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In: Zdeněk Frolík (ed.): Abstracta. 4th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1976. pp. 95–108.

Persistent URL: <http://dml.cz/dmlcz/701052>

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FOURTH WINTER SCHOOL (1976)

A SURVEY ON LIFTING IN MEASURE THEORY

by

Siegfried GRAF

Def.: For a Boolean algebra \mathcal{A} and an ideal $\mathcal{H} \subset \mathcal{A}$ with $1 \notin \mathcal{H}$ a lifting is a map $\varphi: \mathcal{A} \rightarrow \mathcal{A}$, s.t. (i) $A \sim B \Rightarrow \varphi(A) = \varphi(B)$, (ii) $A \sim \varphi(A)$, (iii) $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$, (iv) $\varphi(A \cup B) = \varphi(A) \cup \varphi(B)$, (v) $\varphi(0) = 0, \varphi(1) = 1$ for all $A, B \in \mathcal{A}$. ($A \sim B$ means $A \Delta B \in \mathcal{H}$)

A density is a map $\varphi: \mathcal{A} \rightarrow \mathcal{A}$ which satisfies (i), (ii), (iii), and (v).

A monotone lifting φ only satisfies (i), (ii), and $A \subset B \Rightarrow \varphi(A) \subset \varphi(B)$ for all $A, B \in \mathcal{A}$.

For a set X , a field $\mathcal{A} \subset \mathcal{P}(X)$ and an ideal $\mathcal{H} \subset \mathcal{A}$ with $X \notin \mathcal{H}$ let $\mathcal{L}^\infty(X, \mathcal{A})$ be the space of bounded, real-valued, \mathcal{A} -meas. functions on X with sup-norm.

A lifting for $\mathcal{L}^\infty(X, \mathcal{A})$ w.r.t. \mathcal{H} is a map $\ell: \mathcal{L}^\infty(X, \mathcal{A}) \rightarrow \mathcal{L}^\infty(X, \mathcal{A})$ s.t. (i) $f \sim g \Rightarrow \ell(f) = \ell(g)$, (ii) $\ell(f) \sim f$, (iii) ℓ linear, (iv) ℓ multiplicative, (v) $\forall \alpha \in \mathbb{R}: \ell(\alpha) = \alpha$

A linear lifting only satisfies (i), (ii), and (iii), while a monotone linear lifting is monotone and satisfies (v) in addition.

For a linear lifting $\ell: \mathcal{L}^\infty(X, \mathcal{A}) \rightarrow \mathcal{L}^\infty(X, \mathcal{A})$ we define $\|\ell\| = \sup \{\|\ell(f)\|: \|f\|_{\mathcal{H}} \leq 1\}$, where $\|\ell(f)\|$ denotes

the sup-norm of $\ell(f)$ and $\|f\|_{\mathcal{M}} := \inf \{ \alpha \in \mathbb{R} : |f| \geq \alpha \cdot 3 \in \mathcal{M} \}$.

Prop. 1: (Ionescu-Tulcea 1965) \mathcal{A} σ -field, \mathcal{M} σ -ideal. If L is a lifting for \mathcal{A} w.r.t. \mathcal{M} then there exists exactly one lifting ℓ for $\mathcal{L}^{\infty}(X, \mathcal{A})$, s.t. $1_{L(A)} = \ell(1_A) \forall A \in \mathcal{A}$, and vice versa.

I. On the existence of densities and liftings

Let \mathcal{A} be a Boolean algebra and $\mathcal{M} \subset \mathcal{A}$ an ideal.

Theorem 1: (v. Neumann-Stone 1935)

If \mathcal{M} is k -complete for all $k < \text{card}(\mathcal{A} \setminus \mathcal{M})$, then there is a lifting for \mathcal{A} w.r.t. \mathcal{M} .

