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INDEPENDENT FAMILIES ON COMPLETE BOOLEAN ALGEBRAS

B. Balcar and F. Franěk

We present definitions and lemmas concerning a proof of the following fact, without any set-theoretical assumptions.

Theorem. Every infinite complete Boolean algebra contains a free subalgebra of the same cardinality.

This solves the Question 44 of [vD,M,R]. The history of this problem and a survey of partial solutions ([Ko],[Ky],[M]) is given in [Bla].

The theorem extends the classical result of Hausdorff and Pospíšil concerning complete atomic BA's $(=\mathcal{P}(K))$ to arbitrary cBA's .

Let us summarize some well-known consequences of the Theorem. In what follows, B denotes an infinite cBA and X denotes an infinite extremally disconnected compact (e.d.c.) space.

C1 Let $\mathcal{U}(B)$ be the set of all ultrafilters on B , then $\text{card}(\mathcal{U}(B)) = 2^{\text{card}(B)}$; equivalently, $\text{card}(X) = 2^{w(X)}$, where $w(X)$ is the weight of X .

C2 There are many $(= |\mathcal{U}(B)|)$ ultrafilters on B which have the character (= the least cardinality of a set of generators) equal to $|B|$.

The consequences C1 and C2 solve problems raised by Efimov [Ef] .

C3 If C is a cBA with $|C| \leq |B|$ then there is a homomorfism $f : B \xrightarrow{\text{onto}} C$; equivalently, for an e.d.c. space Y with $w(Y) \leq w(X)$ there is an embedding of Y into X .

C4 There is a continuous mapping $f : X \xrightarrow{\text{onto}} \{0,1\}^{w(X)}$.

C5 The space X contains a copy of itself as a nowhere dense subset and therefore X is not homogeneous. [F].

Notations, definitions

For a BA B let $B^+ = B - \{0\}$. For $u \in B^+$ let B_u denote a "partial subalgebra" of B with the universe $\{v \leq u ; v \in B\}$.

(i) Part $(B) = \{p \subseteq B^+ ; \bigvee p = 1$ and the elements of p are pairwise disjoint $\}$.

(ii) $\mathcal{P} \subseteq$ Part (B) is called an independent family of partitions if for any finite set of partitions $\{p_0, \ldots, p_{n-1}\} \subseteq \mathcal{P}$ and every mapping $f : n \to \bigcup \{p_i , i<n\}$ with $f(i) \in p_i$ we have $\bigwedge \{f(i) , i<n\} \neq 0$.

(iii) B is semifree if there is an independent family of partitions $\mathcal{P}$ on B with $|\mathcal{P}| = |B|$.

Hence the theorem is equivalent to the statement "every infinite cBA is semifree" .

(iv) $D \subseteq B^+$ is dense in B if $(\forall v \in B^+)(\exists u \in D)\ u \leq v$;

$d(B) = \min \{\text{card}(D) ; D$ is dense in $B\}$.

(v) sat $(B) = \min \{\nu ; (\forall p \in \text{Part}(B)) (|p| < \nu)\}$ (! less than)

Trivially, sat $(B) \geq$ sat (B_u) , $d(B) \geq d(B_u)$ for $u \in B^+$. Hence for a cBA B there is a partition p such that

$B = \sum_{u \in p} B_u$ (a product in the category of BA's) and all B_u's are homogeneous in sat and d .

(vi) (Erdös, Tarski). If B is infinite then

sat $(B) =$ κ^+ (κ infinite) or weakly inaccessible ($> \omega$) .

Combinatorial facts

A Let $\{X_i, i \in I\}$ be a family of sets. A set $\mathcal{S} \subseteq \prod_{i \in I} X_i$ is called a finitely distingueshed family (FDF) if for any finite $\mathcal{S}_0 \subseteq \mathcal{S}$ there is an $i \in I$ such that $|\{f(i) ; f \in \mathcal{S}_0\}| = |\mathcal{S}_0|$.

L 1 If X_i's are infinite, then there is a FDF $\mathcal{S} \subseteq \prod X_i$ with $|\mathcal{S}| = |\prod X_i|$.

Consider $B = \mathcal{P}(K)$ for infinite K. We can obtain very easily an independent family $\mathcal{P}_0 \subseteq$ Part (B) such that $|\mathcal{P}_0| = \omega$ and $|p| = K$ for $p \in \mathcal{P}_0$. Using L 1 and $\mathcal{P}_0$ we obtain the well-known fact ([EK],[Ke],[Ku]), namely, there is an independent family of partitions $\mathcal{P} \subseteq$ Part $(\mathcal{P}(K))$ such that $|\mathcal{P}| = 2^K = |B|$ and $(\forall p \in \mathcal{P})\ |p| = K$.

Corollary. If B is a cBA and $B = \sum\{B_u, u \in p\}$ and B_u's are semifree then B is semifree, too.

