Aleksander Błaszczyk; Andrzej Szymański

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A short proof of Parovičenko's theorem

by

A.Błaszczyk and A.Szymański

We shall show a short proof of a theorem of Parovičenko that each compact space of weight at most $\aleph_1$ is a continuous image of the space ω^* ($= \beta\omega - \omega$) of all non-trivial ultrafilters on the set ω. Under CH we shall give a new characterization of ω^*.

We shall use the following properties of ω^*:

(1) ω^* is a zero-dimensional compact space without isolated points,

(2) every two open disjoint F_σ's in ω^* have disjoint closures,

(3) every non-empty G_δ in ω^* has non-empty interior;

for the proof see e.g. Comfort and Negrepontis [1].

Lemma. If f is a continuous map of ω^* onto a compact metric space X and E and F are closed sets covering X, then there exists a closed--open set $U \subset \omega^*$ such that $f(U) = E$ and $f(\omega^* - U) = F$.

Proof. If $E \cap F \neq \emptyset$, choose a countable dense subset D of $E \cap F$. Since the sets $f^{-1}(d)$ are non-empty G_δ's, for each $d \in D$ there exist non-empty closed-open sets U_d and V_d contained in $f^{-1}(d)$. The sets $f^{-1}(X - F) \cup \bigcup\{U_d : d \in D\}$ and $f^{-1}(X - E) \cup \bigcup\{V_d : d \in D\}$ are disjoint open F_σ's in ω^*. Hence, there exists a closed-open set $U \subset \omega^*$ which contains the first of this sets and is disjoint with the second one. It is easy to chack that the set U is the desired one.

Theorem 1 (Parovičenko [3]). Compact spaces of weight at most $\aleph_1$ are continuous images of ω^*.

Proof. Let X be a compact space of weight at most $\aleph_1$. Since the Tychonoff cube $I^{\aleph_1}$ is a continuous image of the Cantor cube $D^{\aleph_1}$, we can assume X to be a closed subspace of $D^{\aleph_1}$. We shall consider $D^{\aleph_1}$ as the limit of the inverse system

$$D \leftarrow D^2 \leftarrow \cdots \leftarrow D^{\alpha} \xleftarrow{p_\alpha^{\alpha+1}} D^{\alpha+1} \leftarrow \cdots \xleftarrow{p_\beta} D^{\aleph_1},$$

where $D = \{0,1\}$, $D^{\alpha+1} = D^{\alpha} \times D$, $D^{\beta} = \varprojlim \{D^{\alpha}, p_\alpha^{\alpha+1}, \alpha<\beta\}$ for limit β and $p_\alpha^{\alpha+1}$ are projections, i.e. $p_\alpha^{\alpha+1}(x) = x|\alpha$ for $\alpha<\aleph_1$. Since $X \subset D^{\aleph_1}$, $X = \varprojlim \{X_\alpha, q_\alpha^{\alpha+1}, \alpha<\aleph_1\}$, where $X_\alpha = p_\alpha(X)$, $q_\alpha^{\alpha+1} = p_\alpha^{\alpha+1}|X_{\alpha+1}$, $\alpha < \aleph_1$. For each $\alpha<\aleph_1$ we shall define a continuous map f_α from ω^* onto X_α in such a way that $f_\alpha = q_\alpha^{\alpha+1} \circ f_{\alpha+1}$ for each $\alpha<\aleph_1$. It suffices to do this for non-limit α's. Assume, we have defined f_α for some $\alpha<\aleph_1$. Since $X_\alpha \subset D^{\alpha}$ and $\alpha<\aleph_1$, X_α is a compact metric space. By the Lemma, we get a closed-open set $U \subset \omega^*$ such that $f_\alpha(U) = q_\alpha^{\alpha+1}(X_{\alpha+1} \cap (X_\alpha \times \{0\}))$ and $f_\alpha(\omega^* - U) = q_\alpha^{\alpha+1}(X_{\alpha+1} \cap (X_\alpha \times \{1\}))$. We define $f_{\alpha+1}$ by setting $f_{\alpha+1}(x) = (f_\alpha(x),0)$ for $x \in U$ and $f_{\alpha+1}(x) = (f_\alpha(x),1)$ for $x \in \omega^* - U$. Clearly, $f_{\alpha+1}$ is a continuous map from ω^* onto $X_{\alpha+1}$ such that $f_\alpha = q_\alpha^{\alpha+1} \circ f_{\alpha+1}$. The limit map induced by all f_α's is the desired one.

It appears that the property formulated in the Lemma characterize the space ω^*. Namely, we get

Theorem 2 (CH). If P is a compact space of weight $\aleph_1$, then P is homeomorphic to ω^* iff it satisfies the following condition:

(4) for each continuous map f from P onto a compact metric space X and each closed sets $E, F \subset X$ covering X there exists a closed-open set $U \subset P$ such that $f(U) = E$ and $f(P - U) = F$.

Corollary (CH). A compact space P of weight $\aleph_1$ is homeomorphic to ω^* iff it satisfies the following condition:

(5) if X and Y are compact metric spaces and $f: P \xrightarrow{\text{onto}} X$ and $g: Y \xrightarrow{\text{onto}} X$ are continuous maps, then there exists a continuous map $h: P \longrightarrow Y$ such that $f = g \circ h$.

Negrepontis [2] has obtained a similar characterization. He has shown that a compact space P of weight $\aleph_1$ is homeomorphic to iff it satisfies the condition (5) and every compact metric space is a continuous image of the space P.

References

[1] W.W.Comfort and S.Negrepontis, The theory of ultrafilters, Springer-Verlag, Berlin - Heidelberg - New York, 1974.

[2] S.Negrepontis, The Stone space of the saturated Boolean algebras, Trans. Amer. Math. Soc. 141 (1969), p.515-527.

[3] I.I.Parovičenko, A universal bicompact of weight $\aleph$, Dokl. Akad. Nauk SSSR 150 (1963), p.36-39.