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The uniform bounded approximation property with respect to the Haar basis

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Let us recall that a Banach space X has the uniform bounded approximation property if and only if there exists a function $k : \mathbb{N} \rightarrow \mathbb{N}$ and a constant $C > 0$ such that for every finite-dimensional subspace $F \subset X$ there exists a subspace $E \subset X$ and a linear operator $S : X \rightarrow E$ such that $E \supset F$, $S|_F = \text{Id}_F$ and $\|S\| \leq C$, $\dim(E) \leq k(\dim(F))$.

Bożejko and Pełczyński [1] showed that the space $L^1(G)$, where G is a compact abelian group, has an analogue of the uniform bounded approximation property. Exactly, they proved this property for the translation invariant operators and translation invariant subspaces.

Now, we shall prove a similar property for the Haar basis in the space L^1 . Let us denote by \mathcal{W} the set of elements of the Haar basis and for $A \subset \mathcal{W}$ let us denote

$$L_A^1 = \left\{ f \in L^1 : f = \sum_{\chi \in \mathcal{W}} a_\chi \cdot \chi \text{ and } a_\chi = 0 \text{ for } \chi \notin A \right\}.$$

For $\chi \in \mathcal{W}$ let $I_\chi = \{x \in I : \chi(x) \neq 0\}$, where $I = [0, 1]$. We shall say that for $\chi_1, \chi_2 \in \mathcal{W}$ element χ_1 is less than element χ_2 ($\chi_1 < \chi_2$) iff $I_{\chi_1} \subset I_{\chi_2}$.

Theorem. For every finite subset $A \subset \mathcal{W}$ there exists a subset $B \subset \mathcal{W}$ and a linear operator $S : L_A^1 \rightarrow L_B^1$ such that $B \supset A$, $|B| \leq 9 \cdot |A|$, $S|_A = \text{Id}_A$ and $\|S\| \leq 5$.

Proof. We shall construct the set B at first. Let us add to the set A all elements χ of the set \mathcal{W} such that the both branches of elements less than χ contain elements of the set A . We obtain some set $B^* = H(B)$. We shall prove, by the induction method, that $|B^*| \leq 3 \cdot |A|$.

Actually, let the element χ_0 be minimal in the set B . We apply the inductive hypothesis to the set $B_0 = B \setminus \{\chi_0\}$. We have $|B_0^*| = |H(B_0)| \leq 3 \cdot |B_0|$. Moreover $H(B) \subset H(B_0) \cup \{\chi_0, \chi_1, \chi_2\}$ where χ_1 and χ_2 are the elements following in the order relation the element χ_0 . So

$$|B^*| \leq |B_0^*| + 3 \leq 3 \cdot |B_0| + 3 = 3 \cdot |B|.$$

Let $B^* = \bigcup_{j=1}^m B_j^*$ be a partition of the set B onto its components (maximal connected subsets in the order relation). Let H_j ($j=1, 2, \dots, m$) be the set of all elements of \mathcal{W} , which are greater than all elements of B_j . Let us take the set G_j of minimal elements of H_j and define the set $B = B^* \cup \bigcup_{j=1}^m G_j$. From the construction it follows that $|B| \leq 3 \cdot |B^*|$.

The set B has the following properties:

1. $B \supset A$, $|B| \leq 9 \cdot |A|$.
2. $B = \bigcup_{j=1}^m B_j$ where the sets B_j are disjoint and connected.
3. For every element B_j $j=1, 2, \dots, m$ both elements χ_1 and χ_2 following χ in the set \mathcal{W} belong to B_j or both these elements don't belong to B_j .
4. For every maximal element χ in B_j only one of two

branches of elements less than χ may contain elements of the set B .

Now we may define an operator S . Let $S(\chi_0) = 0$ for $\chi_0 \notin B$, $S(\chi_0) = \chi_0$ for $\chi_0 \in B_j$ not maximal in B_j . For $\chi_0 \in B_j$ maximal in B_j let us consider two branches of the elements of W less than χ_0 . Let us take this branch, which contains elements of the set B . If such branch does not exist, we choose an arbitrary branch.

Let us take an arbitrary element χ from the chosen

branch and $a = \int_I \chi_0 \cdot e_\chi dt$, where e_χ is the characteristic function of the interval I_χ . We define $S(\chi_0) = -\frac{F(e_\chi)}{a}$, where F is a projection $F(\sum_{\chi \in W} a_\chi \cdot \chi) = \sum_{\chi \in B_j \setminus \{\chi_0\}} a_\chi \cdot \chi$.

This definition does not depend on the element χ . Since the set B_j is connected, we have $\|F\| \leq 2$.

We shall prove that $\|S\| \leq 4$. For this purpose it is sufficient to show that $\|S(e_\chi)\| \leq 4 \cdot \|e_\chi\|$ for $\chi \in W$.

Let us denote $P = \{\chi^* \in B : \int_I \chi^* e_\chi dt \neq 0\}$. We have $S(e_\chi) = \sum_{j=1}^m S(F_j(e_\chi))$ where F_j (for $j=1,2,\dots,m$) are projections $F_j(\sum_{\chi \in W} a_\chi \cdot \chi) = \sum_{\chi \in B_j} a_\chi \cdot \chi$. Moreover, by the definition, $S(F_j(e_\chi)) = 0$ if there is an element χ in P greater than all elements of $B_j \cap P$. But the set P is a chain, so only for one number j it may happen that $S(F_j(e_\chi)) \neq 0$. So let $S(e_\chi) = S(F_{j_0}(e_\chi))$ and let χ_0 be maximal element of $P \cap B_{j_0}$ and $P : L^1 \rightarrow L^1$ be a projection

$$P\left(\sum_{\chi \in W} a_{\chi} \cdot \chi\right) = \sum_{\chi \in B_{j_0} \setminus \{\chi_0\}} a_{\chi} \cdot \chi. \text{ We have } S(F_{j_0}(e_{\chi})) =$$

$$= a \cdot S(\chi_0) + S(P(e_{\chi})) = a \cdot S(\chi_0) + P(e_{\chi}) \text{ where } a =$$

$$= \int_I \chi_0 \cdot e_{\chi} dt. \text{ So}$$

$$\|S(e_{\chi})\| = \|S(F_{j_0}(e_{\chi}))\| \leq a \cdot \|S(\chi_0)\| + \|P(e_{\chi})\| \leq 4 \cdot \|\chi_0\|$$

$$\text{since } \|P\| \leq 2, \quad a \cdot \|\chi_0\| \leq \|e_{\chi}\| \text{ and } \|S(\chi_0)\| \leq 2 \cdot \|\chi_0\|.$$

References

- [1] Bożejko M. and Pełczyński A.: An Analogue in Commutative Harmonic Analysis of the Uniform Bounded Approximation Property of Banach Space. Seminaire D'Analyse Fonctionnelle 1978-1979, Expose No. IX