PROBLEMS CONCERNING WEAK ASPLUND SPACES

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A Banach space E is a <u>weak Asplund (WA) space</u> provided every continuous convex function on E is Gateaux differentiable at each point of some dense G_δ subset of E. The central problem is to give a complete description of the class of spaces with this property. It is convenient to single out two related classes of spaces: Say that E is a <u>Gateaux differentiability space</u> (GDS) or has the <u>Gateaux differentiability property</u> (GDP) if each continuous convex function on E is Gateaux differentiable on a dense set (not necessarily a G_δ). We also say that E has the <u>support function differentiability property</u> (sfDP) if every support function on E is Gateaux differentiable on a dense set. [Call p a <u>support function</u> if there exists a bounded nonempty subset A of E^* such that $p(x) = \sup\{\langle x^*,x\rangle : x^* \in A\}$, $x \in E$; this is clearly convex, continuous and positive homogeneous.] There are obvious inclusions between these three classes of spaces. One reason for introducing the GDP is that we have been unable to resolve the following question.

<u>Problem</u> 1. Is the set of points of Gateaux differentiability of a continuous convex function necessarily a G_δ set? A Borel set? Universally measurable? What if it is assumed to be dense?

The reason for introducing the sfDP is that it can be characterized in a manner completely analogous to the known characterization of Asplund spaces. Recall that $x^* \in K \subseteq E^*$ is a <u>weak* exposed point</u> of K if there exists $x \neq 0$ in E such that

$$\langle x^*,x\rangle > \langle y^*,x\rangle \quad \text{whenever} \quad y^* \in K, \quad y^* \neq x^*.$$

<u>Proposition</u> 1. The Banach space E has the sfDP if and only if every weak* compact convex subset of E^* is the weak* closed convex hull of its weak* exposed points.

Problem 2. Does the sfDP imply the GDP?

One way of resolving this question would use the following result.

Proposition 2. Let R denote the real line. If $E \times R$ has the sfDP, then E has the GDP. (Analogously, if every support function on $E \times R$ is Gateaux differentiable on a dense G_δ set, then E is a weak Asplund space.)

Problem 3. If E has the sfDP, must $E \times R$ have the same property?

Problem 4. Are any of the above three differentiability properties preserved under finite products?

Proposition 3. Each of the following assertions is equivalent to the sfDP: (1) Every nonempty weak* compact convex subset of E^* has at least one weak* exposed point.

(2) Every support functional on E has at least one point of Gateaux differentiability.

The following result yields an interesting necessary condition for the sfDP.

Proposition 4. (Stegall, Larman) If every nonempty weak* compact convex subset K of E^* has at least one extreme point which is a G_δ point of K in the weak* topology (a weak* exposed point has this property), then every bounded sequence in E^* has a weak* convergent subsequence.

Problem 5. Is the conclusion to Proposition 4 a sufficent condition for E to have the sfDP?

The best sufficient condition to date is that due to Edgar Asplund [Acta Math. 1968]:

Proposition 5. If E is a subspace of a weakly compactly generated (WCG) space, then E is a WA space.
(This uses Asplund's theorem and the fact that a subspace of a space whose dual norm is strictly convex has the same property.)

Problem 6. If E is a Lindelöf space in its weak topology, is it a WA space?
(Recall that a subspace of a WCG space is weakly Lindelöf.)

Stegal [The RNP in conjugate Banach spaces, II, Trans. Amer. Math. Soc. (to appear)] has a simple proof that WCG spaces are WA spaces, using the Davis-Figiel-Johnson-Pełczyński factorization theorem for weakly compact operators, and the following result.

Proposition 6. If E is an Asplund space and if there exists a continuous linear map $T:E \to F$ having dense range, then F is a weak Asplund space.

It is easy to prove the following analogue.

Proposition 7. If $T:E \to F$ is bounded, linear and has dense range and if E has the GDP [sfDP], then F has the GDP [sfDP].

Recall Asplund's result:

Proposition 8. If T maps E linearly and continuously onto F, and if E is a WA space, then so is F.

A concrete question we have been unable to resolve is the following:

Problem 7. For which compact Hausdorff spaces X is the continuous function space $C(X)$ a WA space? When will $C(X)$ have the GDP or the sfDP?

A necessary condition is that every nonempty closed subset of X have a dense set of relative G_δ points. (This implies that X is sequentially compact.)

The following is a long-standing problem.

Problem 8. Is the existence of an equivalent norm on E which is Gateaux differentiable (at nonzero points) either necessary or sufficient for E to be a WA space?

The study of these spaces suffers from a lack of examples. The two which we list here are not very surprising.

Examples (1) If Γ is an infinite set, then there is a continuous seminorm on $\ell_\infty(\Gamma)$ which is nowhere Gateaux differentiable. (On ℓ_∞, use $p(x) = \limsup |x_n|$.)

(2) If Γ is uncountable, then the norm in $\ell_1(\Gamma)$ is nowhere Gateaux differentiable.

In regards to the first example, note that every weak* lower semicontinuous convex continuous function on $\ell_\infty(\Gamma)$ is Fréchet differentiable on a dense G_δ, since $\ell_1(\Gamma)$ has the RNP. [Collier, Pacific J. Math. 64 (1976)].

The following "simple" question is still open.

Problem 9. Does a subspace of a WA space or a GDS have the same property?

An affirmative answer to the next question would be surprising.

Problem 10. If E^* is a WA space or a GDS, must E have the same property?