Theorem 2: (v. Neumann-Stone 1935)

If there is a density for \mathcal{A} w.r.t. \mathcal{M} , and if \mathcal{M} is conditionally- k -complete for all $k < \text{card}(\mathcal{A} \setminus \mathcal{M})$, then there is a lifting for \mathcal{A} w.r.t. \mathcal{M} .

Remark: (v. Weizsäcker 1975)

There exists a Boolean algebra \mathcal{A} and an ideal $\mathcal{M} \subset \mathcal{A}$, s.t. there is a density φ for \mathcal{A} w.r.t. \mathcal{M} but no lifting ψ with $\varphi(A) \subset \psi(A) \forall A \in \mathcal{A}$.

Problem: Is there a lifting for \mathcal{A} w.r.t. \mathcal{M} , if there is a density for \mathcal{A} w.r.t. \mathcal{M} ?

Corollary 1: (Gapaillard 1972)

If $\mathcal{A} \subset \mathcal{P}(X)$ is a field and $\mathcal{M} \subset \mathcal{A}$ an ideal in $\mathcal{P}(X)$ with $X \notin \mathcal{M}$, s.t. there exists a density for \mathcal{A} w.r.t. \mathcal{M} , then there is a lifting for \mathcal{A} w.r.t. \mathcal{M} .

Corollary 2: (Graf 1972)

If X is a top. space, \mathcal{O} the σ -field of all sets with Baire property in X and \mathcal{M} the σ -ideal of all sets of first category, then there is a lifting for \mathcal{O} w.r.t. \mathcal{M} .

Theorem 3: (v. Weizsäcker-Graf 1973)

If (X, \mathcal{O}, μ) is a σ -finite measure space and $\mathcal{M} = \{A \in \mathcal{O} : \mu(A) = 0\}$, then there exists a lower density for \mathcal{O} w.r.t. \mathcal{M} .

Corollary: (von Neumann 1931, D. Maharam 1958)

If (X, \mathcal{O}, μ) is complete in addition, then there is a lifting for \mathcal{O} w.r.t. \mathcal{M} .

Remarks: The question of existence of a lifting for the unit interval with Lebesgue measure was raised by Haar and positively answered by von Neumann in 1931. In 1958 D. Maharam generalized the theorem to arbitrary σ -finite measure spaces. In 1974 Erdős showed that there is a finitely additive measure on $\mathcal{P}(\mathbb{N})$, s.t. there is no lifting for $\mathcal{P}(\mathbb{N})$ w.r.t. $\{\mu = 0\}$.

Problems:

Which pairs (\mathcal{O}, μ) admit a lifting (density)?

To be more specific: Let \mathcal{O} be a Boolean σ -algebra, $\mathcal{M} \subset \mathcal{O}$ a σ -ideal, s.t. $\mathcal{O}|\mathcal{M}$ is weakly countably distributive and satisfies the countable chain condition. Does a density (lifting) for \mathcal{O} w.r.t. \mathcal{M} exist?

Does every σ -finite measure space admit a lifting?

II. Conditions for a complete measure space, which are equivalent to the existence of a lifting

Let (X, \mathcal{A}, μ) be a complete measure space, s.t. $\mathcal{A} = \{A \subset X : \forall E \in \mathcal{A} : \mu(E) < \infty \Rightarrow E \cap A \in \mathcal{A}\}$ and define $\mathcal{M} := \{A \in \mathcal{A} : \forall E \in \mathcal{A} : \mu(E) < \infty \Rightarrow \mu(E \cap A) = 0\}$.

(X, \mathcal{A}, μ) has the lifting property (LP), monotone lifting property (MLP) or the density property (DP), iff there is a lifting, monotone lifting, density for \mathcal{A} w.r.t. \mathcal{M} .

a) Decomposition of a measure space

Def.: $\mathcal{Z} \subset \mathcal{A}$ is called a decomposition, iff (i) $\forall Z, Z' \in \mathcal{Z} : Z + Z' \Rightarrow Z \cap Z' = \emptyset$, (ii) $\forall Z \in \mathcal{Z} : 0 < \mu(Z) < \infty$, and (iii) $\forall A \in \mathcal{A} (\mu(A) < \infty \text{ and } \forall Z \in \mathcal{Z} : \mu(A \cap Z) = 0) \Rightarrow \mu(A) = 0$.

