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Mirror Symmetric Monopole Systems

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It has been shown that the imposition of axial and mirror
symmetry on the static magnetic SU(2) Yang-Mills-Higgs system
reduces the usual first order ("self-dual") Bogomolny equati-
ons to five equations for five unknown (gauge-invariant) func-
tions. In this note we show that, for a differentiable Higgs
field, the five field equations and the topology rule out all
charge-distributions along the axis of symmetry except a sing-
le-point distribution, and the boundary conditions at the sing-
le point distribution are discussed.

Ever since the discovery $[1]$ that the static Yang-Mills-
-Higgs equations of unified gauge-theory admit finite-energy
magnetic monopole solutions, there has been interest in the
question of solutions which go beyond the original ones, which
are all spherically symmetric. The next simplest configuration
seems to be that in which the fields are axially symmetric and
symmetric under reflexions in all planes containing the z-axis,
and in this note we wish to consider the field equations for
such systems.

To obtain the simplest possible system of this kind we
do not consider the full second-order YMH equations but rather
the first-order equations

(1) $\qquad\qquad\qquad\qquad \underset{\sim}{F} = \underset{\sim}{D}\bar{\Phi}$,

which are obtained by minimizing the Bogomolny-Prasad-Sommer-field [2] Hamiltonian

$$(2) \qquad H = \frac{1}{2} \int d_3\underset{\sim}{x}(\underset{\sim}{F}^2 + (\underset{\sim}{D}\Phi)^2) \ ,$$

in which the gauge-group is SU(2), the Higgs field Φ is assumed to belong to the adjoint representation, and the scalar potential is set equal to zero and replaced by the boundary condition $(\Phi,\Phi) \to c^2$ as $r \to \infty$.

It has been shown [3,4] that for axial and mirror symmetry the field equations (1) reduce to the five equations

$$(3) \qquad ku_\alpha = b_{,\alpha} + \rho \in_{\alpha\beta} h_{,\beta} \ ,$$

$$(4) \qquad -k_{,\alpha} = bu_\alpha + \rho h \in_{\alpha\beta} \dot{u}_\beta \ ,$$

$$(5) \qquad hk = \rho \in_{\alpha\beta} u_{\alpha\beta} \ ,$$

for the five (gauge-invariant) unknown functions h,k,b,u_α defined as

$$h^2 = (\Phi,\Phi) \ , \ h^2k^2 = (D_\gamma\Phi,D_\gamma\Phi) \ ,$$

$$(6) \qquad h^2k^2(b^2+k^2) = (D_\gamma^2\Phi,D_\gamma^2\Phi) \ , \ h_{,\alpha} h_{,\beta} + h^2 u_\alpha u_\beta =$$
$$= (D_\alpha\Phi,D_\beta\Phi) \ .$$

It was also shown that the first four equations are equivalent to the existence of a master-potential W such that

$$(7) \qquad \frac{\partial W}{\partial z} = hb \ , \ h^2 = c^2 + \triangle W \ ,$$

$$2\rho \frac{\partial W}{\partial \rho} = \rho^2(h^2 - c^2) + n^2 - k^2 - b^2 \ .$$

In particular

$$(8) \qquad (k^2+b^2)_{,\alpha} = \rho^2(h^2)_{,\alpha} - 2\rho \in_{\alpha\beta} (bh)_{,\beta} \ .$$

Furthermore, the ordinary Maxwell magnetic field $\underset{\sim}{B}$, defined as usual [5], to be

$$(9) \qquad B_i = (F_i,\emptyset) - \frac{1}{2}\in_{ijk}(\emptyset,D_j\emptyset \wedge D_k\emptyset) \ , \text{ where } \emptyset = \Phi/h \ .$$

turns out to be just

(10) $\qquad B_\alpha = \frac{1}{\rho} \in_{\alpha\beta} b_{,\beta} \qquad , (B_\psi = 0) .$

Thus the magnetic charge contained in any volume of revolution V around the z-axis and intersecting the axis at two points z_1 and z_2 is just

(11) $\qquad \triangle n = \frac{1}{4\pi} \int d_3x \underset{\sim}{\nabla}.\underset{\sim}{B} = \frac{1}{2} (b(z_1) - b(z_2)) .$

Of course, since (8) is a curl the charge can be located only at points where is not well defined i.e. where the Higgs field ϕ is zero. The boundary conditions for the functions h,b,k, u_α as $r \rightarrow 0$ are

(12) $\qquad k, u_\alpha \rightarrow 0 , h \rightarrow c - \frac{n}{r} , b \rightarrow -n \cos \theta ,$

where n is the total topological charge (in agreement with (9)). The boundary conditions on the z-axis are determined by the finiteness of the energy [2] , which in terms of the variables h,k,b,u_α takes the form

