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SOME THEOREMS ON MEASURABLE AND CONTINUOUS SELECTIONS AND
SOME APPLICATIONS

by

G. MÁGERL

Let X, Y be sets ($\neq \emptyset$), $\mathcal{A} \subseteq \mathcal{P}(X)$, $\mathcal{B} \subseteq \mathcal{P}(Y)$. Call $\Phi : X \rightarrow Y$ a correspondence, iff $\Phi(x)$ is a nonempty subset of Y , $\forall x$. Call a map $f : X \rightarrow Y$ (correspondence $\Phi : X \rightarrow Y$) \mathcal{A} - \mathcal{B} -measurable, iff $f^{-1}(B) \in \mathcal{A}$ ($\Phi^{-1}(B) = \{x \mid \Phi(x) \cap B \neq \emptyset\} \in \mathcal{A}$) $\forall B \in \mathcal{B}$. $f : X \rightarrow Y$ is a selection of Φ , iff $f(x) \in \Phi(x) \forall x$.

Definition. Y topological space, $\sigma : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$ a map such that $\sigma(Y) = Y$. Then $E(X, Y, \sigma) \iff \exists Z \subseteq Y$ dense, \forall coverings $\{A_z \mid z \in Z\}$ of X ($A_z \in \mathcal{A}$) $\exists f : X \rightarrow Y$ \mathcal{A} - \mathcal{U} -measurable (\mathcal{U} = open sets) such that $f(x) \in \sigma\{z \mid x \in A_z\} \forall x \in X$.

Examples are:

(X, \mathcal{A}) measurable space, Y Polish, $\sigma = \text{id}$, then $E(X, Y, \sigma)$.
 X paracompact, $\mathcal{A} = \mathcal{U}$, Y locally convex space, $\sigma(T) = \text{conv}(T)$ ($T \subseteq Y$) then $E(X, Y, \sigma)$.

From this one gets a simultaneous proof of Theorems of KURATOWSKI/RYLL-NARDZEWSKI on measurable selections and MICHAEL on continuous selections, namely.

Suppose \mathcal{A} finite, countable \mathcal{U} -stable, Y complete

metric, (V_n) a fundamental sequence of entourages for Y such that $\sigma(V_n(y)) \subseteq V_n(y) \quad \forall n \quad \forall y, V_n \circ V_n \subseteq V_{n-1}, V_n$ symmetric, $V_n(y)$ open. Suppose $E(X, Y, \sigma)$ is true. Let $\Phi : X \rightarrow Y$ be an \mathcal{U} - \mathcal{V} -measurable correspondence such that $\forall x \quad \forall n \quad \sigma(V_n(\Phi(x))) \subseteq V_n(\Phi(x))$. Then $\exists f : X \rightarrow Y$ \mathcal{U} - \mathcal{V} -measurable, such that $f(x) \in \overline{\Phi(x)} \quad \forall x$.

As a consequence we get:

Theorem (KURATOWSKI/RYLL-NARDZEWSKI).

(X, \mathcal{U}) measurable space, Y Polish, $\Phi : X \rightarrow Y$ \mathcal{U} - \mathcal{V} -measurable (or \mathcal{U} - \mathcal{F} -measurable, \mathcal{F} = closed sets) with closed values, then Φ has an \mathcal{U} - $\mathcal{L}(Y)$ -measurable selection ($\mathcal{L}(Y)$ = Borel subsets of Y).

Corollary. X, Y topological spaces, $\Phi : X \rightarrow Y$, such that $G(\Phi) = \{(x, y) \mid y \in \Phi(x)\}$ is the Hausdorff continuous image of a Polish space, μ a Borel measure on X , then Φ has a $\mathcal{L}(X)^* - \mathcal{L}(Y)$ -measurable selection ($\mathcal{L}(X)^* =$ Caratheodory completion of $\mathcal{L}(X)$); iff X is locally compact and μ a Radon measure, then one can replace $\mathcal{L}(X)^*$ by \mathcal{M} , the μ -measurable sets.

Applications: Implicit function theorems (FILIPPOV's Lemma) and BANG-BANG-principles in control theory, integration of correspondences. Extensions of measures and preimages of measures.

Theorem (YERSHOV, 1970, LUBIN, 1974, IANDERS/ROGGE, 1974).

(1) Suppose X Souslin space, $\mathcal{L} \subseteq \mathcal{L}(X)$ countably generated. Then every measure on \mathcal{L} has an extension to $\mathcal{L}(X)$.

