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SKOROHOD EMBEDDING IN BROWNIAN MOTION IN $\mathbb{R}^n$

by Neil Falkner

Let μ be a measure on $\mathbb{R}^n$ and let $(\Omega, \underline{\underline{B}}, \underline{\underline{B}}_t, B_t, P^\mu)$ be a Brownian motion process in $\mathbb{R}^n$ with initial law μ. We allow the possibility that $\underline{\underline{B}}_t$ may be strictly larger than $\underline{\underline{B}}_t^\circ$ which denotes the usual completion of $\sigma(B_s : 0 \leq s \leq t)$, though of course (B_t) must be Markov with respect to $(\underline{\underline{B}}_t)$. If T is a stopping time (of the filtration $(\underline{\underline{B}}_t)$) then μ_T will denote the measure on $\mathbb{R}^n$ defined by

$$\mu_T(A) = P^\mu(B_T \in A) .$$

In other words, μ_T is the law of B_T (with respect to P^μ) where the mass corresponding to the event $\{T = \infty\}$ is simply discarded. If a measure ν on $\mathbb{R}^n$ is of the form $\nu = \mu_T$ for some stopping time T we say ν is embedded in Brownian motion with initial law μ by means of the stopping time T. It is natural to ask what measures can be embedded in Brownian motion. Skorohod [9, ch. 7] showed that in the case $n = 1$, $\mu = \delta_0$, if $\underline{\underline{B}}_0$ is sufficiently rich in the sense that it admits a continuously distributed random variable independant of $\sigma(B_t : 0 \leq t < \infty)$ then a probability measure ν on $\mathbb{R}$ is of the form $\nu = \mu_T$ for some stopping time T satisfying $E^\mu(T) < \infty$ iff $\int x \, d\nu(x) = 0$ and $\int x^2 \, d\nu(x) < \infty$. Dubins [2] and Root [7] independantly showed that Skorohod's conclusion is valid without the "richness" hypothesis on $\underline{\underline{B}}_0$; thus they showed that such stopping times can be obtained which are stopping times of the natural filtration $(\underline{\underline{B}}_t^\circ)$. The reason for asking that T satisfy a condition of not being too big, such as $E^\mu(T) < \infty$, is that otherwise, in the case $n = 1$, μ_T is virtually unrestricted. To be precise, if $n = 1$ and μ and ν are any probability measures on $\mathbb{R}$ then there is a stopping time T, which is trivial to construct, such that $\mu_T = \nu$. This was noticed by Doob; see [6]. Probably the most natural condition of not being too big is given by the following definition.

<u>Definition 1</u>. A stopping time T is said to be μ-standard iff
$n = 1$ and $(B_{T \wedge t})$ is P^{μ}-uniformly integrable
or $n = 2$ and $(\log^{+}||B_{T \wedge t}||)$ is P^{μ}-uniformly integrable
or $n = 3$.

The curious fact noted by Doob which is mentioned above has to do with the fact that Brownian motion is recurrent when $n = 1$. It is also recurrent when $n = 2$. When $n \geq 3$ it is transient and this is why all stopping times are considered μ-standard when $n \geq 3$. The $\log^{+}$ in the definition of μ-standard stopping times in the case $n = 2$ comes from the logarithmic potential kernel used in 2 dimensions. One can show that when $n = 1$ and $\mu = \delta_0$ then a measure ν on R is of the form $\nu = \mu_T$ for some μ-standard stopping time T iff ν is a probability measure, $\int |x| \, d\nu(x) < \infty$, and $\int x \, d\nu(x) = 0$. For more general initial measures μ and for higher dimensions n, suitable conditions on μ and ν may be formulated in potential theoretic terms. Let us recall the definition of the potential of a measure on R^n. Define $\Phi : R^n \to (-\infty, \infty]$ by

$$\Phi(x) = \begin{cases} -\frac{1}{2}|x| & \text{if } n = 1 \\ -\frac{1}{2\pi} \log||x|| & \text{if } n = 2, \ x \neq 0 \\ \dfrac{1}{(n-2)\sigma_n ||x||^{n-2}} & \text{if } n \geq 3, \ x \neq 0 \\ \infty & \text{if } n \geq 2, \ x = 0 \end{cases}$$