(13) $\qquad H = \frac{1}{2} \int d_3x [(\underset{\sim}{\nabla}h)^2 + h^2\underset{\sim}{u}^2 + (\in_{\alpha\beta}u_{\alpha,\beta})^2 +$
$\qquad + \frac{1}{\rho^2} \{ h^2k^2 + (b_{,\alpha} - ku_\alpha)^2 + (k_{,\alpha} + bu_\alpha)^2 \}] .$

From this equation we see that (subject to uniform behaviour with respect to z) we have

(14) $\qquad \rho^{3/2}\underset{\sim}{\nabla}h \rightarrow 0 , u_\alpha \rightarrow b^{-1}k_{,\alpha} \rightarrow -k^{-1}b_{,\alpha} , hk \rightarrow 0 ,$
$\qquad$ as $\rho \rightarrow 0 .$

From the central equation in (14) and from (10) we have in particular ____ __

(15) $\qquad\qquad b^2 + k^2 = n^2 \qquad$ for $\rho = 0 .$

This equation also follows from (3,4,5) and the first eqn. in (14).

In a previous note [6] it was shown that topological considerations alone implied that for an axially symmetric system

with charges located on the z-axis, the Higgs field h had
to vanish everywhere along the axis joining these points,
and it is interesting to note that this result can be verified
directly from (14) and (15) in a gauge-invariant manner. For
if $h \neq 0$ on the axis, then from (14) we have $k=0$ and hence
from (15) $b = \pm n$. It then follows from (11) that between any
two points where $h \neq 0$ the charge must be either 0, or n.
Since n is the total charge this establishes the result.

The fact that the Higgs field must vanish between charges
suggests that actually there can be no such distribution, ex-
cept at a single point, and we now wish to show that, provided
that the Higgs field h has a continuous derivative such is
indeed the case. Although this condition is quite mild, it is
a little more than is required by finite energy.

The basis for our result will be the two equations

$$(16) \qquad \triangle(hb) = (h^2)_{,z} \quad .$$

$$(17) \qquad \triangle(hk) = \frac{2(f f_{,\alpha} - g \in_{\alpha\beta} f_{,\beta})}{h^2 D^2} (hk)_{,\alpha} +$$

$$+ \left[\frac{D^2}{\rho^2} + \frac{2 \in_{\alpha\beta} g_{,\alpha} f_{,\beta}}{h^2 D^2} \right] (hk) \quad .$$

$$(18) \qquad f \equiv \rho h^2 \ , \ g \equiv hb \ , \ D^2 = b^2 + k^2 + \rho^2 h^2 \ ,$$

which follow directly from (3) - (7). If we now assume that at
any zero, z_0, of the Higgs field on the z-axis, h is C^1
then (17) has the small distance approximation

$$(19) \qquad \triangle(hk) = \frac{n^2}{\rho^2} (hk) \quad .$$

Since finite energy requires that $hk \rightarrow 0$, as $x \rightarrow z_0$, the
solution of (19) is an associated Legendre function of the
first kind. Thus wherever $P_m^n(\cos \theta) \neq 0$ we have

(20) $\qquad hk \rightarrow Kr^m P_m^n(\cos\theta)$, $\quad m \geq n$,

where r, θ are polar coordinates about $\underset{\sim}{z}_0$. Since k is
bounded, equation (20) already shows that h does not vanish
infinitely fast at $\underset{\sim}{z}_0$, but between the powers r and r^m .

Let us now consider those points $\underset{\sim}{z}_0$, where $h \rightarrow 0$, but
$b \neq 0$ for small r and either $\theta = 0$ or π (or both). It fol-
lows from (14) and (15) that such points exist. Also from (15)
we see that, as $r \rightarrow 0$,

(21) $\qquad (hb)^2 + (hk)^2 \rightarrow n^2 k$,

and that, since $b \not\rightarrow 0$, hk can not dominate hb , for all
θ . From (21) we can write (16) in the form

(22) $\qquad \triangle(hb) = \dfrac{1}{n^2}\left[(h^2 b^2)_{,z} + (h^2 k^2)_{,z}\right] + \ldots$.

Since hk does not dominate hb and $hb \rightarrow 0$ as $r \rightarrow 0$, it
follows from (22) that the leading behaviour of hk is a so-
lution of the homogeneous equation $\triangle(hb) = 0$. In other words

(23) $\qquad hb \rightarrow K_s r^s P_s(\cos\theta) \qquad$ for some $s \leq m$,

where K_s is a non-zero constant. It then follows from (21)
that h^2 is of the form

(24) $\qquad h^2 \rightarrow \dfrac{1}{n^2}\left[(K_s r^s P_s(\cos\theta))^2 + (Kr^m P_m^n(\cos\theta))^2\right]$, $s \leq m$,

as $r \rightarrow 0$. Equation (24) shows that h^2 is a regular function
at $\underset{\sim}{z}_0$ and that

(25) $\qquad h^2(r,0) = h^2(r,\pi)$, $\quad r \neq 0$.