Remark: Every Radon-measure-space has a decomposition (of compact sets).

Theorem: The following are equivalent (T.F.A.E.):

- (a) (X, \mathcal{A}, μ) has LP; (b) (X, \mathcal{A}, μ) has DP; (c) (X, \mathcal{A}, μ) has MLP; (d) there is a decomposition for (X, \mathcal{A}, μ) ;
- (e) $\mathcal{L}^\infty(X, \mathcal{A})$ has a linear lifting \mathcal{L} with $\|\mathcal{L}\| < 2$
- (f) $\mathcal{L}^\infty(X, \mathcal{A})$ has a monotone linear lifting.

Remark:

(a) \Leftrightarrow (b) \Leftrightarrow (d) \Leftrightarrow (f) was proved by Ionescu-Tulcea and K6litzow 1968.

(d) \Leftrightarrow (e) was proved by Strauss 1974

(c) \Leftrightarrow (d) was proved by Gapaillard in 1971.

b) Radon-Nikodym theorem

Def.: (X, \mathcal{A}, μ) has the Radon-Nikodym property (RNP), iff for every measure ν on \mathcal{A} , s.t. $\nu(N) = 0$ for all $N \in \mathcal{A}$ with $\mu(N) = 0$ (i.e. ν is μ -continuous), there is an

\mathcal{A} -measurable $f: X \rightarrow [0, +\infty]$, s.t. $\forall A \in \mathcal{A}: \mu(A) < \infty \Rightarrow (\int_A f d\mu < \infty \iff \nu(A) < \infty)$ and $\nu(A) = \int_A f d\mu$.
 f is called a derivative of ν w.r.t. μ .

Remark: (Segal 1951)

(X, \mathcal{A}, μ) RNP $\iff \mathcal{A} \upharpoonright \mu$ complete lattice $\iff L^\infty(X, \mathcal{A}, \mu)$ cond. complete lattice.

Prop. 1: (Köszor 1968)

$$(X, \mathcal{A}, \mu) \text{ LP} \implies (X, \mathcal{A}, \mu) \text{ RNP}$$

Remark: (Fremlin 1973)

The converse of the above theorem does not hold.

Def.: (X, \mathcal{A}, μ) has the monotone (linear) RNP, iff for every measure ν on \mathcal{A} with $\nu \leq \alpha\mu$ for some $\alpha \in \mathbb{R}_+$ there is a derivative f_ν , s.t. for any two of those measures ν_1, ν_2 we have $\nu_1 \leq \nu_2 \implies f_{\nu_1} \leq f_{\nu_2}$

$$(f_{\alpha_1\nu_1 + \alpha_2\nu_2}) = \alpha_1 f_{\nu_1} + \alpha_2 f_{\nu_2} \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}_+.$$

Theorem: (Köszor 1968)

T.F.A.E.: (i) (X, \mathcal{A}, μ) has LP; (ii) (X, \mathcal{A}, μ) has the monotone RNP; (iii) (X, \mathcal{A}, μ) has the linear RNP.

c) Riesz theorem

Def.: (X, \mathcal{A}, μ) has the Riesz property (RP), iff $\forall g \in L^1(X, \mathcal{A}, \mu) \exists f_g \in L^\infty(X, \mathcal{A}, \mu)$, s.t. $\forall f \in \mathfrak{L}^1(X, \mathcal{A}, \mu): \varphi(f) = \int f f_g d\mu$.

The map $\varphi \mapsto f_\varphi$ is called an R-differentiation.

(X, \mathcal{A}, μ) has the monotone (linear) RP, iff there is a monotone (linear) R-differentiation.

