FOURTH WINTER SCHOOL (1976)

ON THE DIFFERENTIATION OF CONVEX FUNCTIONS

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If f is a real continuous convex function which domain D_f is an open convex subset of a Banach space B, we denote by $N(f)$ the set of all points $x \in D_f$ at which f is not Gâteaux differentiable.

Definition: A set $X \subset B$ is called $(C - C)$-hypersurface if there exist a closed hyperplane $H \subset B$, a vector $v \notin H - H$ and two continuous convex functions f, g defined on H such that

$$X = \{x + (f(x) - g(x)) \cdot v, \quad x \in H\}$$

Theorem (i) If f is a continuous convex function in a separable Banach space then $N(f)$ can be covered by a countable union of $(C - C)$-hypersurfaces.

(ii) If $A \subset B$ is a countable union of $(C - C)$-hypersurfaces then there exists a continuous convex function f on B such that $A \subset N(f)$.

Remark: It follows from independent results of N. Aronszajn and R. Phelps on Gâteaux differentiation of Lipschitz functions that $N(f)$ and consequently any $(C - C)$-hypersurface in separable Banach space is of measure zero for any Gaussian measure on B. But I am not able to prove it directly using the above theorem.