where σ_n is the $n-1$ dimensional Lebesgue measure of the surface of the unit ball in R^n. For a measure α on R^n, define U_{+}^{α} and U_{-}^{α} on R^n by

$$U_{\pm}^{\alpha}(x) = \int \Phi^{\pm}(x - y) \, d\alpha(y)$$

and define U^{α}, on the subset of R^n where U_{+}^{α} and U_{-}^{α} are not both infinite, by $U^{\alpha} = U_{+}^{\alpha} - U_{-}^{\alpha}$. U^{α} is called the potential of α. We say α is <u>special</u> iff U^{α} is defined on all of R^n and is superharmonic.

One can show that this happens iff α is finite on compact sets and

$$\int_{||x||\geq 1} |\Phi(x)|\, d\alpha(x) < \infty .$$

More explicitly:

If $n = 1$ then α is special iff α is finite and

$$\int |x|\, d\alpha(x) < \infty .$$

If $n = 2$ then α is special iff α is finite and

$$\int \log^+||x||\, d\alpha(x) < \infty .$$

If $n \geq 3$ then every finite measure on R^n is special and so are many infinite ones.

If α is a special measure on R^n then α is recoverable from U^α ; indeed α is minus the Laplacian of U^α , in the sense of Schwartz distributions.

Theorem 1. Let μ be a special measure on R^n . If $n \geq 2$, assume $\underline{\underline{B}}_0$ admits a continuously distributed random variable independant of $\sigma(B_t : 0 \leq t < \infty)$. Then a measure ν on R^n is of the form $\nu = \mu_T$ for some μ-standard stopping time T iff (ν is special and $U^\mu \geq U^\nu$ and if $n \leq 2$, $\mu(R^n) = \nu(R^n)$).

For $n \geq 3$, this follows from an embedding theorem of Rost [8] which applies to transient Markov processes. (Rost considers only finite measures μ and ν but his method works equally well for measures that are only special.) For $n = 2$, it is proved in [5]. For $n = 1$ it is just about proved in [1] and at any rate is the simplest case of the next theorem.

Now, to dispense with the hypothesis on $\underline{\underline{B}}_0$ when $n \geq 2$ in theorem 1 is not always possible. For example, if μ is the unit point mass at 0 and if ν is the probability measure which has half its mass at 0 and the other half uniformly distributed on the surface of the ball of radius 1 centred at 0 and if $n \geq 2$ then there is no $(\underline{\underline{B}}_t^\circ)$-stopping time T such that $\mu_T = \nu$, even though μ and ν are special and $U^\mu \geq U^\nu$. This is because

$$P^\mu(T > 0) = 0 \text{ or } 1 \text{ if } T \text{ is a } (\underline{\underline{B}}_t^\circ)\text{-stopping time}$$

$$\text{and } P^\mu(B_t = 0 \text{ for some } t > 0) = 0 \text{ if } n \geq 2 .$$

However, in [1] Baxter and Chacon showed that if μ and ν are special probabilities on $\mathbb{R}^n$, if $U^\mu \geq U^\nu$, if U^ν is finite and continuous, and if ($n \geq 3$ or $\lim_{||x|| \to \infty} |U^\mu(x) - U^\nu(x)| = 0$) then there exists a stopping time T for the filtration $(\underline{\underline{B}}_t^\circ)$ such that $\mu_T = \nu$. They do not show that their stopping time is μ-standard, but it is. In [4], the following improvement of their result is proved.

<u>Theorem 2</u>. Let μ and ν be special measures on $\mathbb{R}^n$ such that:

a) $U^\mu \geq U^\nu$ and if $n \leq 2$, $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$;

b) $\mu(Z) \leq \nu(Z)$ for all Borel sets $Z \subseteq \{U^\nu = \infty\}$.

Then there is a μ-standard stopping time T for the filtration $(\underline{\underline{B}}_t^\circ)$ such that $\mu_T = \nu$.