Remark: (Segal 1951)

(X, \mathcal{A}, μ) has RP $\iff (X, \mathcal{A}, \mu)$ has RNP

Theorem: (Közlöw 1968)

T.F.A.E.: (i) (X, \mathcal{A}, μ) has LP; (ii) (X, \mathcal{A}, μ) has the monotone RP; (iii) (X, \mathcal{A}, μ) has the linear RP.

d) Dunford-Pettis theorem

Def.: (X, \mathcal{A}, μ) has the Dunford-Pettis property (DPP), iff

for all Banach spaces B and all bounded linear maps \mathcal{U} :

$: L^1(X, \mathcal{A}, \mu) \rightarrow B'$ there is a weak*-measurable $f_{\mathcal{U}} : X \rightarrow B'$, s.t.

$$\forall f \in L^1(X, \mathcal{A}, \mu) \quad \forall b \in B: [\mathcal{U}(f)](b) = \int f(x) [f_{\mathcal{U}}(x)](b) d\mu(x).$$

The map $\mathcal{U} \mapsto f_{\mathcal{U}}$ is called a DP-differentiation.

(X, \mathcal{A}, μ) has the linear (isometric) DPP, iff there is always a linear (isometric) DP-differentiation. Here $\mathcal{U} \mapsto f_{\mathcal{U}}$ is called isometric, iff

$$\|\mathcal{U}\| = \inf \{ \alpha \in \mathbb{R} : \{ x \in X : \|f_{\mathcal{U}}(x)\| \geq \alpha \} \in \mathcal{N} \}$$

Theorem: (Dieudonné 1951, Ionescu-Tulcea 1962)

T.F.A.E.: (i) (X, \mathcal{A}, μ) has the LP; (ii) (X, \mathcal{A}, μ) has the linear DPP; (iii) (X, \mathcal{A}, μ) has the isometric DPP.

e) Vitali differentiation systems

Def.: For $x \in X$ let $\overline{\mathcal{U}}(x) \subset \{ \mathcal{G} \mid \mathcal{G} \in \mathcal{A} : 0 < \mu(\mathcal{G}) < \infty \}$,

s.t.: $\forall \mathcal{G}' \in \mathcal{A} : 0 < \mu(\mathcal{G}') < \infty$, s.t. $\exists \mathcal{G} \in \overline{\mathcal{U}}(x) : \mathcal{G}' \subset \mathcal{G}$

and $\forall \mathcal{G} \in \overline{\mathcal{U}}(x) \exists \mathcal{G}' \in \mathcal{A} : \mathcal{G}' \subset \mathcal{G}$, then $\mathcal{G}' \in \overline{\mathcal{U}}(x)$. In this case $\overline{\mathcal{U}}(x)$ is called a differentiation system for x .

$\overline{\mathcal{U}} = (\overline{\mathcal{U}}(x))_{x \in D}$ is called a differentiation system for (X, \mathcal{A}, μ) , iff $D \in \mathcal{A}$, $X \setminus D \in \mathcal{N}$ and $\overline{\mathcal{U}}(x)$ is a diff. system for x for all $x \in D$.

For $B \subset X$, $\mathcal{U} \subset \mathcal{A}$ is an $\overline{\mathcal{U}}$ -cover for B, iff there is an

$\mathcal{N} \in \mathcal{U}$ s.t. $B \setminus N \subset D$ and $\forall x \in B \setminus N \exists \mathcal{O}_x \in \overline{\mathcal{U}}(x)$ with $\mathcal{O}_x \cap N = \emptyset$

$\mathcal{M} \subset \mathcal{O}$ is a strong (weak) Vitali cover for B , iff $\forall \varepsilon > 0 \forall C \subset B: 0 < \mu^*(C) < \infty \Rightarrow \exists (V_n)_n \subset \mathcal{M}$ s.t.
 $V_n \cap V_m = \emptyset$ ($n \neq m$), $\mu^*(C \setminus \bigcup_n V_n) = 0$ and
 $\mu^*(\bigcup_n V_n \setminus C) < \varepsilon$ (resp. $\mu^*(C \setminus \bigcup_n V_n) = 0$ and
 $(\sum_n \mu(V_n)) - \mu(\bigcup_n V_n) < \varepsilon$).

