A(G)$ was treated in [4]. The present proof differs from the one in [4] only in that the detailed knowledge of all the prime ideals of a ring $C(X)$ as developed by Kohls [5] is replaced by the topological space

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of *minimal* prime ideals of an arbitrary commutative ring as presented at the first Symposium on General Topology [3].

The symbol G will denote an arbitrary locally compact abelian group. Additional restrictions will be placed on G only as they are needed.

2. Proposition. *A prime ideal in $A(G)$ is contained in the maximal ideal*

$$M_0 = \{f \in A(G) : f(0) = 0\}$$

if and only if it contains I_0.

Proof. Let P be a prime ideal contained in M_0 and let f be any member of I_0. There exists $g \in A(G)$ whose support is contained in $f^{-1}(0)$ and with $g(0) = 1$ [7, Theorem 2.6.2]. Then $fg = 0 \in P$. But g does not belong to P. Since P is prime, $f \in P$. For the converse, we prove, more generally, that if g is any function in $A(G)$ that is not in M_0 then the smallest ideal $I = (I_0, g)$ containing I_0 and g is all of $A(G)$. For, if f is any function in $A(G)$ then by [7, Lemma 7.2.2(a)], there exists $f' \in I$ that agrees with f on some neighborhood of 0. This means that $f - f' \in I_0 \subset I$; whence, $f \in I$.

Remark. It follows from this proposition with 0 replaced by an arbitrary element of G that if a prime ideal of $A(G)$ is contained in a regular maximal ideal then the latter is unique. In case G is not compact, there will be prime ideals that are contained in no regular maximal ideal, namely those prime ideals that contain all functions with compact support.

We recall some definitions from [2]. A *zero-set* in a topological space X is a set of the form $f^{-1}(0)$ for some $f \in C(X)$. A *z-filter* on X is a family $\mathscr{F}$ of zero-sets that satisfies the conditions (i) $\emptyset \notin \mathscr{F}$; (ii) $A, B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$; (iii) $A \in \mathscr{F}$ and $B \supset A$ implies $B \in \mathscr{F}$. A z-filter $\mathscr{F}$ *converges* to $p \in X$ if every neighborhood of p contains a member of $\mathscr{F}$. A z-filter $\mathscr{F}$ is *prime* if $A \cup B \in \mathscr{F}$ implies $A \in \mathscr{F}$ or $B \in \mathscr{F}$. If the intersection of all the members of a prime z-filter $\mathscr{F}$ on a completely regular space is nonempty, then the intersection consists of a single point and $\mathscr{F}$ converges to that point [2, Theorem 3.17]. A maximal z-filter is called a *z-ultrafilter*; it is necessarily prime.

3. Proposition. *For any $\Phi \in C(G)$ and any compact neighborhood V of 0 in G, there exists $f \in A(G)$ such that $f^{-1}(0) \cap V = \Phi^{-1}(0) \cap V$.*

Proof. The set $\Phi^{-1}(0)$ is a closed G_δ in G, for example, $\Phi^{-1}(0) = \bigcap_{n=1}^{\infty} U_n$, where $U_n = \{x : |\Phi(x)| < 1/n\}$. For each n let f_n be a function in $A(G)$ that is 1 on $V - U_n$ (set complement), 0 on $\Phi^{-1}(0) \cap V$, and $\geqq 0$ throughout G [7, p. 49]. Then the function f given by the formula

$$f = \sum_{n=1}^{\infty} 2^{-n} f_n / \|f_n\| ,$$

where $\|f_n\|$ is the L^1-norm of the function whose Fourier transform is f_n, belongs to $A(G)$ which is a Banach algebra under this norm. And f meets the requirement of the proposition.

4. Proposition. *For each prime z-filter* $\mathscr{F}$ *on G converging to* 0, *the set*

$$P(\mathscr{F}) = \{f \in A(G) : f^{-1}(0) \in \mathscr{F}\}$$

is a prime ideal that contains I_0. *The correspondence* $\mathscr{F} \to P(\mathscr{F})$ *is injective and preserves set-theoretic inclusion.*

We omit the routine proof, but make several observations. a) Not every prime ideal containing I_0 is of the form $P(\mathscr{F})$. b) The maximal ideal M_0 is $P(\mathscr{F})$, where $\mathscr{F}$ is the z-ultrafilter consisting of all zero-sets containing the identity. c) Proposition 3 is used to prove that the correspondence is injective.

