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Completeness and continuity of lattice-seminorms

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Let $(G,+,\leqslant)$ be a commutative lattice-ordered group (l-group). For $a, a_n \in G^+$ we write

$$a = \sum_{n=1}^{\infty} a_n \text{ if } a = \bigvee_{k=1}^{\infty} \sum_{i=1}^{k} a_i$$

and

$$a \leqslant' \sum_{n=1}^{\infty} a_n \text{ if } a = \bigvee_{k=1}^{\infty} \left(a \wedge \sum_{i=1}^{k} a_i \right) .$$

Next, for $a, a_n \in G$ we introduce the notatio $a \sim \{a_n\}$ whic means that

$$|a-a_n| \leqslant' \sum_{i=n}^{\infty} |a_{i+1}-a_i| \text{ holds for all } n = 1, \ldots .$$

We define G to be weakly σ-compl te if for every sequence $\{a_n\}$ in G there exists an element a in G satisfying $a \sim \{ _n\}$.

Example 1. If $G = R^S$ (all functions $f : S \to R$ with h pointwise + and $\leqslant$), then $f \sim \{f_n\}$ holds if and only if

$$(*) \quad \sum_{i=1}^{\infty} |f_{i+1}(s)-f_i(s)| < \infty \text{ implies } f(s) = \lim_{n\to\infty} f_n(s) \quad (s \in S).$$

It follows that R^S is weakly σ-complete.

Let L be a subgroup-sublattice (=l-subgroup) of G, and let ν be a lattice-seminorm (= l-seminorm) on L, that is a seminorm $\nu : L \to [0,\infty]$ satisfying also

$$\nu(a) \leqslant \nu(b) \text{ whenever } |a| \leqslant |b|.$$

The l-seminorm ν is called σ-subadditive if

$$|a| = \sum_{n=1}^{\infty} |a_n| \text{ (in G) implies } \nu(a) \leqslant \sum_{n=1}^{\infty} \nu(a_n),$$

and is called C-complete if it is complete (more precisely, if the corresponding semimetric topology is complete) and

$$|a| = \bigvee_{n=1}^{\infty} |a_n| \text{ (in G) and } \nu(a_n) = 0 \text{ for all n imply } \nu(a) = 0.$$

Theorem 1. If ν is C-complete, then ν is σ-subadditive. If ν is σ-subadditive, L = G and G is weakly σ-complete, then ν is C-complete.

Theorem 2. Suppose G is weakly σ-complete. Then ν is σ-subadditive if and only if it can be extended to a C-complete l-seminorm (on an l-subgroup of G).

In the following we assume that G is weakly σ-complete and σ-subadditive, and we consider the canonical C-complete σ-subadditive extensions ν^* and $\bar{\nu}$ of ν defined as follows:

$$\nu^*(a) = \inf \left\{ \sum_{n=1}^{\infty} \nu(a_n) : a_n \in L^+ \ \& \ |a| \leq' \sum_{n=1}^{\infty} |a_n| \right\} \text{ for } a \in G;$$
$$\bar{\nu}(a) = \nu^*(a) \text{ for } a \in \bar{L} \text{ (the } \nu^*\text{-closure of } L \text{ in } G).$$

Theorem 3. An element a of G is in $\bar{L}$ if and only if there exist $a_n \in L$ satisfying $a \sim \{a_n\}$ and $\sum_{n=1}^{\infty} \nu(a_{n+1} - a_n) < \infty$.

Example 2. Let $(S, \mathcal{F}, m)$ be a measure space, $G = R^S$, L the family of all simple $\mathcal{F}$-measurable functions, $\nu(f) = \|f\|_p$ for $f \in L$ and a fixed $p \in [0, \infty]$. Then $\bar{L} = L_p(m)$ and $\bar{\nu}(f) = \|f\|_p$ for all $f \in \bar{L}$. Furthermore, we may assume that $\mathcal{F}$ is merely a field; then, for any $p \in [0, \infty)$, $\bar{L}$ is at once the space $L_p(\bar{m})$, where $\bar{m}$ is the restriction of m^* to the σ-field of all m^*-measurable sets, and $\bar{\nu}(\cdot) = \|\cdot\|_p$. Theorem 3 says that a function $f \in R^S$ is in $L_p(\bar{m})$ if and only if there exist $f_n \in L$ satisfying the condition (*) of Example 1 and $\sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_p < \infty$.