A differentiation system $\overline{\mathcal{U}} = (\overline{\mathcal{U}}(x))_{x \in D}$ for (X, \mathcal{O}, μ) is called strong (weak) Vitali system iff for every $B \subset X$ and for every $\overline{\mathcal{U}}$ -cover \mathcal{M} for B , \mathcal{M} is a strong (weak) Vitali cover.

Theorem: (Köszöw 1968)

T.F.A.E.: (i) (X, \mathcal{O}, μ) has the LP; (ii) \exists strong Vitali system for (X, \mathcal{O}, μ) ; (iii) \exists weak Vitali system for (X, \mathcal{O}, μ) .

Remark:

Applications of Vitali systems to differentiation of semi-group-valued measures and integral representations of operators can be found in Sion: A theory of semigroup valued measures, Lecture Notes 355(1974).

f) Lifting topologies and category measure

Prop.: (Gapaillard 1972)

Let m be a monotone lifting for \mathcal{L}^∞ , $(f_i)_{i \in I}$ a filtering increasing family in \mathcal{L}^∞ , s.t. $f_i \leq m(f_i) \leq g \in \mathcal{L}^\infty$. Then $\sup f_i \in \mathcal{L}^\infty$.

Corollary: (Maharam 1958)

If D is a lower density for \mathcal{O} w.r.t. \mathcal{U} , then $\bigcup_{i \in I} A_i \in \mathcal{O}$

for any family $(A_i)_{i \in I}$ in \mathcal{A} with $A_i \subset D(A_i)$ for all $i \in I$.

Def.: For a density D let $\tau_D = \{A \in \mathcal{A} \mid A \subset D(A)\}$.

Theorem: (A. Ionescu-Tulcea 1967)

τ_D is a topology on X , s.t.

(i) $\tau_D \cap \mathcal{M} = \{\emptyset\}$

(ii) $\forall A \in \mathcal{A} \exists \mathcal{U} \in \tau_D: A \Delta \mathcal{U} \in \mathcal{M}$

(iii) $K \subset X$ is of first category $\iff K$ closed and nowhere dense $\iff K \in \mathcal{M}$

Def.: (X, \mathcal{A}, μ) is called a category measure space iff there exists a topology \mathcal{T} on X , s.t. $\mathcal{A} = \{\text{sets with Baire property w.r.t. } \mathcal{T}\}$ and $\mathcal{M} = \{\text{sets of first category w.r.t. } \mathcal{T}\}$.

Prop.: (Graf 1973)

T.F.A.E.: (i) (X, \mathcal{A}, μ) has the IP; (ii) (X, \mathcal{A}, μ) is a category measure space; (iii) there exists a topology on X , which satisfies (i) & (ii) of the above proposition.

III. Further applications of liftings

a) Disintegration of measures

Let S be a top. space, $\mathcal{B}(S)$ the Borel field of S and (X, \mathcal{A}) a measurable space.

Theorem: (Valadier 1974, Maharam 1973, Saint-Pierre 1975 et al.)

Let $\lambda : \mathcal{A} \otimes \mathcal{B}(S) \rightarrow [0, \infty]$ be a measure, s.t. $\mu = p_X(\lambda)$ and $\nu = p_S(\lambda)$ have the following properties:

(i) (X, \mathcal{A}, μ) has the IP; (ii) ν is a Radon meas. Then there is a family $(\nu_x)_{x \in X}$ of Radon measures on S , s.t.

$x \mapsto \nu_x(B)$ is \mathcal{A} -measurable for all $B \in \mathcal{B}(S)$ and more-

over $\lambda(A \times B) = \int_A \nu_x(B) d\mu(x)$ for all $A \in \mathcal{A}$ with $\mu(A) < \infty$.

