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2-CATEGORICAL TOOLS IN THE THEORY OF CONCRETE CATEGORIES

by

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Let X be a category. Denote by D_X a 2-category objects of which are couples (A,U) where U: A $\longrightarrow$ X is a functor, arrows are couples (F,φ): $(A,U) \longrightarrow (B,V)$ where F: A $\longrightarrow$ B is a functor and $\varphi : U \longrightarrow VF$ a natural transformation and 2-cells $\alpha : (F,\varphi) \longrightarrow (F',\varphi')$ are natural transformations $\alpha : F \longrightarrow F'$ such that $\varphi' = V\alpha \cdot \varphi$. Let E_X be a sub-2-category of D_X having the same objects as D_X such that an arrow (F,φ): $(A,U) \longrightarrow (B,V)$ in D_X belongs to E_X if and only if $VF = U$ and $\varphi = 1_U$ and any 2-cell in D_X between arrows of E_X belongs to E_X.

The 2-category E_X is quite usual, D_X was considered, for instance, by R.Guitart. There are two important choices of X: the one-morphisms category $\mathbb{1}$ and the category Set of sets. $D_1 = E_1 =$ Cat is the 2-category of categories and D_{Set} or E_{Set} contains the 2-category $\bar{D}$ or $\bar{E}$ resp. of concrete categories (i.e. of faithful functors U: A $\longrightarrow$ Set). Our aim is to get consequences for concrete categories by making the theory of categories in D_X. All presented results can be found in [4]. The themes a) and b) are partly considered in [5].

a) <u>Extensions of functors:</u> Let us have two arrows K: M $\longrightarrow$ A and T: M $\longrightarrow$ B in a 2-category $\underline{C}$. A couple L,λ consisting of an arrow L: A $\longrightarrow$ B and a 2-cell λ : T $\longrightarrow$ LK is called a left extension of T along K if for any S: A $\longrightarrow$ B and σ: T $\longrightarrow$ SK there is a unique 2-cell α : L $\longrightarrow$ S such that $\alpha K \cdot \lambda = \sigma$.

Left extensions in Cat are left Kan extensions of functors. Left extensions in E_X are useful for the study of liftings of functors, extensions of full embeddings etc. Constructions of them are described in [3], [4] and [5]. One construction of left extensions in D_X is given in [1]. There is another construction of left extensions in D_X which calculates La for each $a \in A$ by a suitable universal property. For that reason left extensions obtained by it will be called pointwise.

b) <u>Liftings of monads:</u> Let $K: M \longrightarrow A$ be an arrow in a 2-category $\underline{C}$. A left extension of K along K is a comonad in $\underline{C}$ which is called a density comonad. The construction of a density comonad in D_X can be parametrized by comonads in X. This procedure can be adapted for the study of liftings of comonads (or monads by the duality) in the similar manner as we have treated liftings of functors in a).

c) <u>Inductive generation:</u> An arrow $K: M \longrightarrow A$ in a 2-category $\underline{C}$ is called dense if 1_A is the pointwise left extension of K along K. Dense subcategories in $\bar{D}$ are precisely inductive generating subcategories in the sense of [3]. This fact establishes the expected relevance between inductive generation and density in Cat.

d) <u>Pointwise extensions:</u> The preceding theme uses pointwise left extensions which can be defined in any 2-category $\underline{C}$ (see [6]). This definition is based on comma objects. A couple of arrows in $\underline{C}$ with a common domain (codomain) is called a span (an opspan) in $\underline{C}$. A comma object for an opspan $A \xrightarrow{F} C \xleftarrow{G} B$ is a span $A \xleftarrow{D_0} F/G \xrightarrow{D_1} B$ together with a 2-cell $\lambda : FD_0 \longrightarrow GD_1$ which are universal among these data.

Pointwise left extensions in D_X in the sense of a) are pointwise in the sense of the theory of 2-categories if one uses comma objects in D_X for opspans in E_X. Both comma objects in E_X and comma objects in D_X for general opspans give a too strong concept. Pointwise left extensions in D_1 agree with pointwise left Kan extensions.

e) Initial completion: The Yoneda embedding $A \longrightarrow Cat(A^{op}, Set)$ can be internally characterized by a suitable universal property using comma objects (see [7]). The corresponding universal property in D_X (but using the same comma objects as in d) - i.e. comma objects in D_X for opspans in E_X - instead of general ones) determines "a Yoneda embedding" in D_X. The role of the category of functors from A^{op} to Set is in D_X for a given (A,U) played by (A_X, U_X), where A_X has objects (F, δ, x) where $F: A^{op} \longrightarrow Set$ is a functor, $x \in X$ and $\delta: F \longrightarrow X(U-,x)$ is a natural transformation and U_X assigns x to (F, δ, x). If we restrict ourselves to $\bar{D}$, then we have to take only (F, δ, x) such that δ is mono (i.e. F is a subfunctor of $X(U-,x)$) and the corresponding "Yoneda embedding" $(A,U) \longrightarrow (\bar{A}, \bar{U})$ is precisely the initial completion in the sense of [2].

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