Thus the field equations require that h^2 be symmetric about
any point on the z-axis where it vanishes. It is indeed pos-
sible for a single point charge distribution to satisfy this.
However, now consider a charge which is not distributed at a
single point on the z-axis. According to the previous discus-
sion of the topology, h^2 , in this case, must vanish along a

finite section of the z-axis. On the z-axis in a neighbour-hood of an end point z_0 of such a section, $h^2=0$ inside z_0, and $h^2 \neq 0$, $b^2 = n^2 \neq 0$ outside. Thus h^2 can not be symmetric about z_0. Therefore equation (25) rules out all but the single point charge distribution.

Let us now assume that the charge distribution is con-centrated at a single point (superimposed monopoles), which we take to be the origin. The equations derived above for z_0 are still valid, so that near the origin we have the leading behaviour

$$(26) \quad hk \longrightarrow Kr^m P_m^n(\cos \theta) \ , \quad m \geq n \ ,$$

$$hb \longrightarrow K_s r^s P_s(\cos \theta) \ , \quad m \geq s \geq 1 \ ,$$

$$h \longrightarrow \frac{1}{n} r^s \left| K_s P_s(\cos \theta) \right| \ , \quad \text{for} \quad s < m \ ,$$

$$h \longrightarrow \frac{1}{n} r^m \left\{ (K_m P_m(\cos \theta))^2 + (K P_m^n(\cos \theta))^2 \right\}^{1/2}$$
$$\text{for} \quad s=m \ ,$$

where the condition $s \leq m$ comes from the fact that if $s > m$, we have $b \longrightarrow 0$ and hence no topological charge. The most na-tural case (and possibly only stable case) is $m=n$, and then we have

$$(27) \quad hk \longrightarrow K \rho^n \ ,$$

$$hb \longrightarrow K_s r^s P_s(\cos \theta) \ ,$$

$$h \longrightarrow \frac{1}{n} r^s \left| K_s P_s(\cos \theta) \right| \ , \quad \text{for} \quad s < m \ ,$$

$$h \longrightarrow \frac{1}{n} r^n \left\{ (K_n P_n(\cos \theta))^2 + K^2 \rho^{2n} \right\}^{1/2} \ ,$$
$$\text{for} \quad s=m \ .$$

It has been pointed out by P. Rossi that when $s=1$ the beha-viour at the origin can be determined more precisely, namely as

$$(28) \quad h^2 = \left(\sum_{l=1}^{n} K_l \frac{r^l}{n} P_l(\cos \theta) \right)^2 \ ,$$

$$b^2 = (n + \sum_{l=1}^{n} \frac{\sin^2\theta r^{l+1}}{n(l+1)} K_l P_l'(\cos \theta))^2 \quad .$$

This result can be understood in our context, and be slightly generalized, by noting that if $s < m$ then $k \sim r^{m-s}$ and hence from the field equations (3) - (5)

(29) $$b_{,\alpha} + \rho \in_{\alpha\beta} h_{,\beta} = 0(r^{2m-2s-1}) \quad ,$$

which implies that, to the order shown, there exists a harmonic function

(30) $$U = \sum_{l=1}^{n+1} \gamma_l r^l P_l(\cos \theta) \quad ,$$

such that

(31) $$h^2 = (\frac{\partial U}{\partial z})^2 \quad , \quad \text{and} \quad b^2 = \rho^2 (\frac{\partial U}{\partial \rho})^2 \quad .$$

Alternatively, we may note that to the order in which k (and therefore ru_{α}) may be neglected the field equations (7) reduce to the two equations $\triangle h = 0$ and (16) and that these two equations are identically satisfied by (31) if U is harmonic.

A further symmetry which is carried by the field equations (3) - (5) is invariance under reflexion in the (xy)-plane, where h, k are even and b is odd with respect to this reflexion (in agreement with the boundary conditions as $r \rightarrow \infty$). If we impose this condition on the fields then all the integers, m, s, n above must be odd. In particular, only odd topological charges n are permitted. Since this symmetry agrees naturally with the equation and boundary conditions, it suggests that even topological charges might not be allowed [7] (under the assumptions of axial symmetry).

Finally we should mention that the results mentioned here are only very slightly modified when the assumption of mirror

symmetry is removed. Indeed, apart from the complication due to the introduction of four new fields, the results go through for axial symmetry alone. This generalization and a more detailed treatment of the above results, will be given in a forthcoming paper.

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