(2) X, Y Souslin spaces, $f: X \rightarrow Y$ onto, Borel map, μ Borel measure on Y , then \exists Borel measure ν on X , such that $f(\nu) = \mu$.

Proposition (proved for compact metric spaces and continuous f by EISELE, 1975).

X Luzin, Y Souslin, μ Borel measure on Y , $f: X \rightarrow Y$ onto, Borel, if the preimage of μ under f is unique, then $\mu(\{y \mid \text{card } f^{-1}(y) \geq 2\}) = 0$.

The converse holds for compact spaces and continuous maps.

Continuous selections

Theorem (MICHAEL, 1956) (follows from the abstract Lemma).

X paracompact, Y Fréchet, $\Phi: X \rightarrow Y$ lower semicontinuous (i.e. \mathcal{U} - \mathcal{U} -measurable) correspondence with closed convex values, then Φ has a continuous selection.

Corollary. X paracompact, $Y \subseteq$ locally convex space E , compact convex metrizable, $\Phi: X \rightarrow Y$ lower semicontinuous correspondence with closed convex values. Then Φ has a continuous selection.

Remark. Metrizability is essential (v. WEIZSÄCKER, 1975).

Applications

Paracompact spaces are characterized by the above selection property. If X, Y are as above, every continuous function $f: A \rightarrow Y$ ($A \subseteq X$ closed) has a continuous extension to X .

Averaging operator in the sense of KELLEY

X compact metric, $\mathcal{A} \subseteq C(X)$ a subalgebra, R the equivalence relation on X induced by \mathcal{A} ($x \sim y \iff \forall a \in \mathcal{A} : a(x) = a(y)$). Suppose the projection $\Pi : X \rightarrow X/R$ is open and X/R is Hausdorff, then $\exists T : C(X) \rightarrow \mathcal{A}$, $\|T\| = 1$, $T \geq 0$, $T^2 = T$ such that

$$\forall f \in C(X) \quad \forall a \in \mathcal{A} : T(f \cdot a) = T f \cdot a \text{ (averaging equation).}$$

Using selections BLUMENTHAL/LINDENSTRAUSS/PHRIPS (1965) show:

Theorem.

X compact metric, Y compact, $T : C(X) \rightarrow C(Y)$ linear, $\|T\| \leq 1$. Then T is extreme iff $\exists \varphi : Y \rightarrow X$ continuous, $\lambda \in C(Y)$, $\lambda^2 = 1$, such that $\forall f \in C(X) \quad T f = \lambda \cdot (f \circ \varphi)$.

Characterization of a class of compact convex sets.

$E \subseteq$ locally convex space E compact, convex, call K regular, iff there exists $\varphi : K \rightarrow M_+^1(K)$ w^* -continuous, such that $\varphi(x)$ is a maximal representing measure for $x \in K$.

Theorem.

$\dim K \neq 3$, $\text{extr } K \text{ closed} \implies K \text{ regular}$ ($K \text{ regular} \implies \text{extr } K \text{ closed}$, always).

$\dim K = \infty$, $\text{extr } K \text{ closed} \implies K \text{ regular}$.

$\exists K \text{ regular}$, $\dim K = \infty$, K not a Bauer simplex (these are trivially regular).

K is regular iff a generalized Dirichlet problem is solvable, namely $\exists B \subseteq C(K)$ closed subspace with Choquet boundary $\text{extr } K$

such that $\forall f \in C(\text{extr } K) \exists T f \in B$ such that $Tf|_{\text{extr } K} = f$,
 a affine $\implies T(a)|_{\text{extr } K} = a$ ($\implies T$ linear, positive, iso-
 metric).

Metrizible CE-compact convex sets are regular (K is CE-com-
 pact iff the barycenter map $r: M_+^1(K) \rightarrow K$ is open, LIMA,
 O'BRIEN)

E Banach space, $G \subseteq E$, $P_G(x) = \{y \in G \mid \|x - y\| =$
 $= \|x - G\|\}$ (metric projection). If P_G is a correspondence
 (i.e. $P_G(x) \neq \emptyset \forall x$) the existence of continuous selections
 for this correspondence characterizes P_G for certain G (IA-
 ZAR/MORRIS/WULBERT, NÜRNBERGER) and in a certain sense the
 so called Lindenstrauss spaces (products of $L^1(\mu)$ -spaces).