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Realizing Homomorphisms of Category Algebras

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In a series of papers D. Maharam and A.H. Stone investigated the problem of realizing isomorphisms and automorphisms of category algebras by certain point-mappings. It is the purpose of this talk to present a similar result for homomorphisms of category algebras.

Definitions:

For a topological space Z let $k_1(Z)$ denote the \mathcal{C} -ideal of all sets of first category in Z , $\mathcal{L}_p(Z)$ the \mathcal{C} -field $\{B \subset Z \mid \exists U \text{ open: } B \Delta U \in k_1(Z)\}$ of sets with the Baire property, and $\mathcal{L}(Z) = \mathcal{L}_p(Z)/k_1(Z)$ the category algebra of Z . The symbol $\mathcal{B}(Z)$ always stands for the Borel field of Z .

If Z is a Baire topological space (i.e. if an open subset U of Z belongs to $k_1(Z)$ if and only if $U = \emptyset$) then, for every $C \in \mathcal{L}(Z)$, there exists exactly one regular open set $\theta(C)$ in Z (i.e. $\theta(C) = \overline{\overset{\circ}{C}}$) which is contained in the equivalence class C (i.e. $[\theta(C)] = C$).

The map $\theta: \mathcal{L}(Z) \rightarrow \mathcal{B}(Z)$ has the following properties:

- (i) $\theta([\emptyset]) = \emptyset$, $\theta([Z]) = Z$
- (ii) $\theta(C \cap D) = \theta(C) \cap \theta(D) \quad \forall C, D \in \mathcal{L}(Z)$
- (iii) $\theta(\bigvee \{C_i \mid i \in I\}) = \overline{\bigcup \{\theta(C_i) \mid i \in I\}}$ for every family $(C_i)_{i \in I}$ in $\mathcal{L}(Z)$.

Theorem:

Let X be a complete metric space, Y a Baire topological space, and $\Phi: \mathcal{B}(X) \rightarrow \mathcal{L}(Y)$ a \mathcal{C} -homomorphism. Then the following properties of Φ are equivalent:

- (1) For every family $(U_i)_{i \in I}$ of open subsets of X the identity

$$\Phi(\bigcup \{U_i \mid i \in I\}) = \bigvee \{\Phi(U_i) \mid i \in I\}$$

holds.

(ii) For every open covering $(U_i)_{i \in I}$ of X (with $\text{card } I < \text{wt } X$)

$$\bigvee \{ \bar{\Phi}(U_i) \mid i \in I \} = [Y].$$

(iii) There exists a dense G_δ -set $D \subset Y$ and a continuous map

$$f: D \rightarrow X \text{ with } [f^{-1}(B)] = \bar{\Phi}(B) \text{ for all } B \in \mathcal{B}(X).$$

Proof: (i) \Rightarrow (ii) and (iii) \Rightarrow (i) are easy to check.

(ii) \Rightarrow (iii): For $y \in Y$ let $\mathcal{F}_y = \{ B \in \mathcal{B}(X) : y \in \bar{\Phi}(B) \}$. Let D be

the set $\{ y \in Y \mid \mathcal{F}_y \text{ converges} \}$ and define $f: D \rightarrow X$ by $f(y) = \lim \mathcal{F}_y$.

Then (D, f) has the required properties.

Corollary 1:

Let X be a complete metric space, $\mathcal{M} = \text{wt } X$, Y a Baire topological space, and $\bar{\Phi}: \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$ a \mathcal{G} -homomorphism. Then the following conditions are equivalent:

(i) $\bar{\Phi}$ is an \mathcal{M} -homomorphism.

(ii) $\bar{\Phi}$ is a complete homomorphism.

(iii) There exists a dense G_δ -set $D \subset Y$ and a continuous map $f: D \rightarrow X$ with $[f^{-1}(B)] = \bar{\Phi}([B])$ for all $B \in \mathcal{B}_p(X)$.

Proof: The corollary is obtained from the theorem by considering the \mathcal{G} -homomorphism $\mathcal{B}(X) \rightarrow \mathcal{L}(Y)$, $B \mapsto \bar{\Phi}([B])$.

Corollary 2: (Maharam - Stone [3])

Let X and Y be complete metric spaces and $\bar{\Phi}: \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$ an isomorphism onto. Then there exist dense G_δ -sets $D \subset Y$ and $E \subset X$ and a homeomorphism f from D onto E such that $[f^{-1}(A)] = \bar{\Phi}([A])$ and $[f(B)] = \bar{\Phi}^{-1}([B])$ for all $A \in \mathcal{B}_p(X)$, $B \in \mathcal{B}_p(Y)$.

Corollary 3:

Let X be a separable metric space, Y a Baire topological space, and $f: Y \rightarrow X$ an arbitrary mapping. Then f is $\mathcal{B}_p(Y)$ - $\mathcal{B}(X)$ -measurable if and only if there exists a dense G_δ -set $D \subset Y$ such that $f|_D$ is continuous.

Corollary 4: (Fort [1]) (cf. Namioka [6], Theorem 1.2)

Let Y be a Baire topological space, Z a locally compact separable metric space, and R a separable metric space. If $f: Y \times Z \rightarrow R$ is continuous in each variable separately then there exists a dense G_δ -subset D of Y such that f is continuous at each point of $D \times Z$.

Let us recall that a finite measure ν on the Borel field $\mathcal{B}(X)$ of a topological space X is called τ -continuous if, for every filtering decreasing family $(A_i)_{i \in I}$ of closed sets in X with $\bigcap_{i \in I} A_i = \emptyset$, the equality $\inf\{\nu(A_i) \mid i \in I\} = 0$ is satisfied. For a measure space (Y, \mathcal{A}, μ) let \mathcal{A}/μ be the quotient of the σ -field \mathcal{A} w.r.t. the σ -ideal of μ -nullsets.

Corollary 5:

Let X be a complete metric space, (Y, \mathcal{A}, μ) a complete finite measure space, and $\Phi: \mathcal{B}(X) \rightarrow \mathcal{A}/\mu$ a σ -homomorphism such that $\mu \circ \Phi$ is a τ -continuous measure on $\mathcal{B}(X)$. Then there exists an \mathcal{A} - $\mathcal{B}(X)$ -measurable map such that $[\hat{f}^{-1}(B)] = \Phi(B)$ for all $B \in \mathcal{B}(X)$.

Proof: It follows from Ionescu Tulcea [2], p. 54, Proposition 1 that there exists a topology \mathcal{T} on Y such that (Y, \mathcal{T}) is a Baire topological space, \mathcal{A} equals the σ -field of sets with the Baire property w.r.t. \mathcal{T} , and the μ -nullsets are just the sets of first category in (Y, \mathcal{T}) . Thus \mathcal{A}/μ is the category algebra of (Y, \mathcal{T}) . Using the τ -continuity of $\mu \circ \Phi$ we obtain immediately that Φ satisfies condition (ii) in the theorem. Hence the theorem implies the existence of an \mathcal{A} - $\mathcal{B}(X)$ -measurable map with the desired properties.

References

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