Now assume, for simplicity, that ν takes only finite values, and let us consider the following possible properties of ν on L (here $a, a_n, b \in L^+$):

(F) $a = \bigvee_{n=1}^{\infty} \uparrow a_n$ (in G) implies $\nu(a_n) \uparrow \nu(a)$ (Fatou property),

(D) $0 = \bigwedge_{n=1}^{\infty} \downarrow a_n$ (in G) implies $\nu(a_n) \downarrow 0$ (Daniell property),

(S) $\forall_n \sum_{i=1}^{n} a_i \leq a$ implies $\nu(a_n) \downarrow 0$ (saturability property),

(BL) $\sup_k \nu\left(\sum_{i=1}^{k} a_i\right) < \infty$ implies $\nu(a_n) \to 0$ (Beppo Levi prop.),

(A) $\nu(a+b) = \nu(a) + \nu(b)$ (additivity property).

Theorem 4. (F) $\underset{\not\leftarrow}{\rightarrow}$ (D) $\underset{\not\rightarrow}{\leftarrow}$ (S) $\underset{\not\rightarrow}{\leftarrow}$ (BL) $\underset{\not\rightarrow}{\leftarrow}$ (A).

For instance (excluding some trivial cases) $\|\cdot\|_\infty \in (F) \setminus (D)$, $\|\cdot\|_\infty / C[0,1] \in (D) \setminus (S)$, $\|\cdot\|_0 \in (S) \setminus (BL)$, $\|\cdot\|_p \in (BL) \setminus (A)$ for $p \in (0,1) \cup (1,\infty)$, $\|\cdot\|_1 \in (A)$.

Theorem 5. Every one of the above five properties of ν is inherited by the extension $\bar{\nu}$.

Theorem 6. If ν satisfies (F), then for any a in $\bar{L}$

$$\bar{\nu}(a) = \inf \left\{ \lim_{n\to\infty} \uparrow \nu(a_n) \ : \ a_n \in L^+ \ \& \ |a| \leqslant' \bigvee_{n=1}^{\infty} \uparrow a_n \right\}.$$

If ν satisfies (D), then for any $a \in \bar{L}$

$$\bar{\nu}(a) = \sup \left\{ \lim_{n\to\infty} \downarrow \nu(a_n) \ : \ a_n \in L^+ \ \& \ |a| \geqslant' \bigwedge_{n=1}^{\infty} \downarrow a_n \right\}.$$

Theorem 7. If ν satisfies (S), $a_n, b \in \bar{L}$, $|a_n| \leqslant b$ for all n and $G \ni a = \text{o-}\lim_{n\to\infty} a_n$, then $a \in \bar{L}$ and $\bar{\nu}(a-a_n) \to 0$.
If ν satisfies (BL), $a_n \in \bar{L}$, $\sup_k \bar{\nu}\left(\bigvee_{i=1}^{k} |a_i|\right) < \infty$ and $G \ni a = \text{o-}\lim_{n\to\infty} a_n$, then $a \in \bar{L}$ and $\bar{\nu}(a-a_n) \to 0$.

Some details on the subject (concerning the case of function spaces) can be found in the author's papers

"Integration of functions with values in a normed group", Bull. Acad. Polon. Sci. 20 (1972), 911 - 916,

"Fubini-type theorems for integrable spaces", Bull. Acad. Polon. Sci. 22 (1974), 257 - 261,

"Real integrable spaces", Colloquium Mathematicum 32 (1975), 233 - 248.

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