Paul Ressel

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POSITIVE DEFINITE FUNCTIONS ON ABELIAN SEMIGROUPS

by

Paul RESSEL

The lecture concerns common work, done in København by Christian BERG, Jens Peter Reus CHRISTENSEN and myself.

Let $(S,+)$ be an abelian semigroup with neutral element 0.

Def. $f: S \longrightarrow \mathbb{R}$ is positive definite iff f is bounded and

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j f(t_i + t_j) \geq 0 \qquad \forall (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$$
$$\forall (t_1,\dots,t_n) \in S^n$$
$$\forall n \in \mathbb{N} .$$

$\varrho : S \to [-1,1]$ is a semicharacter:

$$*\begin{cases} (1) & \varrho(0) = 1 \\ (2) & \varrho(s+t) = \varrho(s)\varrho(t) \quad \forall s,t \in S . \end{cases}$$

$\hat{S} := \{\varrho : \varrho$ is semicharacter on $S\} \subseteq [-1,1]^S$ is a compact abelian semigroup in the topology of pointwise convergence.

Example: $S = \mathbb{N}_0 := \{0,1,2,\dots\}$ with addition.

$$\begin{aligned} [-1,1] &\longrightarrow \hat{\mathbb{N}}_0 \\ a &\longmapsto (n \longrightarrow a^n) \end{aligned}$$ is a topol. semigroup isomorphism.

$\mathcal{P} = \mathcal{P}(S) := \{f : f$ is positive definite on $S\}$

$\mathcal{P}_1 := \{f \in \mathcal{P} : f(0) = 1\}$

Lemma: $f \in \mathcal{P} \succ \sup_{s\ S} |f(s)| = f(0)$. In particular we get that $\mathcal{P}$ is closed and $\mathcal{P}_1$ is compact. Of course $\hat{S} \subseteq \mathcal{P}_1$.

Theorem. $\mathcal{P}_1$ is a Choquet simplex and extr $(\mathcal{P}_1) = \hat{S}$. In particular $\forall f \in \mathcal{P} \exists !$ Radon measure $\mu \in M_+(\hat{S})$ giving the desintegration

$$f(s) = \int_{\hat{S}} \rho(s)\, d\mu(\rho) \qquad \forall\ s \in S.$$

Def. $\psi : S \longrightarrow [0,\infty[$ is called negative definite iff

$$(\psi(s_i) + \psi(s_j) - \psi(s_i + s_j))_{i,j=1,\ldots,n}$$

is pos. semidef. $\forall (s_1,\ldots,s_n) \in S^n$, $\forall n \in \mathbb{N}$.

Proposition. Let $\psi : S \longrightarrow [0,\infty[$. Then the following are equivalent:

(i) $\psi \in \mathcal{N}$

(ii) $e^{-t\psi} \in \mathcal{P} \qquad \forall\ t > 0$

(iii) $\sum_1^m \alpha_i = 0 \succ \sum_{i,j} \alpha_i \alpha_j \psi(s_i + s_j) \leq 0.$

Here $\mathcal{N}$ denotes the cone of all neg. def. functions.

Theorem. Let $\psi \in \mathcal{N}$. Then there are uniquely determined

1) $c \in [0, \infty[$

2) $h: S \longrightarrow [0, \infty[$ additive

3) a non-negative Radon measure μ on $\hat{S} - \{1\}$ such that

$$\psi(s) = c + h(s) + \int_{\hat{S}\setminus\{1\}} (1 - \rho(s))\, d\mu(\rho) \quad \forall s \in S.$$

Here $c = \psi(0)$ and $h(s) = \lim_{n \to \infty} \frac{\psi(ns)}{n}$.

Let $f: S \longrightarrow [0, \infty[$, $a_1, \ldots, a_n \in S$.

$$\nabla_1 f(s; a_1) := f(s) - f(s + a_1)$$

$$\nabla_n f(s; a_1, \ldots, a_n) := \nabla_{n-1} f(s\ a_1, \ldots, a_{n-1}) -$$

$$- \nabla_{n-1}(s + a_n; a_1, \ldots, a_{n-1})$$

Def. (CHOQUET)

f is called monotone of infinite order:

$* \ \nabla_n f(s; a_1, \ldots, a_n) \geq 0$

f is called alternating of infinite order:

$* \ \nabla_n f(s; a_1, \ldots, a_n) \leq 0$

$\forall$ s, $a_1, \ldots, a_n \in S$ and $\forall\, n \in \mathbb{N}$.

Theorem. a) $\mathcal{M} \subseteq \mathcal{P}$, $\mathcal{M}$ is an extreme subcone of $\mathcal{P}$.

b) $\mathcal{A} \subseteq \mathcal{N}$, $\mathcal{A}$ " " " " " $\mathcal{N}$.

c) If S is 2-divisible (i.e. $\forall\, s \in S\ \exists\ t \in S$: $s = 2t$) then

$$\mathcal{M} = \mathcal{P} \quad \text{and} \quad \mathcal{A} = \mathcal{N}.$$

Here $\mathcal{M}(\mathcal{A})$ stands for the cone of monotone (alterating) functions of infinite order.

Theorem. Let $\psi \in \mathcal{N}$ have the representation

$$\psi(s) = c + h(s) + \int_{\hat{S}\setminus\{1\}} (1 - \varphi(s))\, d\mu(\varphi).$$

Then $\psi \in \mathcal{A}$ iff μ is concentrated on $(\hat{S} - \{1\})_+$.

Applications.

1) The classical Laplace-Transformation.

Theorem. $f: \mathbb{R}_+^n \longrightarrow \mathbb{R}$ is Laplace-Transform of a finite non-negative measure on $\mathbb{R}_+^n$ iff f is continuous and positive definite.

2) The semigroup $([0,1], \wedge)$.

Proposition. a) f is positive definite $\Leftrightarrow$ $f \geq 0$ and f is increasing

b) f is negative definite $\Leftrightarrow$ $f \geq 0$ and f is decreasing.

3) The semigroup $(L_1^\infty([0,1]), \cdot)$.

We mean the unit ball in L^∞ with multiplication of equivalence classes and the $\sigma(L^\infty, L^1)$ - topology. It is a compact metrizable space, but the semigroup operation is only separately continuous.

$$\varphi : L_1^\infty([0,1]) \longrightarrow \mathbb{R} \quad , \quad \varphi(f) := \int_0^1 f(t)\, dt$$

is continuous and pos. def., but the unique representing prob. measure on $\widehat{L_1^\infty}$ can be shown to be concentrated on a compact subset of the semicharacters, none of which is continuous in the neutral element of L_1^∞ .

Open Problem: Is this pathology impossible, if the semigroup is for ex. compact (or locally compact) and the addition is jointly continuous ?