Corollary:

Let $\nu: \mathcal{B}(S) \rightarrow [0, \infty]$ be a Radon measure and $f: S \rightarrow X$ a $\mathcal{B}(S)$ - \mathcal{A} -measurable map, s.t. (X, \mathcal{A}, μ) with $\mu = f(\nu)$ has the LP.

Then there is a family $(\nu_x)_{x \in X}$ of Radon measures on S , s.t. $x \mapsto \nu_x(B)$ is \mathcal{A} -measurable for all $B \in \mathcal{B}(S)$ and moreover $\int_A \nu_x(B) d\mu(x) = \nu(B \cap f^{-1}(A))$ for all $A \in \mathcal{A}$ with $\mu(A) < \infty$.

b) Strassen's theorem

Let B be a Banach space, $p: B \rightarrow \mathbb{R}$ sublinear. Then p is continuous if and only if $\|p\| = \sup\{|p(b)| : b \in B \text{ and } \|b\| \leq 1\} < \infty$.

Let ℓ be a lifting for $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$.

Theorem: (Strassen 1965, Ionescu-Tulcea 1968)

Let $(p_x)_{x \in X}$ be a family of continuous sublinear functionals on B , s.t.

- (a) $\forall b \in B: x \mapsto p_x(b)$ is \mathcal{A} -measurable
- (b) $\int \|p_x\| d\mu(x) < \infty$ and $x \mapsto \|p_x\|$ in \mathcal{L}^∞ .
- (c) $\exists N \in \mathcal{N} \forall b \in B \forall x \in X \setminus N: \ell(t \mapsto p_x(b))(x) \leq p_x(b)$

Define $q: B \rightarrow \mathbb{R}$ by $q(b) := \int p_x(b) d\mu(x)$.

Then q is a continuous sublinear functional and for $q \in B'$ we have:

$q \leq q' \iff \exists$ family $(\lambda_x)_{x \in X}$ in B' , s.t. $x \mapsto \lambda_x(b)$ is \mathcal{A} -measurable, $x \mapsto \|\lambda_x\|$ is in \mathcal{L}^∞ and $\lambda_x \leq p_x$ for all $x \in X$.

c) Endomorphisms of L^∞ induced by point-mappings

Let X, Y be locally compact spaces, μ Radon-measure on X and ν a Radon measure on Y . Define $\mu': \mathcal{B}(X) \rightarrow [0, \infty]$ by $\mu'(B) = \sup \{ \mu(K) : K \subset B \}$ and ν' in an analogous way:

Def.: Let $T: L^\infty(X, \mu) \rightarrow L^\infty(Y, \nu)$ be a Banach algebra homomorphism. T is called normal iff $T1 = 1$ and $T(\sup_{i \in I} f_i) = \sup_{i \in I} T(f_i)$ for all families $(f_i)_{i \in I}$ in $L^\infty(X, \mu)$.

Theorem: (Ionescu-Tulcea 1965, Vesterström-Vils 1968).

Let $T: L^\infty(X, \mu) \rightarrow L^\infty(Y, \nu)$ be a normal Banach algebra homomorphism. Then there is a $\mu: Y \rightarrow X$, s.t.

- (i) $\forall f: X \rightarrow \mathbb{R}$ with compact support and continuous $f \circ \mu$ is μ' -measurable
- (ii) $\forall N \subset Y$ ν' -nullset: $\mu^{-1}(N)$ is a μ' -nullset
- (iii) $\forall \tilde{f} \in L^\infty(X, \mu): \tilde{T}\tilde{f} = \widetilde{f \circ \mu}$.

IV. Liftings with additional properties

a) Strong liftings

Let (X, \mathcal{T}) be a top. space, \mathcal{A} a σ -field on X with $\mathcal{T} \subset \mathcal{A}$ and $\mathcal{M} \subset \mathcal{A}$ an ideal.