5. Proposition. *Every minimal prime ideal contained in* M_0 *is of the form* $P(\mathscr{F})$ *for some minimal prime z-filter* $\mathscr{F}$.

Proof. Given a minimal prime ideal $P \subset M_0$, let $\mathscr{F}$ be the z-filter consisting of all zero-sets that contain any set of the form $V \cap f^{-1}(0)$, V a neighborhood of 0 and $f \in P$. We prove first that $P = P(\mathscr{F})$. Obviously, $P \subset P(\mathscr{F})$. Suppose $g \in P(\mathscr{F})$. Then there exist $f \in P$ and a neighborhood V of 0 such that $g^{-1}(0) \supset V \cap f^{-1}(0)$. Since P is a minimal prime ideal, there exists [3, Lemma 1.1] h in $A(G)$, $h \notin P$, with $fh = 0$. It follows that gh vanishes on the neighborhood V; whence $gh \in I_0 \subset P$. Since P is a prime ideal, we have $g \in P$. Next, we prove that $\mathscr{F}$ is prime. Suppose, then that A and B are zero-sets and $A \cup B \supset V \cap f^{-1}(0)$, V a neighborhood of 0, which we may assume to be a compact zero-set, and $f \in P$. By Proposition 3, there exist $g, h \in A(G)$ such that $A \cap V = g^{-1}(0) \cap V$ and $B \cap V = h^{-1}(0) \cap V$. Then $(A \cup B) \cap \cap V = (gh)^{-1}(0) \cap V \supset V \cap f^{-1}(0) \in \mathscr{F}$. It follows that $gh \in P(\mathscr{F}) = P$. Therefore, either $g \in P$ or $h \in P$. Consequently, either $A \in \mathscr{F}$ or $B \in \mathscr{F}$. It is clear that $\mathscr{F}$ converges to 0, and minimality of $\mathscr{F}$ follows immediately from Proposition 4.

6. Theorem. *Let G be metrizable and nondiscrete. In the topological space* $\mathscr{P}$ *of minimal prime ideals of the algebra* $A(G)/I_0(G)$, *the relation* $P_1 \sim P_2$ *defined by the condition: the ideal* (P_1, P_2) *generated by* P_1 *and* P_2 *is not the (unique) maximal ideal in* $A(G)/I_0(G)$, *is an equivalence relation. The quotient space of* $\mathscr{P}$ *relative to this relation is homeomorphic with the space* $Y = \beta(V - \{0\}) - (V - \{0\})$, *where V is any compact neighborhood of* 0 *in G.*

Proof. The first step is to pass from the space $\mathscr{P}$ of minimal prime ideals of $A(G)/I_0(G)$ to the space $\mathscr{P}'$ of minimal prime ideals of $A(G)$ that contain $I_0(G)$. By [3, Theorem 2.1] the natural mapping $\tau : \mathscr{P}' \to \mathscr{P}$ by $\tau(P) = P/I_0$ is a homeomorphism of $\mathscr{P}'$ onto a subset of $\mathscr{P}$. In this case, $\tau[\mathscr{P}']$ is all of $\mathscr{P}$, however. For, any member of $\mathscr{P}$ has the form P/I_0, where P is a prime ideal in $A(G)$ containing I_0. The issue is whether P is minimal. But, if P_0 is a prime ideal contained in P, then $P_0 \subset M_0$, whence $P_0 \supset I_0$. Therefore, P_0/I_0 is defined and is a prime ideal in $A(G)/I_0(G)$ contained in P/I_0. Since P/I_0 is minimal, $P_0/I_0 = P/I_0$, so that $P_0 = P$ and P is minimal.