B The following lemma is a straightforward reformulation of a result of Vladimirov and Monk ([V],[M]).

L 2 Let B be a cBA and $\mathcal{P} \subseteq$ Part (B). For $p \in \mathcal{P}$ let $p^{\Sigma} = \{\bigvee p_1 ; p_1 \subseteq p\}$. Let $(\mathcal{P}^{\Sigma})^{\Pi} = \{\bigwedge a ; a$ is a selector of $\{p^{\Sigma} ; p \in \mathcal{P}\}\}$.

If for every $u \in \bigcup\{p ; p \in \mathcal{P}\}$ the set $\{x \leq u ; x \in (\mathcal{P}^{\Sigma})^{\Pi} - \{0\}\}$ is not dense in B_u, then there is a partition $q = \{x_0, x_1\}$ such that $x \wedge u \neq 0$ for every $x \in q$ and $u \in \bigcup \mathcal{P}$

C In the sequel we assume that all BA's are homogeneous in sat.

We use the following "disjoint refinement lemma" from [BV] in the proof of L 3. Let ν be a cardinal, $\nu^+ < \text{sat}(B)$. Then for any family $\{u_\alpha ; \alpha < \nu\} \subseteq B^+$ there is a disjoint refinement, i.e. a family

$\{v_\alpha \;;\; \alpha < \nu\} \subseteq B^+$ such that $v_\alpha \leq u_\alpha$ and $v_\alpha \wedge v_\beta =$ $= 0$ if $\alpha \neq \beta$.

L 3 Let sat (B) = K be a weakly inaccessible cardinal. Then there is an independent family $\mathscr{P}$ of partitions on B such that

(i) $|\mathscr{P}| = K$

(ii) $\sup\{|p| \;;\; p \in \mathscr{P}\} = K$.

For a proof of the theorem it is sufficient to deal only with atomless cBA's . If B is not atomless then $B = B_1 \oplus B_2$, where B_1 is atomic and $B_2 = 0$ or B_2 is atomless. If $|B| = |B_1|$, B is then semifree because B_1 is by the classical result. Otherwise $|B| = |B_2|$ and B is semifree iff B_2 is.

Let $B = \sum\{B_u \;;\; u \in p\}$ be a decomposition of an atomless cBA B into factors homogeneous in the both cardinal characteristics sat and d . Then it is sufficient to prove that B_u's are semifree.

Thus, let B be an atomless cBA homogeneous in sat and d .

Case 1. (Well-known before [Ky])

sat (B) = K^+ and d(B) = λ.

Then $|B| = \lambda^K$ and we can use L 1, L 2 .

Case 2. sat (B) = K , K is weakly inaccess.

d(B) = λ .

Then $|B| = \lambda^K$ and we can use L 1, L 2, L 3.

References

[BV] B. B car, P. Vo : Refining systems on Boolean alg br L c u Note n M ch. 619, 5-5

[Bla] A. Blaszczyk: O mapp gs of extremally diccon ected compact spaces onto Cantor cubes, Proceedings of Colloquium on Topology (Budapest 1978) (to appear)

[CN] W.W. Comfort, S. Negrepontis: The theory of ultrafilters, Springer-Verlag 19 4

[vD,M,R] E. van Douwen, I.D. Monk, M. Rubin: Some questions about Boolean algebras (preprint)

[E] B.A. Efinov: Extremally disconnected compact spaces and absolutes, Trudy Moskov.Mat.Obsc. 23 (1970), 235-276 (Russian)

[EK] R. Engelkin, M. Karlowicz: Some theorems of set theory and their topological consequences, Fund.Math. 57 (1965), 275-285

[ET] P. Erdös, A. Tarski: On families of mutually exclusive sets, Ann. of Math. 44 (1943), 315-329

[H] F. Hausdorff: Über zwei Sätze von G. Fichtenholz und L. Kantorowich, Studia Math. 6 (1936), 18-19

[F] Z. Frolik: Fixed points of maps of extremally disconnected spaces and complete Boolean algebras. Bull.Acad.Polon. Sci. XVI, 4 (1968), 269-275

[Ky] S. Keslyakov: Free subalgebras of complete Boolean algebras and spaces of continous functions, Sibirski Mat. Zh. 14 (1973), 569-581

[Ke] J. Ketonen: Everything you wanted to know about ultrafilters - ... , Doctoral dissertation University of Wisconsin 1971

[Ko] S. Koppelberg: Free subalgebras of complete Boolean algebras, Notices Amer.Math.Soc. 20 (1973), A-418

[Ku] K. Kunen: Ultrafilters and independent sets, Trans.Amer. Math.Soc. 172 (1972), 299-306

[M] I.D. Monk: On free subalgebras of complete Boolean algebras, Arch. der Math. 29 (1977), 113-115

[P] B. Pospíšil: Remark on bicompact spaces, Ann. of Math. 38 (1937), 845-846

[V] D. Vladimirov: Boolean algebras, Nauka, 1969

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