(<u>Remark</u>. It follows that actually, for every Borel polar set $Z \subseteq \mathbb{R}^n$, $\nu(Z) = \mu(Z \cap \{U^\nu = \infty\})$ since $P^\mu(B_t \in Z$ for some $t > 0) = 0$ and since $\nu(Z \cap \{U^\nu < \infty\}) = 0$..)

<u>Corollary 1</u>. Let μ be a special measure on $\mathbb{R}^n$ which does not charge polar sets. Then a measure ν on $\mathbb{R}^n$ is of the form $\nu = \mu_T$ for some μ-standard $(\underline{\underline{B}}_t^\circ)$-stopping time T iff (ν is special and $U^\mu \geq U^\nu$ and if $n \leq 2$, then $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$).

In particular, given a special measure μ on $\mathbb{R}^n$, if μ does not charge polar sets then considering Brownian motion processes with filtrations larger than the natural one does not enlarge the range of possibilities for μ_T where T is a μ-standard stopping time. We recall that a set is said to be polar iff it is contained in set of the form $\{U^\alpha = \infty\}$ for some special measure α. Polar sets are the small sets of potential theory. Every polar set has Lebesgue measure 0 (but not conversely). Thus if μ is absolutely continuous with respect to Lebesgue measure then μ does not charge polar sets.

<u>Corollary 2</u>. Let ν be a special measure on $\mathbb{R}^n$ such that U^ν is finite. Then the following are equivalent for a special measure μ on $\mathbb{R}^n$:

a) $U^\mu \geq U^\nu$ and if $n \leq 2$, $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$.

b) There exists a μ-standard stopping time T for the filtration $(\underline{\underline{B}}_t^\circ)$ such that $\mu_T = \nu$.

We remark that theorem 2 is not the best result one could hope for, since one can have special measures μ and ν on R^n such that $\mu(\{U^\nu = \infty\}) > \nu(\{U^\nu = \infty\}) = 0$ but there exists a μ-standard stopping time T for the filtration $(\underline{\underline{B}}_t^\circ)$ such that $\mu_T = \nu$. Indeed if we take $\mu = \delta_0$, and take $p_k \in [0,1]$ with $\sum_k p_k = 1$ and distinct $r_k \in (0,\infty)$ and let ν be the spherically symmetric probability measure on R^n which assigns mass p_k to $\{x : ||x|| = r_k\}$ then with the right choice of the p_k's and r_k's we can have $U^\nu(0) = \infty$, but using the beautiful theorem 2 of [3] one can show the existence of a stopping time T, μ-standard if ν is special, which is actually a stopping time of the natural filtration of the process $(||B_t||)$, as one might have hoped in view of the spherical symmetry, such that $\mu_T = \nu$. For the details and also for a simplified proof of the key theorem 2 of [3], see [4].

<u>Conjecture</u>. Let μ be a special measure on R^n. Then a) and b) below are equivalent for a measure ν on R^n:

a) There exists a μ-standard $(\underline{\underline{B}}_t^\circ)$-stopping time T such that $\mu_T = \nu$.

b) The following conditions hold:

i) ν is special and $U^\mu \geq U^\nu$ and if $n \leq 2$, $\mu(R^n) = \nu(R^n)$;

ii) there exists a Borel set C such that for every Borel polar set $Z \subseteq R^n$, $\nu(Z) = \mu(Z \cap C)$.

That i) is necessary for a) follows from the forward implication in theorem 1. That ii) is necessary for a) follows from the fact that $\{T = 0\} \in \underline{\underline{B}}_0^\circ$ so there is a Borel set $C \subseteq R^n$ such that $P^\mu(\{T=0\} \Delta B_0^{-1}[C]) = 0$; for such a C one has for every Borel polar set $Z \subseteq R^n$, $P^\mu(B_T \in Z, B_0 \notin C) = 0$ since $P^\mu(B_t \in Z$ for some $t > 0) = 0$.

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This is as far as I got in my talk at the winter school and as far as I had gotten in my research on this problem until I came to writing up this summary. In the course of writing this summary, I began thinking once again about how to prove the conjecture just stated. I am delighted to report that after working on this off and on for quite a long time, I have finally solved it. The conjecture is true. The proof of this will be published elsewhere.

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