Def.: A lifting (lower density) φ for \mathcal{A} w.r.t. μ is called strong, iff $\mathcal{U} \subset \varphi(\mathcal{U})$ for all $\mathcal{U} \in \mathcal{T}$.

Lemma: (Ionescu-Tulcea)

Let (X, \mathcal{T}) be completely regular, L a lifting for \mathcal{A} w.r.t. μ and \mathcal{L} the corresponding lifting for $\mathcal{L}^\infty(X, \mathcal{A})$. Then:

L is strong if and only if $\forall f \in \mathcal{L}_D(X): \mathcal{L}(f) = f$.

Theorem: (Graf 1974)

Let μ be a σ -finite measure on \mathcal{A} , s.t. $\mu(U) > 0$
 $\forall U \in \mathcal{T} \setminus \{\emptyset\}$, and (X, \mathcal{T}) second countable. Then there
 is a strong lower density for (X, \mathcal{A}, μ) .

Corollary:

If (X, \mathcal{A}, μ) is complete in addition, then there is a strong
 lifting for (X, \mathcal{A}, μ) .

Remark: In the case where (X, \mathcal{T}) is completely regular or
 metrizable, the above corollary was proved by several people,
 for instance by Ionescu-Tulcea, Sion, and Kellerer.

Prop.: (Ionescu-Tulcea 1969)

Let X be a locally compact space and ν a Radon measure on X .
 There exists a strong lifting for (X, ν) if and only if there
 is a decomposition $(K_j)_{j \in J}$ of (X, ν) , s.t. K_j is compact, $K_j =$
 $= \text{supp } \nu|_{K_j}$, and $(K_j, \nu|_{K_j})$ has a strong lifting.

Corollary: (Ionescu-Tulcea)

If X is a metrizable locally compact space and ν Radon mea-
 sure on X with $\text{supp } \nu = X$, then there is a strong lifting for
 (X, ν) .

Problem: Let X be a locally compact space, ν a Radon measu-
 re on X with $\text{supp } \nu = X$. Does (X, ν) have a strong lifting?
 Due to the above proposition it is enough to solve the problem
 for compact spaces. Bichteler and C. Ionescu-Tulcea even redu-
 ced the problem to products of two-point-spaces and products
 of unit intervals resp.

Application: Strict disintegration of measures

Theorem: (Ionescu-Tulcea)

Let X, S be compact and $f: S \rightarrow X$ continuous, onto.

Furthermore let ν be a Radon measure on S , ($\mu = f(\nu)$).

If there is a strong lifting for $(X, \mathcal{B}_{\mu}(X), \mu)$, then there is a family $(\nu_x)_{x \in X}$ of Radon measures on S , s.t. $x \mapsto \nu_x(B)$ is $\mathcal{B}_{\mu}(X)$ -measurable for all $B \in \mathcal{B}(S)$, $\text{supp } \nu_x \subset f^{-1}(X)$ and $\forall A \in \mathcal{B}(X): \nu(B \cap f^{-1}(A)) = \int_A \nu_x(B) d\mu(x)$.

Remark: The above result generalizes to the case where S and X are locally compact and f is Luzin-measurable.

b) Borel liftings

Let X be a top. space and $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ a measure.

A lifting for $(X, \mathcal{B}(X), \mu)$ is called a Borel lifting.

Theorem: (v. Neumann-Stone 1935 using continuum hypothesis)

If X is a second countable top. space, then there is a Borel lifting for $(X, \mathcal{B}(X), \mu)$.

Problem:

Does every Radon measure on a compact space have a Borel lifting?

Maher proved that the problem can be reduced to the products of unit intervals.

c) Invariant liftings

Let (X, \mathcal{A}, μ) be a measure space and S a set of bijective, bi-measurable mappings $g: X \rightarrow X$ with $g^{-1}(\mu) = \mu$.