$\mathscr{P}'$ is evidently homeomorphic with the space $\mathscr{Q}$ of all minimal prime z-filters on G that converge to 0, topologized as follows: the set of all z-filters containing a given zero-set is a basic closed set.

Given any compact neighborhood V of 0 in G, let $V' = V - \{0\}$. If a prime z-filter $\mathscr{F}$ belongs to $\mathscr{Q}$ then each member of $\mathscr{F}$ meets V'. For, consider any $f \in P(\mathscr{F})$. Since $P(\mathscr{F})$ is a minimal prime ideal in $A(G)$, there exists [3, Lemma 1.1] $g \in A(G)$ with $g \notin P(\mathscr{F})$, and $fg = 0$. But then $g \notin I_0$, so there exists $x \in V$ with $g(x) \neq 0$. Since 0 is not an isolated point of G and g is continuous, there exists such an $x \neq 0$. And $x \in V' \cap f^{-1}(0)$. As a consequence, the trace of $\mathscr{F}$ on V' generates a z-filter on V'. As a matter of fact, the trace of $\mathscr{F}$ on V' is itself a *minimal prime z-filter, distinct members of $\mathscr{Q}$ have distinct traces on V', and every minimal prime z-filter on V' that converges to $0 \in V$ is the trace of a member of $\mathscr{Q}$.* All of this follows from one simple circumstance, namely that every zero-set in V' is the intersection with V' of a zero-set in V. It is precisely for this purpose that G is assumed metrizable (cf. also [2, p. 48, 3C.2]).

Each minimal prime z-filter on V' is contained in a unique z-ultrafilter on V', and the z-ultrafilters are in one-one correspondence with the points of $\beta V'$. Moreover, because V is the one-point compactification of V', the minimal prime z-filters that converge to $0 \in V$ are contained in those z-ultrafilters on V' that correspond to the points of $\beta V' - V' = Y$. Let $\psi : \mathscr{P} \to Y$ denote the composition of all the natural mappings that we have been discussing:

$$\mathscr{P} \to \mathscr{P}' \to \mathscr{Q} \to \mathscr{Q}|_{V'} \to Y.$$

The first two of these mappings are homeomorphisms, as has already been observed. The third is also a homeomorphism with respect to the topology for spaces of prime z-filters described above. And the last mapping is continuous because the topology. of $\beta V'$ corresponds to the above topology in the space of z-ultrafilters [2, p. 87]. Thus, ψ is a continuous mapping of $\mathscr{P}$ onto Y. Two ideals P_1 and P_2 in $\mathscr{P}$ satisfy $\psi(P_1) = \psi(P_2)$ if and only if the corresponding minimal prime z-filters on V' are contained in the same z-ultrafilter on V'. The set P' of all functions $f \in A(G)$ such that $f^{-1}(0) \cap V$ belongs to such a z-ultrafilter is a prime ideal in $A(G)$, contained in M_0. But $P' \neq M_0$ because there is a function in $A(G)$ whose only zero within V is at 0. ($\{0\}$ is a zero-set in G.) Clearly, $(P_1, P_2) \subset P'/I_0$. Therefore, $\psi(P_1) = \psi(P_2)$ if and only if $P_1 \sim P_2$. Moreover, the space $\mathscr{P}$ is compact (cf. [3, Theorem 5.6]). Consequently, Y is homeomorphic with the quotient space of $\mathscr{P}$ as stated.

7. Theorem. *Let G be metrizable and locally connected. If $A(G)/I_0(G)$ is isomorphic with $A(T^n)/I_0(T^n)$ for some $1 \leqq n \leqq \omega$, then G is locally isomorphic with T^n.*

Proof. By applying Theorem 6 to both G and T^n, we draw the conclusion that the space Y for the group G is homeomorphic with the space $S_n = \beta(T^n - \{0\}) - (T^n - \{0\})$. By [4, Theorem 2] dim $S_n = n$ if n is finite, and the same argument shows that S_ω is infinite dimensional. According to Pontrjagin's structure theorem [6, p. 170], every locally connected metrizable group is locally isomorphic with a torus group of some dimension. We just have seen that the algebra of germs $A(G)/I_0(G)$ determines that dimension.

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