Def.: A lifting (density) $\varphi: \mathcal{A} \rightarrow \mathcal{A}$ is called S -invariant iff $\forall A \in \mathcal{A} \forall g \in S: \varphi(g(A)) = g(\varphi(A))$.

Theorem: (A. Ionescu-Tulcea)

Let S be an amenable group. Then (X, \mathcal{A}, μ) has an S -invariant density if and only if (X, \mathcal{A}, μ) has an S -invariant lifting.

Corollary:

If S is a countable, amenable group, then (X, \mathcal{A}, μ) has an S -invariant lifting.

Theorem: (Ionescu-Tulcea 1967)

Let X be a loc. comp. group, μ Haar measure on X , and S the group of left (resp. right) translations on X (i.e. $X \cong S$). Then there exists an S -invariant lifting for (X, μ) . Such a lifting is always strong.

Remark: (v. Weizsäcker 1975)

Let (X, μ) be as in the theorem, $S \subsetneq S'$. Then there is no S' -invariant lifting for (X, μ) .

V. Theorems on the non-existence of liftings

Theorem: (von Neumann 1931, Ionescu-Tulcea)

If (X, \mathcal{A}, μ) is a measure space, which is not atomic, then there is no monotone linear lifting $\ell : \mathcal{L}_p(X, \mathcal{A}, \mu) \rightarrow \mathcal{L}_p(X, \mathcal{A}, \mu)$.

Theorem:

If (X, \mathcal{A}, μ) is as in the above theorem, then there is no lifting L for \mathcal{A} , s.t. $\forall (A_m)_{m \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}} : \bigcap_{m \in \mathbb{N}} L(A_m) = L(\bigcap_{m \in \mathbb{N}} A_m)$.

VI. Liftings for mappings with values in a top. space

Let (X, \mathcal{A}, μ) be a measure space and E a completely regular space.

Def.: $f: X \rightarrow E$ measurable: $\iff \forall g \in \mathcal{C}_b(E) : g \circ f$ measurable
 $\mathcal{L}_E^{\infty} := \mathcal{L}_E^{\infty}(X, \mathcal{A}, \mu) := \{f \in E^X \mid f \text{ meas.}, f(X) \text{ relatively comp.}\}$

Theorem: (Ionescu-Tulcea 1969)

Let l be a lifting for $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$.

Then there is a uniquely determined map $l_E: \mathcal{L}_E^\infty \rightarrow \mathcal{L}_E^\infty$,
s.t.

$$(i) \quad \forall g \in \mathcal{C}_b(E) \quad \forall f \in \mathcal{L}_E^\infty: g \circ f \sim g \circ l_E(f)$$

$$(ii) \quad (\forall g \in \mathcal{C}_b(E): g \circ f \sim g \circ f') \implies l_E(f) = l_E(f') \\ \forall f, f' \in \mathcal{L}_E^\infty$$

$$(iii) \quad \forall f \in \mathcal{L}_E^\infty \quad \forall g \in \mathcal{C}_b(E): l(g \circ f) = g \circ l_E(f).$$

Application:

Theorem: (Ionescu-Tulcea 1969)

If $(X_t)_{t \in T} \subset \mathcal{L}_E^\infty$ is an E-valued stochastic process on (X, \mathcal{A}, μ) ($T \subset \mathbb{R}$ interval), l, l_E as above. Then $(Y_t)_{t \in T}$ with $Y_t = l_E(X_t)$ is a separable modification of $(X_t)_{t \in T}$.

Problems:

Let E be a Banach space (ordered Banach space)

$\mathcal{L}_E^\infty(X, \mathcal{A}, \mu)$ = Vector space of Bochner-measurable E-valued functions on X.

Is there a linear (monotone linear) lifting for

$\mathcal{L}_E^\infty(X, \mathcal{A}, \mu)$?

Can the Banach spaces be characterized, s.t. such a lifting always exists?

What about the analogous questions for bounded weakly measurable E-valued maps?